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Solution of Interface Problems for the Linearised Benjamin–Bona–Mahony Equation on Star-Shaped Networks

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1 Abstract

This report presents an analytic solution to the Linearised Benjamin–Bona–Mahony (LBBM) equation on an arbitrary star-shaped network of semi-infinite leads. Using the unified transform method we solve the interface problem where only the initial wave heights at time $t = 0$ are known. This technique allows us to determine the full solution for the network at any time and location without the need for any intermediate steps. This work has wide applicability, with potential applications in modelling wave propagation in many different physical networks, facilitating asymptotic analysis of such systems, and enabling the development of efficient split-step algorithms for numerically solving the fully non-linear problem.

2 Introduction

Wave propagation models across networks are essential for studying a plethora of phenomena, such as water channels, structures of flexible elements [7], the cardiovascular system [10], and optical communication systems [13]. These systems are often mathematically represented as star-shaped networks with many leads connected to a central interface.

Designed as an improvement over the Korteweg–de Vries (KdV) equation governing shallow waves, the Benjamin–Bona–Mahony (BBM) equation offers substantial enhancements in regularity and well-posedness properties. Alongside its theoretical advantages, the BBM equation has been shown to far surpass shallow waves in its modelling potential, working well for an incredible variety of physical applications. While no analytic solution is known for the fully non-linear BBM problem, the linearised problem (LBBM) (1):

$$u_t - u_{xxt} + cu_x = 0, \quad c > 0 \tag{1}$$

offers a compelling approximation while retaining many well-posedness and stability features of the original system, and has additionally been shown to be well-posed on star-shaped networks [2, 15]. Furthermore, an analytic solution to the linearised system would facilitate the development of split-step algorithms for efficiently computing solutions of the non-linear problem [8].

The study of networks of dispersive long waves is highly motivated by a variety of applications, including in the study of linked flexible elements, channels, the human arterial tree, and optical communication systems.

Linked flexible elements such as networks of strings or cables are a widely used structural element due to their minimal cost and material requirements. For example, cable net structures are commonly used to cover large outdoor areas — such as stadiums and outdoor venues — from natural phenomena such as wind, rain, and snow. These structures are subject to the effects of wave propagation through stress and deformation, thus knowing the propagation of waves through these structures is of significant importance to their design. The BBM equation has been successfully applied to model these propagations in a network of linked flexible elements [7].

Canals and channels — both anthropogenic and natural — have floods, tides, tsunamis, and the wake from ships propagating through them regularly. These propagations are subject to the effects of the geometry of the channel, and as such can be amplified dramatically. For example, wave heights of over 40 metres were observed in the 2011 Tohoku tsunami in Japan due to the geometry of a natural channel [14]. Apart from extreme amplification events, the wake of ships and cycle of the tide can cause recurring amplification events, which causes erosion of the channel walls or banks. Knowing the propagation of waves through channels is therefore of significant importance for engineering applications. Further, studying how these wave phenomena move material would allow for better management of erosion and improve the design of channels. The movement of particles in waves governed by the BBM equation has been successfully modelled [9].

In the human arterial tree, the flow of blood can be modelled as a wave propagating through a network of arteries. The geometry of branching points is of significant importance, with geometric properties being associated with high risk of disease, aneurysms, and plaque formation [11, 1, 6]. For these reasons, knowing the long term behaviour of the flow of blood through the arterial tree could be of significant importance for medical applications. The BBM equation has been shown to capture the wave propagation phenomena present in the flow of blood through the arterial tree [4].

The BBM equation has been successfully used to model a variety of wave phenomena, including the aforementioned tsunami waves, linked flexible elements, shallow waves in channels, and the human arterial tree [12, 7, 3, 10]. For each of these applications, having an analytic solution of the linearised problem would provide several significant advantages beyond providing an improved numerical solution algorithm, including allowing for non-trivial interference effects to be separated and analysed, and long term behaviour to be studied via asymptotic analysis. These methods of analysis would enable better design of canals for tsunami protection, improved understanding of the human arterial tree, and better informed design of flexible elements in engineering applications.

This research project successfully utilises the unified transform method to solve the interface problem for the LBBM equation (1) on an arbitrary star-shaped network with $n + m$ leads. This solution is analytic, meaning that given appropriate initial data, the exact solution for the entire network at any time t can be determined. This enables asymptotic analysis of this solution to be performed to study the limiting behaviour of the solution [5], and for the development of improved numerical solution algorithms [8].

3 Acknowledgements

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4 Problem Definition

Governing the wave propagation across the network is the LBBM equation with an introduced positive global parameter c to model the speed of the wave:

$$u_t - u_{xxt} + cu_x = 0, \quad c > 0 \tag{1}$$

This is applied to a star-shaped network of n incoming semi-infinite leads $\{u_{-j}(x,t) : x \leq 0,\ t \in (0,T]\}$ and m outgoing semi-infinite leads $\{u_{+j}(x,t) : x \geq 0,\ t \in (0,T]\}$. The incoming leads are indexed $u_i(x,t)$, $i \in \{-n, \dots, -1\}$ and the outgoing leads are indexed $u_i(x,t)$, $i \in \{+1, \dots, +m\}$. This is illustrated in Figure 1.

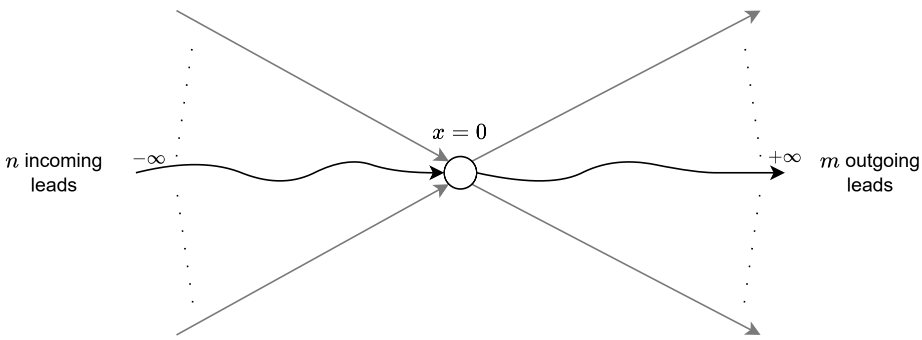

Figure 1: Diagram of the star-shaped network, showing the incoming and outgoing leads.

The network is subject to the boundary conditions:

$$b_j u_j(0,t) = b_i u_i(0,t) \qquad \text{for } j, i \in \{-n, \dots, -1, +1, \dots, +m\},\ t > 0,\ b_i \in \mathbb{C} \setminus \{0\} \tag{2}$$

$$\sum_{i=-n}^{-1} \beta_i \partial_x u_i(0,t) = \sum_{i=+1}^{+m} \beta_i \partial_x u_i(0,t) \qquad t > 0,\ \beta_i \in \mathbb{C} \setminus \{0\}. \tag{3}$$

We assume that the full initial state of the system at $t = 0$ is known, and that these data have sufficient regularity. In the following sections, we will derive the solution to the system. We show that the solution exists and is well-posed for all network parameters, including the singular case where $\alpha + \gamma = 0$; this is non-trivial and is explored in Appendix 11.4.

5 Global Relation Derivation

To apply the unified transform method, we seek to derive a global relation for the network. This is done by taking the Fourier transform of the LBBM (1) with respect to the spatial variable x separately for the outgoing and incoming lead regions $\{x : x \geq 0\}$ and $\{x : x \leq 0\}$ respectively (see Appendix 11.2 for an explicit derivation). It is found through this process that the system satisfies the following global relations:

$$e^{\omega t} U_{+j}(\lambda, t) = U_{+j}(\lambda, 0) + \frac{cG_{+j}(\lambda, t)}{1 + \lambda^2}, \qquad \mathrm{Im}(\lambda) \leq 0, \lambda \neq -i,\ +j \in \{+1, \dots, +m\} \tag{4}$$

$$e^{\omega t}U_{-j}(\lambda,t) = U_{-j}(\lambda,0) - \frac{cG_{-j}(\lambda,t)}{1+\lambda^2}, \qquad \text{Im}(\lambda)\geq 0, \lambda\neq i, \; -j\in\{-n,\dots,-1\} \tag{5}$$

with

$$\begin{aligned} f_{\pm j,0}(\omega,t) &= \int_0^t e^{\omega s}u_{\pm j}(0,s)\,ds, & f_{\pm j,1}(\omega,t) &= \int_0^t e^{\omega s}\partial_x u_{\pm j}(0,s)\,ds \\ G_{\pm j}(\lambda,t) &= f_{\pm j,0}(\omega,t) + i\lambda f_{\pm j,1}(\omega,t) & \omega &= \frac{ci\lambda}{1+\lambda^2} \end{aligned} \tag{6}$$

$$U_{+j}(\lambda,t) = (1+\lambda^2)\hat{u}_{+j}(\lambda,t) + \partial_x u_{+j}(0,t) + i\lambda u_{+j}(0,t)$$
$$U_{-j}(\lambda,t) = (1+\lambda^2)\hat{u}_{-j}(\lambda,t) - \partial_x u_{-j}(0,t) - i\lambda u_{-j}(0,t)$$

with $\hat{u}_{\pm j}(\lambda,t)$ denoting the Fourier transforms of the solutions of the leads at time t, these are given by:

$$\hat{u}_{+j}(\lambda,t) = \int_0^\infty e^{-i\lambda x}u_{+j}(x,t)\,dx, \qquad \hat{u}_{-j}(\lambda,t) = \int_{-\infty}^0 e^{-i\lambda x}u_{-j}(x,t)\,dx$$

Remark. *Despite being a function of λ, we denote $\omega(\lambda)$ simply by ω to highlight its invariance under the map $\lambda\to\frac{1}{\lambda}$. This abuse of notation is a common practice within the unified transform method.*

6 Solution Involving Unknown Boundary Data

6.1 Outgoing Lead Solution Involving Unknown Boundary Data

Solving the global relation (4) for $\hat{u}_{+j}(\lambda,t)$ and subsequently applying the inverse Fourier transform, the integral expression for the outgoing lead solution is found:

$$\begin{aligned} \hat{u}_{+j}(\lambda,t) = {}& \frac{ce^{-t\omega}G_{+j}(\lambda,t)}{(1+\lambda^2)^2} + e^{-t\omega}\hat{u}_{+j}(\lambda,0) + \frac{e^{-t\omega}}{1+\lambda^2}\left[\partial_x u_{+j}(0,0) + i\lambda u_{+j}(0,0)\right] \\ & - \frac{1}{1+\lambda^2}\left[\partial_x u_{+j}(0,t) + i\lambda u_{+j}(0,t)\right] \end{aligned}$$

by applying the inverse Fourier transform to both sides of the equation, we find that

$$\begin{aligned} \implies u_{+j}(x,t) = {}& \frac{1}{2\pi}\int_{-\infty}^\infty e^{i\lambda x - t\omega}\hat{u}_{+j}(\lambda,0)\,d\lambda + \frac{1}{2\pi}\oint_{\mathcal{C}^+}\frac{ce^{i\lambda x-t\omega}G_{+j}(\lambda,t)}{(1+\lambda^2)^2}\,d\lambda \\ & + \frac{1}{2\pi}\oint_{\mathcal{C}^+}\frac{e^{i\lambda x - t\omega}}{1+\lambda^2}\left[\partial_x u_{+j}(0,0) + i\lambda u_{+j}(0,0)\right]d\lambda + \frac{e^{-x}}{2}\left(u_{+j}(0,t) - \partial_x u_{+j}(0,t)\right), \end{aligned} \tag{7}$$

Remark. *Notice the unknown boundary data terms $u_{+j}(0,t)$, these present a problem in the singular case (as discussed further in 11.4) but are ultimately eliminated by using the interface conditions and the invariance of ω under the transformation $\lambda\to 1/\lambda$.*

where $\mathcal{C}^+$ is a CCW contour in the upper half plane enclosing i. This deformation is possible due to the decay of the integrands in the upper half plane and the analyticity of the functions involved; this is justified in Appendix 11.3.

6.2 Incoming Lead Solution Involving Unknown Boundary Data

Solving the global relation (5) for $\hat{u}_{-j}(\lambda, t)$ and subsequently applying the inverse Fourier transform, the integral expression for the incoming lead solution is found:

$$\begin{aligned}\hat{u}_{-j}(\lambda,t) = &-\frac{ce^{-t\omega}G_{-j}(\lambda,t)}{(1+\lambda^2)^2} + e^{-t\omega}\hat{u}_{-j}(\lambda,0) - \frac{e^{-t\omega}}{1+\lambda^2}\left[\partial_x u_{-j}(0,0) + i\lambda u_{-j}(0,0)\right] \\ &+ \frac{1}{1+\lambda^2}\left[\partial_x u_{-j}(0,t) + i\lambda u_{-j}(0,t)\right]\end{aligned}$$

by applying the inverse Fourier transform to both sides of the equation, we find that

$$\begin{aligned}\implies u_{-j}(x,t) = &\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x - t\omega}\hat{u}_{-j}(\lambda,0)\,d\lambda + \frac{1}{2\pi}\oint_{\mathcal{C}^-}\frac{ce^{i\lambda x - t\omega}G_{-j}(\lambda,t)}{(1+\lambda^2)^2}\,d\lambda \\ &+ \frac{1}{2\pi}\oint_{\mathcal{C}^-}\frac{e^{i\lambda x - t\omega}}{1+\lambda^2}\left[\partial_x u_{-j}(0,0) + i\lambda u_{-j}(0,0)\right]d\lambda + \frac{e^x}{2}\left(u_{-j}(0,t) + \partial_x u_{-j}(0,t)\right), \qquad (8)\end{aligned}$$

where $\mathcal{C}^-$ is a CCW contour in the lower half plane enclosing $-i$. Similarly to the outgoing lead solution, the deformation is possible due to the decay of the integrands in the lower half plane and their analyticity; this is justified in Appendix 11.3.

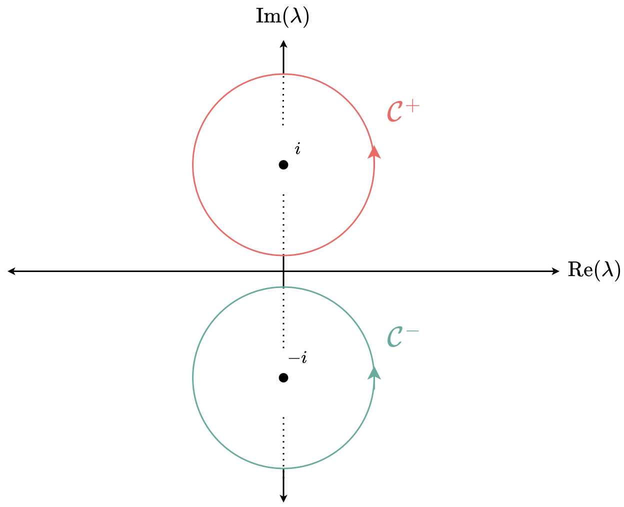

Figure 2: The contours $\mathcal{C}^+$ and $\mathcal{C}^-$.

7 Solving for The Unknown Boundary Data

The expressions for the outgoing and incoming lead solutions (7) (8) have multiple issues in their current form. Firstly, they contain unknown boundary data of the interface $x = 0$ at times $t > 0$. Secondly, it can be observed that the functions G_{+j} are currently defined for $\mathrm{Im}(\lambda) \le 0$, however they appear inside of integrands requiring $\mathrm{Im}(\lambda) \ge 0$. This problem is also similarly present for the functions G_{-j}. Almost remarkably, following the unified transform method the functions G_{+j} and G_{-j} can be extended to the necessary half planes, and in doing so the unknown boundary data will be completely eliminated by use of residue calculus. In order to extend the G functions, a system of equations valid entirely on one half of the plane must be constructed and then the transformed boundary data functions f solved for by inverting this system. To obtain this system, the invariance of ω under the transformation $\lambda \to 1/\lambda$ is used. Under this transformation, the global relations (4) (5) become:

$$e^{\omega t}U_{+j}(1/\lambda,t) = U_{+j}(1/\lambda,0) + \frac{cG_{+j}(1/\lambda,t)}{1+1/\lambda^2}, \quad \mathrm{Im}(\lambda) \geq 0, \lambda \neq i, \ +j \in \{+1,\dots,+m\}$$
$$e^{\omega t}U_{-j}(1/\lambda,t) = U_{-j}(1/\lambda,0) - \frac{cG_{-j}(1/\lambda,t)}{1+1/\lambda^2}, \quad \mathrm{Im}(\lambda) \leq 0, \lambda \neq -i, \ -j \in \{-n,\dots,-1\}$$

with

$$G_{\pm j}(1/\lambda,t) = f_{\pm j,0}(\omega,t) + \frac{i}{\lambda} f_{\pm j,1}(\omega,t). \tag{9}$$

The interface conditions (2) (3) are used to obtain relations between the transformed boundary data functions $f_{\pm j,\cdot}(\omega,t)$:

$$b_{\pm j}u_{\pm j}(0,t) = b_{-n}u_{-n}(0,t)$$
$$\implies f_{\pm j,0}(\omega,t) = \frac{b_{-n}}{b_{\pm j}} f_{-n,0}(\omega,t), \tag{10}$$

and

$$\partial_x u_{-n}(0,t) = \sum_{i=+1}^{+m} \frac{\beta_i}{\beta_{-n}} \partial_x u_{+i}(0,t) - \sum_{i=-(n-1)}^{-1} \frac{\beta_i}{\beta_{-n}} \partial_x u_i(0,t)$$
$$\implies f_{-n,1}(\omega,t) = \sum_{i=+1}^{+m} \frac{\beta_i}{\beta_{-n}} f_{+i,1}(\omega,t) - \sum_{i=-(n-1)}^{-1} \frac{\beta_i}{\beta_{-n}} f_{i,1}(\omega,t). \tag{11}$$

By now substituting the relations (10) (11) into the global relations valid specifically in the upper half plane, the following system of equations for $G_{\pm j}$ is found that is valid entirely on the upper half plane:

$$\begin{bmatrix} G_{-n}(\lambda,t) \\ G_{-(n-1)}(\lambda,t) \\ \vdots \\ G_{-1}(\lambda,t) \\ G_{+1}\left(\frac{1}{\lambda},t\right) \\ \vdots \\ G_{+(m-1)}\left(\frac{1}{\lambda},t\right) \\ G_{+m}\left(\frac{1}{\lambda},t\right) \end{bmatrix} = \begin{bmatrix} 1 & \frac{-i\lambda\beta_{-(n-1)}}{\beta_{-n}} & \cdots & \frac{-i\lambda\beta_{-1}}{\beta_{-n}} & \frac{i\lambda\beta_{+1}}{\beta_{-n}} & \cdots & \frac{i\lambda\beta_{+(m-1)}}{\beta_{-n}} & \frac{i\lambda\beta_{+m}}{\beta_{-n}} \\ \frac{b_{-n}}{b_{-(n-1)}} & i\lambda & & & & & & \\ \vdots & & \ddots & & & & & \\ \frac{b_{-n}}{b_{-1}} & & & i\lambda & & & & \\ \frac{b_{-n}}{b_{+1}} & & & & \frac{i}{\lambda} & & & \\ \vdots & & & & & \ddots & & \\ \frac{b_{-n}}{b_{+(m-1)}} & & & & & & \frac{i}{\lambda} & \\ \frac{b_{-n}}{b_{+m}} & & & & & & & \frac{i}{\lambda} \end{bmatrix} \begin{bmatrix} f_{-n,0}(\omega,t) \\ f_{-(n-1),1}(\omega,t) \\ \vdots \\ f_{-1,1}(\omega,t) \\ f_{+1,1}(\omega,t) \\ \vdots \\ f_{+(m-1),1}(\omega,t) \\ f_{+m,1}(\omega,t) \end{bmatrix}. \tag{12}$$

Written in block matrix form to aid in the calculation of the determinant and inversion of the system, (12) can be rewritten as $\vec{b} = A\vec{x}$, where:

$$A = \begin{bmatrix} 1 & \vec{u} \\ \vec{v}^T & D \end{bmatrix} \tag{13}$$

with

$$\vec{b} = \begin{bmatrix} G_{-n}(\lambda,t) & \cdots & G_{-1}(\lambda,t) & G_{+1}\left(\frac{1}{\lambda},t\right) & \cdots & G_{+m}\left(\frac{1}{\lambda},t\right)\end{bmatrix}^T$$

$$\vec{x} = \begin{bmatrix} f_{-n,0}(\omega,t) & f_{-(n-1),1}(\omega,t) & \cdots & f_{-1,1}(\omega,t) & f_{+1,1}(\omega,t) & \cdots & f_{+m,1}(\omega,t) & \end{bmatrix}^T$$

$$\vec{u} = \begin{bmatrix} \frac{-i\lambda\beta_{-(n-1)}}{\beta_{-n}} & \cdots & \frac{-i\lambda\beta_{-1}}{\beta_{-n}} & \frac{i\lambda\beta_{+1}}{\beta_{-n}} & \cdots & \frac{i\lambda\beta_{+m}}{\beta_{-n}}\end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} \frac{b_{-n}}{b_{-(n-1)}} & \cdots & \frac{b_{-n}}{b_{-1}} & \frac{b_{-n}}{b_{+1}} & \cdots & \frac{b_{-n}}{b_{+(m-1)}} & \frac{b_{-n}}{b_{+m}}\end{bmatrix}$$

$$D = \operatorname{diag}\left(\underbrace{i\lambda,\cdots,i\lambda}_{n-1 \text{ times}}, \underbrace{\frac{i}{\lambda},\cdots,\frac{i}{\lambda}}_{m \text{ times}}\right)$$

7.1 Conversion to Lower Triangular Form

By subtracting multiples of each row from the first, (13) is transformed to the lower triangular system $\vec{b}' = A'\vec{x}$, with

$$\vec{b}' = \begin{bmatrix} \left(\sum_{i=-n}^{-1} \frac{\beta_i}{\beta_{-n}} G_i(\lambda,t) - \lambda^2 \sum_{i=+1}^{+m} \frac{\beta_i}{\beta_{-n}} G_i\left(\frac{1}{\lambda},t\right)\right) & \cdots & G_{-1}(\lambda,t) & G_{+1}\left(\frac{1}{\lambda},t\right) & \cdots & G_{+m}\left(\frac{1}{\lambda},t\right)\end{bmatrix}^T$$

$$A' = \begin{bmatrix} \frac{b_{-n}}{\beta_{-n}}\left(\sum_{i=-n}^{-1} \frac{\beta_i}{b_i} - \lambda^2 \sum_{i=+1}^{+m} \frac{\beta_i}{b_i}\right) & 0 \\ \vec{v}^T & D \end{bmatrix} \tag{14}$$

7.2 Determinant Calculation and Singular Case

The lower triangular block matrix (14) makes it easy to calculate the determinant by multiplying the entries of the main diagonal D by the upper left entry. Because the only row operations used were subtracting multiples of the each row from the first, the determinant of the original matrix A is the same as the determinant of the lower triangular matrix A':

$$\begin{aligned} \det(A) = \det(A') &= \frac{b_{-n}}{\beta_{-n}}\left(\sum_{i=-n}^{-1} \frac{\beta_i}{b_i} - \lambda^2 \sum_{i=+1}^{+m} \frac{\beta_i}{b_i}\right)\left(i^{m+n-1}\lambda^{n-m-1}\right) \\ &= \frac{b_{-n}}{\beta_{-n}}\left(\alpha - \lambda^2\gamma\right)\left(i^{m+n-1}\lambda^{n-m-1}\right) \end{aligned}$$

with

$$\alpha = \sum_{i=-n}^{-1} \frac{\beta_i}{b_i}, \qquad \gamma = \sum_{i=+1}^{+m} \frac{\beta_i}{b_i}$$

Observing the determinant, we see that it vanishes at the critical points $\lambda = \pm i$ if and only if $\alpha + \gamma = 0$. We refer to this as the singular case. However, as shown in Appendix 11.4, the solution remains well-posed in this case.

7.3 Inverting the Matrix

Finally, by forward substitution, the inverse of the matrix A', and therefore the functions f valid entirely on the upper half plane are found to be

$$\begin{aligned}
f_{-n,0}(\omega,t) &= \frac{\sum_{i=-n}^{-1}\beta_i G_i(\lambda,t) - \lambda^2\sum_{i=+1}^{+m}\beta_i G_i\left(\frac{1}{\lambda},t\right)}{b_{-n}(\alpha-\lambda^2\gamma)} && \text{Im}(\lambda)\geq 0,\ \lambda\neq i,\\
f_{j,1}(\omega,t) &= \frac{G_j(\lambda,t)-\frac{b_{-n}}{b_j}f_{-n,0}(\omega,t)}{i\lambda} && \text{Im}(\lambda)\geq 0,\ \lambda\neq i,\ -j\in\{-(n-1),\dots,-1\},\\
f_{j,1}(\omega,t) &= \frac{\lambda\left(G_j\left(\frac{1}{\lambda},t\right)-\frac{b_{-n}}{b_j}f_{-n,0}(\omega,t)\right)}{i} && \text{Im}(\lambda)\geq 0,\ \lambda\neq i,\ +j\in\{+1,\dots,+m\}.
\end{aligned} \tag{15}$$

Extension of $G_{+j}(\lambda,t)$ to the Upper Half Plane

By substituting (15) into (6) and only considering the outgoing leads, it is found that

$$\begin{aligned}
G_{+j}(\lambda,t) &= \frac{b_{-n}}{b_{+j}}f_{-n,0}(\omega,t)+i\lambda f_{+j,1}(\omega,t)\\
&= \frac{(1-\lambda^2)}{(\alpha-\lambda^2\gamma)b_{+j}}\left[\sum_{i=-n}^{-1}\beta_i G_i(\lambda,t)\right]-\frac{\lambda^2(1-\lambda^2)}{(\alpha-\lambda^2\gamma)b_{+j}}\left[\sum_{i=+1}^{+m}\beta_i G_i\left(\frac{1}{\lambda},t\right)\right]\\
&\quad+\lambda^2 G_{+j}\left(\frac{1}{\lambda},t\right) \qquad \text{Im}(\lambda)\geq 0,\lambda\neq i
\end{aligned} \tag{16}$$

Extension of $G_{-j}(\lambda,t)$ to the Lower Half Plane

By first making a $\lambda\to 1/\lambda$ substitution in (15) to make them valid entirely on the lower half plane, the following set of solutions to the system valid on the lower half plane are found:

$$\begin{aligned}
f_{-n,0}(\omega,t) &= \frac{\lambda^2\sum_{i=-n}^{-1}\beta_i G_i\left(\frac{1}{\lambda},t\right) - \sum_{i=+1}^{+m}\beta_i G_i(\lambda,t)}{b_{-n}(\lambda^2\alpha-\gamma)} && \text{Im}(\lambda)\leq 0,\ \lambda\neq -i,\\
f_{j,1}(\omega,t) &= \frac{\lambda\left(G_j\left(\frac{1}{\lambda},t\right)-\frac{b_{-n}}{b_j}f_{-n,0}(\omega,t)\right)}{i} && \text{Im}(\lambda)\leq 0,\ \lambda\neq -i,\ -j\in\{-n,\dots,-1\},\\
f_{j,1}(\omega,t) &= \frac{G_j(\lambda,t)-\frac{b_{-n}}{b_j}f_{-n,0}(\omega,t)}{i\lambda} && \text{Im}(\lambda)\leq 0,\ \lambda\neq -i,\ +j\in\{+1,\dots,+m\},
\end{aligned} \tag{17}$$

then substituting these new identities (17) into (6) and only considering the incoming leads, it is found that

$$\begin{aligned}
G_{-j}(\lambda,t) &= \frac{b_{-n}}{b_{-j}}f_{-n,0}(\omega,t)+i\lambda f_{-j,1}(\omega,t)\\
&= -\frac{(1-\lambda^2)}{(\alpha\lambda^2-\gamma)b_{-j}}\left[\sum_{i=+1}^{+m}\beta_i G_i(\lambda,t)\right]+\frac{\lambda^2(1-\lambda^2)}{(\alpha\lambda^2-\gamma)b_{-j}}\left[\sum_{i=-n}^{-1}\beta_i G_i\left(\frac{1}{\lambda},t\right)\right]\\
&\quad+\lambda^2 G_{-j}\left(\frac{1}{\lambda},t\right) \qquad \text{Im}(\lambda)\leq 0,\lambda\neq -i
\end{aligned} \tag{18}$$

The case $G_{-n}(\lambda,t)$ must also be considered separately as it has a different identity to the other $G_{-j}(\lambda,t)$ functions, but it is shown nonetheless to be equivalent to the general $G_{-j}(\lambda,t)$ case extended to include $-n$:

$$
\begin{aligned}
G_{-n}(\lambda,t) &= f_{-n,0}(\omega,t) - i\lambda\left[\sum_{i=-(n-1)}^{-1}\frac{\beta_i}{\beta_{-n}}f_{i,1}(\omega,t)\right] + i\lambda\left[\sum_{i=+1}^{+m}\frac{\beta_i}{\beta_{-n}}f_{i,1}(\omega,t)\right]\\
&= f_{-n,0}(\omega,t) + i\lambda f_{-n,1}(\omega,t) && \text{See (11)}\\
&= \frac{b_{-n}}{b_{-n}}f_{-n,0}(\omega,t) + i\lambda f_{-n,1}(\omega,t),
\end{aligned}
$$

which is equivalent to (18).

Remark. *This is why the definition of $f_{j,1}(\omega,t)$ for the incoming leads (17) has been extended beyond the original system of equations to include $j=-n$ and not simply $\{-(n-1)\ldots-1\}$.*

7.4 Solving for Outgoing Leads ($\alpha+\gamma\neq 0$)

Note. *In this section, the solution is obtained while assuming $\alpha+\gamma\neq 0$; however it is shown in Appendix 11.4 that when $\alpha+\gamma=0$, the same solution is found.*

Substituting the found equation for $G_{+j}(\lambda,t)$ (16) into the outgoing lead solution involving unknown boundary data (7) we obtain

$$
\begin{aligned}
&\frac{1}{2\pi}\oint_{\mathcal{C}^+}\frac{ce^{i\lambda x-t\omega}G_{+j}(\lambda,t)}{(1+\lambda^2)^2}\,d\lambda = A_{+j}-B_{+j}+C_{+j}\\
&\qquad\Longrightarrow\ u_{+j}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\lambda x-t\omega}\hat{u}_{+j}(\lambda,0)\,d\lambda + A_{+j}-B_{+j}+C_{+j}\\
&\qquad\qquad + \frac{1}{2\pi}\oint_{\mathcal{C}^+}\frac{e^{i\lambda x-t\omega}}{1+\lambda^2}\left[\partial_x u_{+j}(0,0)+i\lambda u_{+j}(0,0)\right]d\lambda + \frac{e^{-x}}{2}\left(u_{+j}(0,t)-\partial_x u_{+j}(0,t)\right)
\end{aligned}
\tag{19}
$$

where

$$
\begin{aligned}
A_{+j} = {}&\frac{1}{2\pi}\oint_{\mathcal{C}^+}\frac{e^{i\lambda x-t\omega}(1-\lambda^2)}{(1+\lambda^2)(\alpha-\lambda^2\gamma)b_{+j}}\left[\sum_{i=-n}^{-1}-\beta_i(\partial_x u_i(0,0)+i\lambda u_i(0,0))\right] + \frac{e^{i\lambda x-t\omega}(1-\lambda^2)}{(\alpha-\lambda^2\gamma)b_{+j}}\left[\sum_{i=-n}^{-1}\beta_i\hat{u}_i(\lambda,0)\right]d\lambda\\
&+\frac{1}{2\pi}\oint_{\mathcal{C}^+}\frac{e^{i\lambda x}(1-\lambda^2)}{(1+\lambda^2)(\alpha-\lambda^2\gamma)b_{+j}}\left[\sum_{i=-n}^{-1}\beta_i\left(\partial_x u_i(0,t)+i\lambda u_i(0,t)\right)\right] - \frac{e^{i\lambda x}(1-\lambda^2)}{(\alpha-\lambda^2\gamma)b_{+j}}\left[\sum_{i=-n}^{-1}\beta_i\hat{u}_i(\lambda,t)\right]d\lambda,
\end{aligned}
$$

$$
\begin{aligned}
B_{+j} = {}&\frac{1}{2\pi}\oint_{\mathcal{C}^+}\frac{e^{i\lambda x-t\omega}(1-\lambda^2)}{(1+\lambda^2)(\alpha-\lambda^2\gamma)b_{+j}}\left[\sum_{i=+1}^{+m}-\beta_i\left(\partial_x u_i(0,0)+\frac{i}{\lambda}u_i(0,0)\right)\right] - \frac{e^{i\lambda x-t\omega}(1-\lambda^2)}{\lambda^2(\alpha-\lambda^2\gamma)b_{+j}}\left[\sum_{i=+1}^{+m}\beta_i\hat{u}\left(\frac{1}{\lambda},0\right)\right]d\lambda\\
&+\frac{1}{2\pi}\oint_{\mathcal{C}^+}\frac{e^{i\lambda x}(1-\lambda^2)}{(1+\lambda^2)(\alpha-\lambda^2\gamma)b_{+j}}\left[\sum_{i=+1}^{+m}\beta_i\left(\partial_x u_i(0,t)+\frac{i}{\lambda}u_i(0,t)\right)\right] + \frac{e^{i\lambda x}(1-\lambda^2)}{\lambda^2(\alpha-\lambda^2\gamma)b_{+j}}\left[\sum_{i=+1}^{+m}\beta_i\hat{u}_i\left(\frac{1}{\lambda},t\right)\right]d\lambda,
\end{aligned}
$$

$$
\begin{aligned}
C_{+j} = {}&\frac{1}{2\pi}\oint_{\mathcal{C}^+}\frac{e^{i\lambda x-t\omega}}{(1+\lambda^2)}\left[-\left(\partial_x u_{+j}(0,0)+\frac{i}{\lambda}u_{+j}(0,0)\right)\right] - \frac{e^{i\lambda x-t\omega}}{\lambda^2}\hat{u}_{+j}\left(\frac{1}{\lambda},0\right)d\lambda\\
&+\frac{1}{2\pi}\oint_{\mathcal{C}^+}\frac{e^{i\lambda x}}{(1+\lambda^2)}\left(\partial_x u_{+j}(0,t)+\frac{i}{\lambda}u_{+j}(0,t)\right) + \frac{e^{i\lambda x}}{\lambda^2}\hat{u}_{+j}\left(\frac{1}{\lambda},t\right)d\lambda.
\end{aligned}
$$

Recalling that $\alpha+\gamma\neq 0$ (and therefore the $(\alpha-\lambda^2\gamma)^{-1}$ terms cannot change the order of the poles at $\pm i$), by deforming the contour $\mathcal{C}^+$ *before* substituting the unknown boundary terms, the only singularity left inside the contour is guaranteed to be $\lambda = i$. Then, by application of Cauchy's integral formula and using the fact that the Fourier transform is analytic around $\lambda = i$, the auxiliary functions A_{+j}, B_{+j}, C_{+j} are further simplified to

$$\begin{aligned}
A_{+j} &= \frac{1}{2\pi}\oint_{\mathcal{C}^+} \frac{e^{i\lambda x - t\omega}(1-\lambda^2)}{(1+\lambda^2)(\alpha-\lambda^2\gamma)b_{+j}}\left[\sum_{i=-n}^{-1} -\beta_i(\partial_x u_i(0,0)+i\lambda u_i(0,0))\right] \\
&\quad + \frac{e^{i\lambda x - t\omega}(1-\lambda^2)}{(\alpha-\lambda^2\gamma)b_{+j}}\left[\sum_{i=-n}^{-1}\beta_i\hat{u}_i(\lambda,0)\right] d\lambda + \frac{e^{-x}}{(\alpha+\gamma)b_{+j}}\left[\sum_{i=-n}^{-1}\beta_i(\partial_x u_i(0,t)-u_i(0,t))\right], \\
B_{+j} &= \frac{1}{2\pi}\oint_{\mathcal{C}^+} \frac{e^{i\lambda x - t\omega}(1-\lambda^2)}{(1+\lambda^2)(\alpha-\lambda^2\gamma)b_{+j}}\left[\sum_{i=+1}^{+m} -\beta_i\left(\partial_x u_i(0,0)+\frac{i}{\lambda}u_i(0,0)\right)\right] \\
&\quad - \frac{e^{i\lambda x - t\omega}(1-\lambda^2)}{\lambda^2(\alpha-\lambda^2\gamma)b_{+j}}\left[\sum_{i=+1}^{+m}\beta_i\hat{u}\left(\frac{1}{\lambda},0\right)\right] d\lambda + \frac{e^{-x}}{(\alpha+\gamma)b_{+j}}\left[\sum_{i=+1}^{+m}\beta_i(\partial_x u_i(0,t)+u_i(0,t))\right], \\
C_{+j} &= \frac{1}{2\pi}\oint_{\mathcal{C}^+} \frac{e^{i\lambda x - t\omega}}{(1+\lambda^2)}\left[-\left(\partial_x u_{+j}(0,0)+\frac{i}{\lambda}u_{+j}(0,0)\right)\right] \\
&\quad - \frac{e^{i\lambda x - t\omega}}{\lambda^2}\hat{u}_{+j}\left(\frac{1}{\lambda},0\right) d\lambda + \frac{e^{-x}}{2}(\partial_x u_{+j}(0,t)+u_{+j}(0,t)).
\end{aligned}$$

Now, isolating the terms involving unknown data, denoted $A'_{+j}, B'_{+j}, C'_{+j}$, and using the interface conditions to simplify, it is found that

$$A'_{+j} - B'_{+j} + C'_{+j} = \frac{e^{-x}}{2}\left(\partial_x u_{+j}(0,t) - u_{+j}(0,t)\right),$$

which successfully cancels with the rest of the unknown data present in the solution.

7.5 Solving for Incoming Leads $(\alpha+\gamma\neq 0)$

Substituting the found equation for $G_{-j}(\lambda,t)$ (18) into the incoming lead solution involving unknown boundary data (8) we obtain

$$\begin{aligned}
\frac{1}{2\pi}\oint_{\mathcal{C}^-} \frac{ce^{i\lambda x - t\omega}G_{-j}(\lambda,t)}{(1+\lambda^2)^2}\,d\lambda &= -A_{-j}+B_{-j}+C_{-j} \\
u_{-j}(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x - t\omega}\hat{u}_{-j}(\lambda,0)\,d\lambda - A_{-j}+B_{-j}+C_{-j} \\
&\quad + \frac{1}{2\pi}\oint_{\mathcal{C}^-}\frac{e^{i\lambda x - t\omega}}{1+\lambda^2}\left[\partial_x u_{-j}(0,0)+i\lambda u_{-j}(0,0)\right] d\lambda + \frac{e^{x}}{2}\left(u_{-j}(0,t)+\partial_x u_{-j}(0,t)\right)
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
A_{-j} &= \frac{1}{2\pi}\oint_{\mathcal{C}^-} \frac{e^{i\lambda x - t\omega}(1-\lambda^2)}{(1+\lambda^2)(\alpha\lambda^2-\gamma)b_{-j}}\left[\sum_{i=+1}^{+m} -\beta_i(\partial_x u_i(0,0)+i\lambda u_i(0,0))\right] - \frac{e^{i\lambda x - t\omega}(1-\lambda^2)}{(\alpha\lambda^2-\gamma)b_{-j}}\left[\sum_{i=+1}^{+m}\beta_i\hat{u}_i(\lambda,0)\right] d\lambda \\
&\quad + \frac{1}{2\pi}\oint_{\mathcal{C}^-} \frac{e^{i\lambda x}(1-\lambda^2)}{(1+\lambda^2)(\alpha\lambda^2-\gamma)b_{-j}}\left[\sum_{i=+1}^{+m}\beta_i\left(\partial_x u_i(0,t)+i\lambda u_i(0,t)\right)\right] + \frac{e^{i\lambda x}(1-\lambda^2)}{(\alpha\lambda^2-\gamma)b_{-j}}\left[\sum_{i=+1}^{+m}\beta_i\hat{u}_i(\lambda,t)\right] d\lambda,
\end{aligned}$$

$$B_{-j} = \frac{1}{2\pi}\oint_{\mathcal{C}^-} \frac{e^{i\lambda x - t\omega}(1-\lambda^2)}{(1+\lambda^2)(\alpha\lambda^2-\gamma)b_{-j}}\left[\sum_{i=-n}^{-1} -\beta_i\left(\partial_x u_i(0,0) + \frac{i}{\lambda}u_i(0,0)\right)\right] + \frac{e^{i\lambda x - t\omega}(1-\lambda^2)}{\lambda^2(\alpha\lambda^2-\gamma)b_{-j}}\left[\sum_{i=-n}^{-1}\beta_i\hat{u}\left(\frac{1}{\lambda},0\right)\right] d\lambda$$

$$+ \frac{1}{2\pi}\oint_{\mathcal{C}^-} \frac{e^{i\lambda x}(1-\lambda^2)}{(1+\lambda^2)(\alpha\lambda^2-\gamma)b_{-j}}\left[\sum_{i=-n}^{-1}\beta_i\left(\partial_x u_i(0,t) + \frac{i}{\lambda}u_i(0,t)\right)\right] - \frac{e^{i\lambda x}(1-\lambda^2)}{\lambda^2(\alpha\lambda^2-\gamma)b_{-j}}\left[\sum_{i=-n}^{-1}\beta_i\hat{u}_i\left(\frac{1}{\lambda},t\right)\right] d\lambda,$$

$$C_{-j} = \frac{1}{2\pi}\oint_{\mathcal{C}^-} \frac{e^{i\lambda x - t\omega}}{1+\lambda^2}\left[-\left(\partial_x u_{-j}(0,0) + \frac{i}{\lambda}u_{-j}(0,0)\right)\right] + \frac{e^{i\lambda x - t\omega}}{\lambda^2}\hat{u}_{-j}\left(\frac{1}{\lambda},0\right) d\lambda$$

$$+ \frac{1}{2\pi}\oint_{\mathcal{C}^-} \frac{e^{i\lambda x}}{1+\lambda^2}\left(\partial_x u_{-j}(0,t) + \frac{i}{\lambda}u_{-j}(0,t)\right) - \frac{e^{i\lambda x}}{\lambda^2}\hat{u}_{-j}\left(\frac{1}{\lambda},t\right) d\lambda.$$

Recalling that $\alpha + \gamma \neq 0$, by deforming the contour $\mathcal{C}^-$ *before* substituting the unknown boundary terms, the only singularity left inside the contour is guaranteed to be $\lambda = -i$. Then, by application of Cauchy's integral formula and using the fact that the Fourier transform is analytic around $\lambda = -i$, the auxiliary functions A_{-j}, B_{-j}, C_{-j} are further simplified to

$$A_{-j} = \frac{1}{2\pi}\oint_{\mathcal{C}^-} \frac{e^{i\lambda x - t\omega}(1-\lambda^2)}{(1+\lambda^2)(\alpha\lambda^2-\gamma)b_{-j}}\left[\sum_{i=+1}^{+m} -\beta_i(\partial_x u_i(0,0) + i\lambda u_i(0,0))\right]$$

$$- \frac{e^{i\lambda x - t\omega}(1-\lambda^2)}{(\alpha\lambda^2-\gamma)b_{-j}}\left[\sum_{i=+1}^{+m}\beta_i\hat{u}_i(\lambda,0)\right] d\lambda + \frac{e^x}{(\alpha+\gamma)b_{-j}}\left[\sum_{i=+1}^{+m}\beta_i(\partial_x u_i(0,t) + u_i(0,t))\right],$$

$$B_{-j} = \frac{1}{2\pi}\oint_{\mathcal{C}^-} \frac{e^{i\lambda x - t\omega}(1-\lambda^2)}{(1+\lambda^2)(\alpha\lambda^2-\gamma)b_{-j}}\left[\sum_{i=-n}^{-1} -\beta_i\left(\partial_x u_i(0,0) + \frac{i}{\lambda}u_i(0,0)\right)\right]$$

$$+ \frac{e^{i\lambda x - t\omega}(1-\lambda^2)}{\lambda^2(\alpha\lambda^2-\gamma)b_{-j}}\left[\sum_{i=-n}^{-1}\beta_i\hat{u}\left(\frac{1}{\lambda},0\right)\right] d\lambda + \frac{e^x}{(\alpha+\gamma)b_{-j}}\left[\sum_{i=-n}^{-1}\beta_i(\partial_x u_i(0,t) - u_i(0,t))\right],$$

$$C_{-j} = \frac{1}{2\pi}\oint_{\mathcal{C}^-} \frac{e^{i\lambda x - t\omega}}{1+\lambda^2}\left[-\left(\partial_x u_{-j}(0,0) + \frac{i}{\lambda}u_{-j}(0,0)\right)\right]$$

$$+ \frac{e^{i\lambda x - t\omega}}{\lambda^2}\hat{u}_{-j}\left(\frac{1}{\lambda},0\right) d\lambda - \frac{e^x}{2}(\partial_x u_{-j}(0,t) - u_{-j}(0,t)).$$

Now, isolating the terms involving unknown data, denoted $A'_{-j}, B'_{-j}, C'_{-j}$, and using the interface conditions to simplify, it is found that

$$-A'_{+j} + B'_{+j} + C'_{+j} = \frac{e^x}{2}\left(\partial_x u_{-j}(0,t) + u_{-j}(0,t)\right),$$

which successfully cancels with the rest of the unknown data present in the solution.

8 Solution

We recall for the reader's convenience the definitions of α, γ, and ω:

$$\alpha = \sum_{i=-n}^{-1}\frac{\beta_i}{b_i}, \qquad \gamma = \sum_{i=+1}^{+m}\frac{\beta_i}{b_i}, \qquad \omega = \frac{ci\lambda}{1+\lambda^2}$$

8.1 Unsimplified Solutions

Expanding terms and simplifying the above expression for the outgoing lead solution (19), and then further carefully considering the case when $\alpha + \gamma = 0$ (Appendix 11.4), it is found that for outgoing leads the solution for all α, γ is given by

$$
\begin{aligned}
u_{+j}(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\lambda x - t\omega}\hat{u}_{+j}(\lambda,0)\,d\lambda + \frac{1}{2\pi}\oint_{\mathcal{C}+} \frac{e^{i\lambda x-t\omega}(\partial_x u_{+j}(0,0)+i\lambda u_{+j}(0,0))}{1+\lambda^2}\,d\lambda \\
&+ \frac{1}{2\pi}\oint_{\mathcal{C}+} \frac{e^{i\lambda x-t\omega}(1-\lambda^2)}{(1+\lambda^2)(\alpha-\lambda^2\gamma)b_{+j}}\left[\sum_{i=-n}^{-1}(-\beta_i)(\partial_x u_i(0,0)+i\lambda u_i(0,0))\right] + \frac{e^{i\lambda x-t\omega}(1-\lambda^2)}{(\alpha-\lambda^2\gamma)b_{+j}}\left[\sum_{i=-n}^{-1}\beta_i\hat{u}_i(\lambda,0)\right]d\lambda \\
&- \frac{1}{2\pi}\oint_{\mathcal{C}+} \frac{e^{i\lambda x-t\omega}(1-\lambda^2)}{(1+\lambda^2)(\alpha-\lambda^2\gamma)b_{+j}}\left[\sum_{i=+1}^{+m}(-\beta_i)(\partial_x u_i(0,0)+\frac{i}{\lambda}u_i(0,0))\right] - \frac{e^{i\lambda x-t\omega}(1-\lambda^2)}{(\alpha-\lambda^2\gamma)b_{+j}}\left[\sum_{i=+1}^{+m}\beta_i\hat{u}_i\left(\frac{1}{\lambda},0\right)\right]d\lambda \\
&+ \frac{1}{2\pi}\oint_{\mathcal{C}+} \frac{e^{i\lambda x-t\omega}}{1+\lambda^2}\left[-(\partial_x u_{+j}(0,0)+\frac{i}{\lambda}u_{+j}(0,0))\right] - \frac{e^{i\lambda x-t\omega}}{\lambda^2}\hat{u}_{+j}\left(\frac{1}{\lambda},0\right)d\lambda,
\end{aligned}
$$

where $\mathcal{C}^+$ is a CCW contour in the upper half plane enclosing i.

And then similarly for the incoming leads, the solution is found to be

$$
\begin{aligned}
u_{-j}(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\lambda x - t\omega}\hat{u}_{-j}(\lambda,0)\,d\lambda + \frac{1}{2\pi}\oint_{\mathcal{C}-} \frac{e^{i\lambda x-t\omega}}{1+\lambda^2}(\partial_x u_{-j}(0,0)+i\lambda u_{-j}(0,0))\,d\lambda \\
&- \frac{1}{2\pi}\oint_{\mathcal{C}-} \frac{e^{i\lambda x-t\omega}(1-\lambda^2)}{(1+\lambda^2)(\lambda^2\alpha-\gamma)b_{-j}}\left[\sum_{i=+1}^{+m}(-\beta_i)(\partial_x u_i(0,0)+i\lambda u_i(0,0))\right] - \frac{e^{i\lambda x-t\omega}(1-\lambda^2)}{(\lambda^2\alpha-\gamma)b_{-j}}\left[\sum_{i=+1}^{+m}\beta_i\hat{u}_i(\lambda,0)\right]d\lambda \\
&+ \frac{1}{2\pi}\oint_{\mathcal{C}-} \frac{e^{i\lambda x-t\omega}(1-\lambda^2)}{(1+\lambda^2)(\lambda^2\alpha-\gamma)b_{-j}}\left[\sum_{i=-n}^{-1}(-\beta_i)(\partial_x u_i(0,0)+\frac{i}{\lambda}u_i(0,0))\right] + \frac{e^{i\lambda x-t\omega}(1-\lambda^2)}{(\lambda^2\alpha-\gamma)b_{-j}}\left[\sum_{i=-n}^{-1}\beta_i\hat{u}_i\left(\frac{1}{\lambda},0\right)\right]d\lambda \\
&+ \frac{1}{2\pi}\oint_{\mathcal{C}-} \frac{e^{i\lambda x-t\omega}}{1+\lambda^2}(-1)\left[\partial_x u_{-j}(0,0)+\frac{i}{\lambda}u_{-j}(0,0)\right] + \frac{e^{i\lambda x-t\omega}}{\lambda^2}\hat{u}_{-j}\left(\frac{1}{\lambda},0\right)d\lambda,
\end{aligned}
$$

where $\mathcal{C}^-$ is a CCW contour in the lower half plane enclosing $-i$.

These solutions can be further simplified by considering the interface conditions (3) relating to the partial derivatives of the initial data. By using these relations, it is found that the outgoing leads can be further reduced to

$$
\begin{aligned}
u_{+j}(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\lambda x - t\omega}\hat{u}_{+j}(\lambda,0)\,d\lambda - \frac{1}{2\pi}\oint_{\mathcal{C}+} \frac{ie^{i\lambda x-t\omega}(1-\lambda^2)}{\lambda(1+\lambda^2)}(u_{+j}(0,0))\,d\lambda - \frac{1}{2\pi}\oint_{\mathcal{C}+}\frac{e^{i\lambda x-t\omega}}{\lambda^2}\hat{u}_{+j}\left(\frac{1}{\lambda},0\right)d\lambda \\
&- \frac{1}{2\pi}\oint_{\mathcal{C}+} \frac{ie^{i\lambda x-t\omega}(1-\lambda^2)\lambda}{(1+\lambda^2)(\alpha-\lambda^2\gamma)b_{+j}}\left[\sum_{i=-n}^{-1}\beta_i u_i(0,0)\right] - \frac{e^{i\lambda x-t\omega}(1-\lambda^2)}{(\alpha-\lambda^2\gamma)b_{+j}}\left[\sum_{i=-n}^{-1}\beta_i\hat{u}_i(\lambda,0)\right]d\lambda \\
&+ \frac{1}{2\pi}\oint_{\mathcal{C}+} \frac{ie^{i\lambda x-t\omega}(1-\lambda^2)}{(1+\lambda^2)(\alpha-\lambda^2\gamma)\lambda b_{+j}}\left[\sum_{i=+1}^{+m}\beta_i u_i(0,0)\right] + \frac{e^{i\lambda x-t\omega}(1-\lambda^2)}{(\alpha-\lambda^2\gamma)\lambda^2 b_{+j}}\left[\sum_{i=+1}^{+m}\beta_i\hat{u}_i\left(\frac{1}{\lambda},0\right)\right]d\lambda, \quad (21)
\end{aligned}
$$

and similarly for the incoming leads, it is found that

$$
\begin{aligned}
u_{-j}(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\lambda x - t\omega}\hat{u}_{-j}(\lambda,0)\,d\lambda - \frac{1}{2\pi}\oint_{\mathcal{C}-} \frac{ie^{i\lambda x-t\omega}(1-\lambda^2)}{\lambda(1+\lambda^2)}(u_{-j}(0,0))\,d\lambda + \frac{1}{2\pi}\oint_{\mathcal{C}-}\frac{e^{i\lambda x-t\omega}}{\lambda^2}\hat{u}_{-j}\left(\frac{1}{\lambda},0\right)d\lambda \\
&+ \frac{1}{2\pi}\oint_{\mathcal{C}-} \frac{ie^{i\lambda x-t\omega}(1-\lambda^2)\lambda}{(1+\lambda^2)(\lambda^2\alpha-\gamma)b_{-j}}\left[\sum_{i=+1}^{+m}\beta_i u_i(0,0)\right] + \frac{e^{i\lambda x-t\omega}(1-\lambda^2)}{(\lambda^2\alpha-\gamma)b_{-j}}\left[\sum_{i=+1}^{+m}\beta_i\hat{u}_i(\lambda,0)\right]d\lambda \\
&- \frac{1}{2\pi}\oint_{\mathcal{C}-} \frac{ie^{i\lambda x-t\omega}(1-\lambda^2)}{(1+\lambda^2)(\lambda^2\alpha-\gamma)\lambda b_{-j}}\left[\sum_{i=-n}^{-1}\beta_i u_i(0,0)\right] - \frac{e^{i\lambda x-t\omega}(1-\lambda^2)}{(\lambda^2\alpha-\gamma)\lambda^2 b_{-j}}\left[\sum_{i=-n}^{-1}\beta_i\hat{u}_i\left(\frac{1}{\lambda},0\right)\right]d\lambda,
\end{aligned}
\quad (22)
$$

Remark. *Notice that there are now no terms involving the spatial derivatives of the initial data despite their presence in the interface conditions! This is an attractive property for numerical evaluation.*

These solutions can be further simplified by considering the interface conditions (2),

$$\text{Note that } \frac{\beta_i}{b_{\pm j}} u_i(0,0) = \frac{\beta_i}{b_i} u_{\pm j}(0,0)$$

$$\implies \frac{ie^{i\lambda x - t\omega}(1-\lambda^2)}{(1+\lambda^2)(\alpha-\lambda^2\gamma)b_{+j}} \left[-\lambda \sum_{i=-n}^{-1} \beta_i u_i(0,0) + \frac{1}{\lambda} \sum_{i=+1}^{+m} \beta_i u_i(0,0) \right] = \frac{ie^{i\lambda x - t\omega}(1-\lambda^2)(\gamma - \alpha\lambda^2)}{(1+\lambda^2)(\alpha-\lambda^2\gamma)\lambda} (u_{+j}(0,0)), \tag{23}$$

and so by plugging (23) into (21), it is found that the outgoing lead solution can be simplified to

$$u_{+j}(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\lambda x - t\omega} \hat{u}_{+j}(\lambda,0)\, d\lambda + \frac{1}{2\pi} \oint_{\mathcal{C}+} \frac{ie^{i\lambda x - t\omega}(1-\lambda^2)(\gamma-\alpha)}{\lambda(\alpha-\lambda^2\gamma)} (u_{+j}(0,0))\, d\lambda - \frac{1}{2\pi} \oint_{\mathcal{C}+} \frac{e^{i\lambda x - t\omega}}{\lambda^2} \hat{u}_{+j}\left(\frac{1}{\lambda},0\right) d\lambda$$
$$+ \frac{1}{2\pi} \oint_{\mathcal{C}+} \frac{e^{i\lambda x - t\omega}(1-\lambda^2)}{(\alpha-\lambda^2\gamma)b_{+j}} \left[\sum_{i=-n}^{-1} \beta_i \hat{u}_i(\lambda,0) \right] + \frac{e^{i\lambda x - t\omega}(1-\lambda^2)}{(\alpha-\lambda^2\gamma)\lambda^2 b_{+j}} \left[\sum_{i=+1}^{+m} \beta_i \hat{u}_i\left(\frac{1}{\lambda},0\right) \right] d\lambda, \tag{24}$$

similarly for the incoming lead solution, it is found that

$$u_{-j}(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\lambda x - t\omega} \hat{u}_{-j}(\lambda,0)\, d\lambda + \frac{1}{2\pi} \oint_{\mathcal{C}-} \frac{ie^{i\lambda x - t\omega}(1-\lambda^2)(\gamma-\alpha)}{\lambda(\lambda^2\alpha-\gamma)} (u_{-j}(0,0))\, d\lambda + \frac{1}{2\pi} \oint_{\mathcal{C}-} \frac{e^{i\lambda x - t\omega}}{\lambda^2} \hat{u}_{-j}\left(\frac{1}{\lambda},0\right) d\lambda$$
$$+ \frac{1}{2\pi} \oint_{\mathcal{C}-} \frac{e^{i\lambda x - t\omega}(1-\lambda^2)}{(\lambda^2\alpha-\gamma)b_{-j}} \left[\sum_{i=+1}^{+m} \beta_i \hat{u}_i(\lambda,0) \right] + \frac{e^{i\lambda x - t\omega}(1-\lambda^2)}{(\lambda^2\alpha-\gamma)\lambda^2 b_{-j}} \left[\sum_{i=-n}^{-1} \beta_i \hat{u}_i\left(\frac{1}{\lambda},0\right) \right] d\lambda, \tag{25}$$

8.2 General Simplified Solutions

By now grouping the contour integrals and factoring out the common coefficients, the solution for the outgoing leads condenses to

$$\begin{aligned} u_{+j}(x,t) = {} & \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\lambda x - t\omega} \hat{u}_{+j}(\lambda,0)\, d\lambda \\ & + \frac{1}{2\pi} \oint_{\mathcal{C}+} e^{i\lambda x - t\omega} \Bigg[-\frac{1}{\lambda^2} \hat{u}_{+j}\left(\frac{1}{\lambda},0\right) + \left(\frac{1-\lambda^2}{\alpha-\lambda^2\gamma}\right) \Bigg(\frac{i(\gamma-\alpha)}{\lambda} (u_{+j}(0,0)) \\ & \qquad + \frac{1}{b_{+j}} \left[\sum_{i=-n}^{-1} \beta_i \hat{u}_i(\lambda,0) \right] + \frac{1}{\lambda^2 b_{+j}} \left[\sum_{i=+1}^{+m} \beta_i \hat{u}_i\left(\frac{1}{\lambda},0\right) \right] \Bigg) \Bigg] d\lambda, \end{aligned} \tag{26}$$

Applying the exact same grouping to the incoming leads yields a highly symmetric counterpart

$$\begin{aligned} u_{-j}(x,t) = {} & \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\lambda x - t\omega} \hat{u}_{-j}(\lambda,0)\, d\lambda \\ & + \frac{1}{2\pi} \oint_{\mathcal{C}-} e^{i\lambda x - t\omega} \Bigg[\frac{1}{\lambda^2} \hat{u}_{-j}\left(\frac{1}{\lambda},0\right) + \left(\frac{1-\lambda^2}{\lambda^2\alpha-\gamma}\right) \Bigg(\frac{i(\gamma-\alpha)}{\lambda} (u_{-j}(0,0)) \\ & \qquad + \frac{1}{b_{-j}} \left[\sum_{i=+1}^{+m} \beta_i \hat{u}_i(\lambda,0) \right] + \frac{1}{\lambda^2 b_{-j}} \left[\sum_{i=-n}^{-1} \beta_i \hat{u}_i\left(\frac{1}{\lambda},0\right) \right] \Bigg) \Bigg] d\lambda. \end{aligned} \tag{27}$$

9 Final Discussion and Conclusion

9.1 Results

We successfully applied the unified transform method to the Linearised Benjamin–Bona–Mahony (LBBM) equation on an arbitrary star-shaped network. This approach yielded explicit, analytic solutions for the waves on both incoming and outgoing leads, as presented in equations (26) and (27). These formulae allow for the determination of the state of the network at any time $t > 0$ given only the initial conditions, without requiring intermediate time-stepping. Additionally, we identified the condition $\alpha + \gamma = 0$ under which the problem becomes singular, and demonstrated that the solution nonetheless remains well-posed (see Appendix 11.4). These results allow for the creation of a split-step algorithm to efficiently compute numerical solutions to the fully non-linear BBM equation [8].

9.2 Future Work

Direct numerical integration of the solution encounters exponential instability for large times t, as illustrated in Figure 3 (see Appendix 11.5 for details). This could be resolved utilising numerical and analytical methods. Filon quadrature — a numerical integration scheme for highly oscillatory integrals — would perform well numerically. Alternatively, by analytically computing residue contributions and evaluating series expansions, a series representation could be constructed avoiding the need for numerical integration altogether. Either of the aforementioned methods would be an ideal alternative to naïve quadrature methods. And, while the numerical instability does not present challenges for asymptotic analysis or for a split step algorithm, it increases the computational complexity of the solution by requiring a change in working precision for large values of $t \gg 0$.

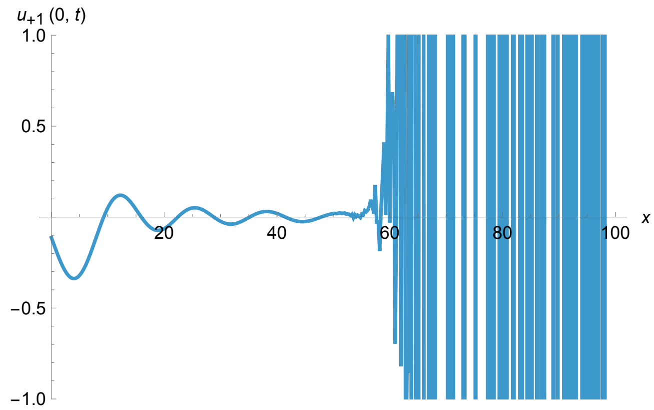

Figure 3: Plot of $u_{+1}(0, t)$ with same initial conditions as in Figure 6, showing the numerical instability of the solution over time.

10 Statement of Authorship

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11 Appendix

11.1 Numerical Simulations

The solutions were simulated numerically in Mathematica using a variety of initial and boundary conditions.

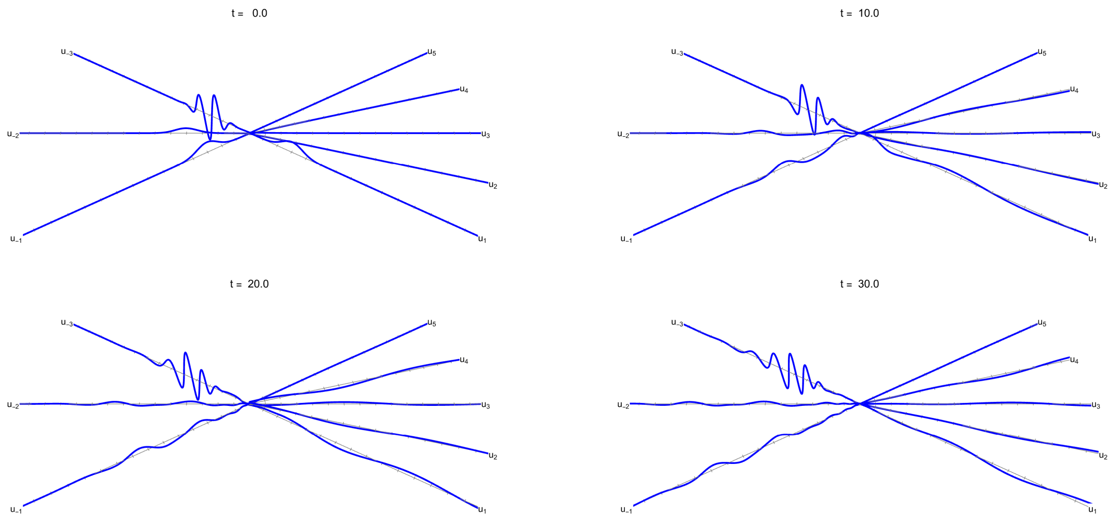

Figure 4: Numerical simulation for the case $n = 3, m = 5$.

As we placed no constraints on the values of b_i and β_i outside of being non-zero, the interface conditions can accept complex coefficients and effectively 'rotate' waves through the complex plane as they pass through the interface. For example, here is the propagation of a gaussian through the interface with coefficients

$$b_{+1} = i, \quad b_{-1} = 1, \quad \beta_{+1} = i, \quad \beta_{-1} = 1.$$

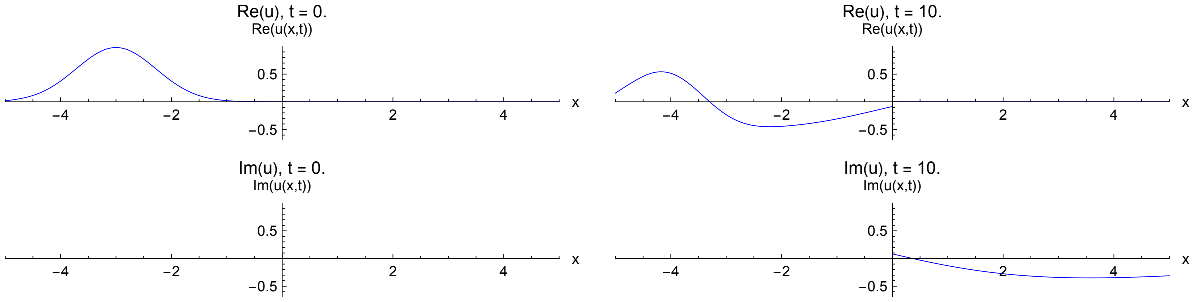

Figure 5: Numerical simulation for the case $n = 1, m = 1$ with complex coefficients. The leads u_{-1} and u_{+1} have been stitched together at the interface $x = 0$ to complete the real line.

Finally, exploring the singular case with 2 leads and coefficients

$$b_{+1} = 1, \quad b_{-1} = -1, \quad \beta_{+1} = 1, \quad \beta_{-1} = 1.$$

such that $\alpha + \gamma = 0$,

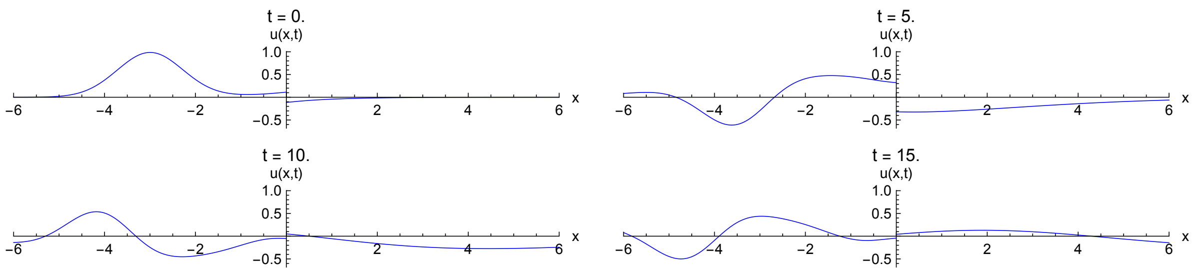

Figure 6: Numerical simulation for the case $n = 1, m = 1$ with coefficients such that $\alpha + \gamma = 0$. The leads u_{-1} and u_{+1} have been stitched together at the interface $x = 0$ to complete the real line.

11.2 Global Relation Derivations

We seek to use the LBBM equation to derive an identity for the Fourier transform of the solution in terms of initial and unknown data. This identity is known as the *global relation*.

Outgoing Global Relation Derivation

Assuming $x \geq 0$:

$$\begin{aligned}
0 &= [\partial_t - \partial_{xxt} + c\partial_x]\, u(x,t) \\
\implies 0 &= \int_0^\infty e^{-i\lambda x}\, [\partial_t - \partial_{xxt} + c\partial_x]\, u(x,t)\, dx \\
&= \widehat{u_t}(\lambda,t) + c\widehat{u_x}(\lambda,t) - \widehat{u_{xxt}}(\lambda,t) \\
&= \widehat{u_t}(\lambda,t) + c\widehat{u_x}(\lambda,t) - \Big[e^{-i\lambda x} u_{xt}(x,t) + i\lambda e^{-i\lambda x} u_t(x,t)\Big]_{x=0}^{x=\infty} + \lambda^2 \widehat{u_t}(\lambda,t),
\end{aligned}$$

if $\lambda = a + ib$, then $e^{-i\lambda x} = e^{-iax}e^{bx}$. For the boundary terms to vanish at $x = \infty$ and for the half-line Fourier transforms to converge, we require $\mathrm{Im}(\lambda) \le 0$.

$$= \widehat{u_t}(\lambda,t)(1+\lambda^2) - cu(0,t) + ci\lambda\hat{u}(\lambda,t) + i\lambda u_t(0,t) + u_{xt}(0,t) \tag{28}$$

$$\implies \widehat{u_t}(\lambda,t) + \frac{ci\lambda}{1+\lambda^2}\hat{u}(\lambda,t) = \frac{1}{1+\lambda^2}\left(cu(0,t) - u_{xt}(0,t) - i\lambda u_t(0,t)\right) \qquad (\lambda \ne \pm i),$$

by using the integrating factor $\mu(t) = \exp\left(\frac{ci\lambda}{1+\lambda^2}t\right) = \exp(\omega t)$,

$$\begin{aligned}
\implies e^{\omega t}\hat{u}(\lambda,t) &= \frac{1}{1+\lambda^2}\int_0^t e^{\omega s}\left(cu(0,s) - u_{xs}(0,s) - i\lambda u_s(0,s)\right) ds + \hat{u}(\lambda,0)\, ds \\
\implies e^{\omega t}U(\lambda,t) &= U(\lambda,0) + \frac{c\int_0^t \left[e^{\omega s}u(0,s) + i\lambda e^{\omega s}u_x(0,s)\right] ds}{1+\lambda^2} \\
&= U(\lambda,0) + \frac{cG(\lambda,t)}{1+\lambda^2}
\end{aligned}$$

with

$$f_0(\omega,t) = \int_0^t e^{\omega s}u(0,s)\,ds, \qquad f_1(\omega,t) = \int_0^t e^{\omega s}u_x(0,s)\,ds$$

$$G(\lambda,t) = f_0(\omega,t) + i\lambda f_1(\omega,t) \qquad \omega = \frac{ci\lambda}{1+\lambda^2}$$

$$U(\lambda,t) = (1+\lambda^2)\hat{u}(\lambda,t) + u_x(0,t) + i\lambda u(0,t).$$

Incoming Global Relation Derivation

Assuming $x \le 0$:

$$\begin{aligned}
0 &= \left[\partial_t - \partial_{xxt} + c\partial_x\right]u(x,t)\,dx \\
\implies 0 &= \int_{-\infty}^0 e^{-i\lambda x}\left[\partial_t - \partial_{xxt} + c\partial_x\right]u(x,t)\,dx \\
&= \widehat{u_t}(\lambda,t) + c\widehat{u_x}(\lambda,t) - \widehat{u_{xxt}}(\lambda,t) \\
&= \widehat{u_t}(\lambda,t) + c\widehat{u_x}(\lambda,t) - \left[e^{-i\lambda x}u_{xt}(x,t) + i\lambda e^{-i\lambda x}u_t(x,t)\right]_{x=-\infty}^{x=0} + \lambda^2\widehat{u_t}(\lambda,t),
\end{aligned}$$

if $\lambda = a + ib$, then $e^{-i\lambda x} = e^{-iax}e^{bx}$. For the boundary terms to vanish at $x = -\infty$ and for the half-line Fourier transforms to converge, we require $\mathrm{Im}(\lambda) \ge 0$.

$$= \widehat{u_t}(\lambda,t)(1+\lambda^2) + cu(0,t) + ci\lambda\hat{u}(\lambda,t) - i\lambda u_t(0,t) - u_{xt}(0,t) \tag{29}$$

$$\implies \widehat{u_t}(\lambda,t) + \frac{ci\lambda}{1+\lambda^2}\hat{u}(\lambda,t) = \frac{1}{1+\lambda^2}\left(-cu(0,t) + u_{xt}(0,t) + i\lambda u_t(0,t)\right) \qquad (\lambda \ne \pm i),$$

by using the integrating factor $\mu(t) = \exp\left(\frac{ci\lambda}{1+\lambda^2}t\right) = \exp(\omega t)$,

$$\implies e^{\omega t}\hat{u}(\lambda,t) = \frac{1}{1+\lambda^2}\int_0^t e^{\omega s}\left(-cu(0,s) + u_{xs}(0,s) + i\lambda u_s(0,s)\right) ds + \hat{u}(\lambda,0)\, ds$$

$$\implies e^{\omega t}U(\lambda,t) = U(\lambda,0) - \frac{c\int_0^t \left[e^{\omega s}u(0,s) + i\lambda e^{\omega s}u_x(0,s)\right]\,ds}{1+\lambda^2}$$
$$= U(\lambda,0) - \frac{cG(\lambda,t)}{1+\lambda^2}$$

with

$$f_0(\omega,t) = \int_0^t e^{\omega s}u(0,s)\,ds, \qquad f_1(\omega,t) = \int_0^t e^{\omega s}u_x(0,s)\,ds$$
$$G(\lambda,t) = f_0(\omega,t) + i\lambda f_1(\omega,t) \qquad \omega = \frac{ci\lambda}{1+\lambda^2}$$
$$U(\lambda,t) = (1+\lambda^2)\hat{u}(\lambda,t) - u_x(0,t) - i\lambda u(0,t).$$

11.3 Justifying Contour Deformation

In solving for the outgoing and incoming leads, the inverse Fourier transform requires evaluating integrals over the real line, $\lambda \in (-\infty,\infty)$. To apply the residue theorem, we must close these contours with a semicircular arc of radius $R \to \infty$ in either the upper or lower half of the complex plane.

Analysis of the Integrand Behavior at Infinity

All integrands deformed in (7) and (8) contain the exponential factors $e^{i\lambda x - t\omega}$ or $e^{i\lambda x}$. Let $\lambda = a + ib$. First, noting that

$$\lim_{|\lambda|\to\infty} \omega = \lim_{|\lambda|\to\infty} \frac{ci\lambda}{1+\lambda^2} = 0 \implies \lim_{|\lambda|\to\infty} e^{i\lambda x - t\omega} = \lim_{|\lambda|\to\infty} e^{i\lambda x},$$

the exponential term governing spatial decay expands to:

$$\left|e^{i\lambda x}\right| = \left|e^{i(a+ib)x}\right| = e^{-bx}.$$

To guarantee exponential decay along the bounding arc as $R \to \infty$:

- For the **outgoing leads**, the spatial domain is $x \in [0,\infty)$. Therefore, to ensure $e^{-bx} \to 0$, we require $\text{Im}(\lambda) > 0$. The contour must be closed in the **upper half-plane** ($\mathcal{C}^+$).
- For the **incoming leads**, the spatial domain is $x \in (-\infty, 0]$. Therefore, to ensure $e^{-bx} \to 0$, we require $\text{Im}(\lambda) < 0$. The contour must be closed in the **lower half-plane** ($\mathcal{C}^-$).

The Rational Terms $F(\lambda)$

1. **The Boundary Initial Data Terms:**

$$F(\lambda) = \frac{\partial_x u_{\pm j}(0,0) + i\lambda u_{\pm j}(0,0)}{1+\lambda^2} = \mathcal{O}\left(\frac{1}{\lambda}\right) \text{ as } |\lambda| \to \infty \text{ uniformly in the argument of } \lambda.$$

2. **The $G_{\pm j}(\lambda,t)$ Terms:** Recalling that $G(\lambda,t) = f_0(\omega,t) + i\lambda f_1(\omega,t)$, this term scales linearly with λ.

$$F(\lambda) = \frac{G_{\pm j}(\lambda,t)}{(1+\lambda^2)^2} = \mathcal{O}\left(\frac{\lambda}{\lambda^4}\right) = \mathcal{O}\left(\frac{1}{\lambda^3}\right) \text{ as } |\lambda| \to \infty \text{ uniformly in the argument of } \lambda.$$

In both cases, $F(\lambda) \to 0$ uniformly as $|\lambda| \to \infty$. Therefore, the conditions for Jordan's Lemma are satisfied. The integrals over the semicircular arcs at infinity vanish, justifying the deformations from the real line into $\mathcal{C}^+$ and $\mathcal{C}^-$ respectively. Additionally, since the only poles and singularities not at infinity are at $\lambda = \pm i$, the contours can be deformed to be precisely about $\lambda = \pm i$.

11.4 Exploring Singular Case

As the only non-data terms depending on the value of $\alpha+\gamma$ within the solutions (20) (19) are contained entirely within the auxiliary $A_{\pm j}, B_{\pm j}, C_{\pm j}$ variables, it suffices to check that the terms containing non-data within these cancel with the non-data terms present in (7) and (8). If so, then the singular case is well-posed.

Note the following results, obtained during the derivation of the global relations (See (29) and (28)):

$$\begin{aligned}(1+\lambda^2)\partial_t \hat{u}_{+j}(\lambda,t) + ci\lambda\hat{u}_{+j}(\lambda,t) &= cu_{+j}(0,t) - \partial_{xt}u_{+j}(0,t) - i\lambda\partial_t u_{+j}(0,t)\\ \implies \hat{u}_{+j}(-i,t) &= u_{+j}(0,t) - \frac{\partial_{xt}u_{+j}(0,t)}{c} - \frac{\partial_t u_{+j}(0,t)}{c}, \qquad (30)\\ (1+\lambda^2)\partial_t \hat{u}_{-j}(\lambda,t) + ci\lambda\hat{u}_{-j}(\lambda,t) &= -cu_{-j}(0,t) + \partial_{xt}u_{-j}(0,t) + i\lambda\partial_t u_{-j}(0,t)\\ \implies \hat{u}_{-j}(i,t) &= u_{-j}(0,t) - \frac{\partial_{xt}u_{-j}(0,t)}{c} + \frac{\partial_t u_{-j}(0,t)}{c}. \qquad (31)\end{aligned}$$

In the singular case when $\alpha+\gamma=0$, by taking $\gamma=-\alpha$ and looking at just the terms of $A_{\pm j}, B_{\pm j}, C_{\pm j}$ that contain unprescribed boundary data, denoted $A'_{\pm j}, B'_{\pm j}, C'_{\pm j}$ respectively, it can be shown that all the unknown boundary data once again cancels out with the terms present in (7) and (8), and thus the well-posed property of the non-singular case is preserved.

Outgoing Singular Case

$$\begin{aligned}A'_{+j} &= \frac{1}{2\pi}\oint_{\mathcal{C}^+} \frac{e^{i\lambda x}(1-\lambda^2)}{\alpha b_{+j}(1+\lambda^2)^2}\left[\sum_{i=-n}^{-1}\beta_i(\partial_x u_i(0,t) + i\lambda u_i(0,t))\right]\\ &\quad - \frac{e^{i\lambda x}(1-\lambda^2)}{\alpha b_{+j}(1+\lambda^2)}\left[\sum_{i=-n}^{-1}\beta_i\hat{u}_i(\lambda,t)\right]d\lambda\\ &= \frac{e^{-x}}{2\alpha b_{+j}}\sum_{i=-n}^{-1}\left[\beta_i\left(u_i(0,t) + x\partial_x u_i(0,t) - xu_i(0,t)\right)\right] - \frac{e^{-x}}{\alpha b_{+j}}\left[\sum_{i=-n}^{-1}\beta_i\hat{u}_i(i,t)\right],\end{aligned}$$

$$\begin{aligned}B'_{+j} &= \frac{1}{2\pi}\oint_{\mathcal{C}^+} \frac{e^{i\lambda x}(1-\lambda^2)}{\alpha b_{+j}(1+\lambda^2)^2}\left[\sum_{i=+1}^{+m}\beta_i\left(\partial_x u_i(0,t) + \frac{i}{\lambda}u_i(0,t)\right)\right]\\ &\quad + \frac{e^{i\lambda x}(1-\lambda^2)}{\alpha b_{+j}\lambda^2(1+\lambda^2)}\left[\sum_{i=+1}^{+m}\beta_i\hat{u}_i\left(\frac{1}{\lambda},t\right)\right]d\lambda\\ &= \frac{e^{-x}}{2\alpha b_{+j}}\sum_{i=+1}^{+m}\left[\beta_i\left(u_i(0,t) + x\partial_x u_i(0,t) + xu_i(0,t)\right)\right] - \frac{e^{-x}}{\alpha b_{+j}}\left[\sum_{i=+1}^{+m}\beta_i\hat{u}_i(-i,t)\right].\end{aligned}$$

Now, combining these results and using the interface conditions (2) to simplify the sums:

$$\begin{aligned}
A'_{+j} - B'_{+j} &= \frac{e^{-x}}{2\alpha b_{+j}} \left[\sum_{i=-n}^{-1} \beta_i (1-x) u_i(0,t) - \sum_{i=+1}^{+m} \beta_i (1+x) u_i(0,t) \right] \\
&\quad - \frac{e^{-x}}{\alpha b_{+j}} \left[\sum_{i=-n}^{-1} \beta_i \hat{u}_i(i,t) - \sum_{i=+1}^{+m} \beta_i \hat{u}_i(-i,t) \right] \\
&= \frac{e^{-x}}{2\alpha b_{+j}} \left[b_{-n} u_{-n}(0,t) \underbrace{(\alpha - \gamma)}_{=2\alpha} - b_{-n} u_{-n}(0,t) x \underbrace{(\alpha + \gamma)}_{=0} \right] - \frac{e^{-x}}{\alpha b_{+j}} \left[\sum_{i=-n}^{-1} \beta_i \hat{u}_i(i,t) - \sum_{i=+1}^{+m} \beta_i \hat{u}_i(-i,t) \right] \\
&= \frac{e^{-x}}{b_{+j}} \left[b_{-n} u_{-n}(0,t) \right] - \frac{e^{-x}}{\alpha b_{+j}} \left[\sum_{i=-n}^{-1} \beta_i \hat{u}_i(i,t) - \sum_{i=+1}^{+m} \beta_i \hat{u}_i(-i,t) \right].
\end{aligned}$$

Substituting the identities (30) and (31), and using the linearity of the derivative alongside the interface conditions (3), the derivative terms cancel:

$$\begin{aligned}
&= e^{-x} u_{+j}(0,t) - \frac{e^{-x}}{\alpha b_{+j}} \left[\sum_{i=-n}^{-1} \beta_i \left(u_i(0,t) - \frac{\partial_{xt} u_i(0,t)}{c} + \frac{\partial_t u_i(0,t)}{c} \right) \right. \\
&\qquad \left. - \sum_{i=+1}^{+m} \beta_i \left(u_i(0,t) - \frac{\partial_{xt} u_i(0,t)}{c} - \frac{\partial_t u_i(0,t)}{c} \right) \right] \\
&= e^{-x} u_{+j}(0,t) - \frac{e^{-x}}{\alpha b_{+j}} \left[2\alpha b_{+j} u_{+j}(0,t) \right] \\
&= -e^{-x} u_{+j}(0,t)
\end{aligned}$$

So $A'_{+j} - B'_{+j} + C'_{+j} = \frac{e^{-x}}{2}(\partial_x u_{+j}(0,t) - u_{+j}(0,t))$ (C does not depend on α, γ, so is unchanged).

This will then cancel with the non-data terms present inside of (19) just as the non-singular case. And so the solution is still well posed, and is identical to the non-singular $\alpha + \gamma \neq 0$ case.

Incoming Singular Case

$$\begin{aligned}
A'_{-j} &= \frac{1}{2\pi} \oint_{\mathcal{C}^-} \frac{e^{i\lambda x}(1-\lambda^2)}{\alpha b_{-j}(1+\lambda^2)^2} \left[\sum_{i=+1}^{+m} \beta_i (\partial_x u_i(0,t) + i\lambda u_i(0,t)) \right] \\
&\quad + \frac{e^{i\lambda x}(1-\lambda^2)}{\alpha b_{-j}(1+\lambda^2)} \left[\sum_{i=+1}^{+m} \beta_i \hat{u}_i(\lambda,t) \right] d\lambda \\
&= \frac{e^{x}}{2\alpha b_{-j}} \sum_{i=+1}^{+m} \left[\beta_i \left(u_i(0,t) + x \partial_x u_i(0,t) + x u_i(0,t) \right) \right] - \frac{e^{x}}{\alpha b_{-j}} \left[\sum_{i=+1}^{+m} \beta_i \hat{u}_i(-i,t) \right],
\end{aligned}$$

$$\begin{aligned}
B'_{-j} &= \frac{1}{2\pi} \oint_{\mathcal{C}^-} \frac{e^{i\lambda x}(1-\lambda^2)}{\alpha b_{-j}(1+\lambda^2)^2} \left[\sum_{i=-n}^{-1} \beta_i \left(\partial_x u_i(0,t) + \frac{i}{\lambda} u_i(0,t) \right) \right] \\
&\quad - \frac{e^{i\lambda x}(1-\lambda^2)}{\alpha b_{-j} \lambda^2 (1+\lambda^2)} \left[\sum_{i=-n}^{-1} \beta_i \hat{u}_i \left(\frac{1}{\lambda}, t \right) \right] d\lambda
\end{aligned}$$

$$= \frac{e^x}{2\alpha b_{-j}} \sum_{i=-n}^{-1} \left[\beta_i \left(u_i(0,t) + x\partial_x u_i(0,t) - x u_i(0,t)\right)\right] - \frac{e^x}{\alpha b_{-j}} \left[\sum_{i=-n}^{-1} \beta_i \hat{u}_i(i,t)\right].$$

Combining these results and using the interface conditions (2) to simplify the sums:

$$\begin{aligned} -A'_{-j} + B'_{-j} &= \frac{e^x}{2\alpha b_{-j}} \left[\sum_{i=-n}^{-1} \beta_i (1-x) u_i(0,t) - \sum_{i=+1}^{+m} \beta_i (1+x) u_i(0,t)\right] \\ &\quad - \frac{e^x}{\alpha b_{-j}} \left[\sum_{i=-n}^{-1} \beta_i \hat{u}_i(i,t) - \sum_{i=+1}^{+m} \beta_i \hat{u}_i(-i,t)\right] \\ &= \frac{e^x}{2\alpha b_{-j}} \left[b_{-n} u_{-n}(0,t) \underbrace{(\alpha - \gamma)}_{=2\alpha} - b_{-n} u_{-n}(0,t) x \underbrace{(\alpha + \gamma)}_{=0}\right] - \frac{e^x}{\alpha b_{-j}} \left[\sum_{i=-n}^{-1} \beta_i \hat{u}_i(i,t) - \sum_{i=+1}^{+m} \beta_i \hat{u}_i(-i,t)\right] \\ &= \frac{e^x}{b_{-j}} \left[b_{-n} u_{-n}(0,t)\right] - \frac{e^x}{\alpha b_{-j}} \left[\sum_{i=-n}^{-1} \beta_i \hat{u}_i(i,t) - \sum_{i=+1}^{+m} \beta_i \hat{u}_i(-i,t)\right]. \end{aligned}$$

Substituting the identities (30) and (31), and using the linearity of the derivative alongside the interface conditions (3), the derivative terms cancel:

$$\begin{aligned} &= e^x u_{-j}(0,t) - \frac{e^x}{\alpha b_{-j}} \left[\sum_{i=-n}^{-1} \beta_i \left(u_i(0,t) - \frac{\partial_{xt} u_i(0,t)}{c} + \frac{\partial_t u_i(0,t)}{c}\right)\right. \\ &\qquad\qquad \left. - \sum_{i=+1}^{+m} \beta_i \left(u_i(0,t) - \frac{\partial_{xt} u_i(0,t)}{c} - \frac{\partial_t u_i(0,t)}{c}\right)\right] \\ &= e^x u_{-j}(0,t) - \frac{e^x}{\alpha b_{-j}} \left[2\alpha b_{-j} u_{-j}(0,t)\right] \\ &= -e^x u_{-j}(0,t) \end{aligned}$$

So $A'_{-j} - B'_{-j} + C'_{-j} = -\frac{e^x}{2}(\partial_x u_{-j}(0,t) + u_{-j}(0,t))$ (C does not depend on α, γ, so is unchanged).

This will then cancel with the non-data terms present inside of (20) just as the non-singular case. And so the solution is still well posed, and is identical to the non-singular $\alpha + \gamma \neq 0$ case.

11.5 Note on Numerics

When numerically evaluating the solutions, the $e^{-t\omega}$ terms are problematic in the regions where $\mathrm{Re}(\omega) < 0$ (See Figure 7) as they grow exponentially with time, but decay in the regions where $\mathrm{Re}(\omega) > 0$. So it is essential for numerical stability to ensure as much of the contour as possible is within the latter region. It is recommended to make a change of variables $\lambda \to \frac{1}{\lambda}$ on the incoming lead solutions so that the contour is about $\lambda = i$, and then use a circular contour integral of an appropriate radius whilst ensuring that the singularities present at $\lambda = 0$ inside of $\hat{u}\left(\frac{1}{\lambda}, 0\right)$ and $e^{i/\lambda}$ are not enclosed (it was found that a radius of ≈0.5 performed sufficiently). Additionally, when $\alpha + \gamma \neq 0$ the roots of $\alpha - \lambda^2\gamma = 0$ must be identified, and any poles introduced within the contour must be accounted for via residue calculus. These adjustments were found to extend the range of stability of the numerical evaluation from having $t \approx \frac{20}{c}$ to $t \approx \frac{50}{c}$ without the need for additional working precision. While the method of steepest descent would theoretically improve the stability, currently the terms containing $e^{-t\omega}\hat{u}\left(\frac{1}{\lambda},0\right)$ force the contour through regions where $\mathrm{Re}(\omega)<0$, presenting a significant challenge. If the state of the system at arbitrarily large times t is required, we recommend evaluating the system over smaller, stable time intervals, then using the result as the initial condition of a subsequent system, effectively implementing a time-stepping scheme.

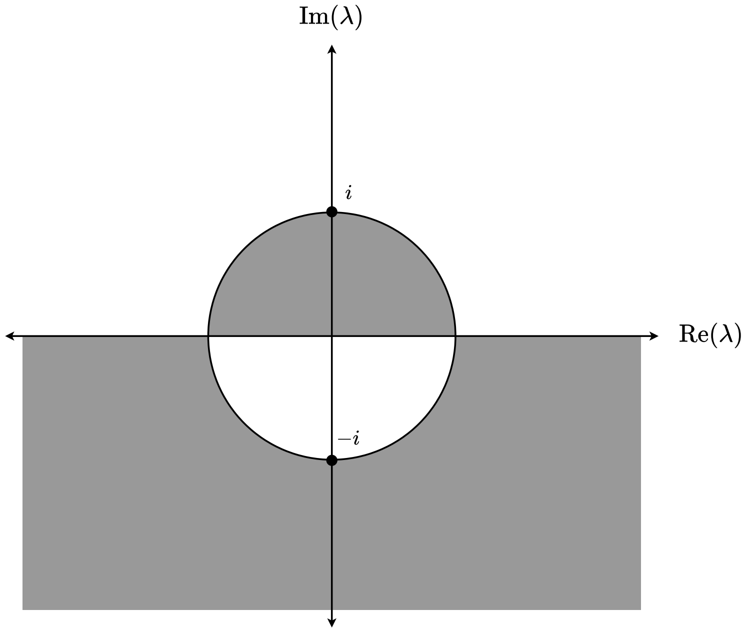

Figure 7: Regions of the complex plane where $\mathrm{Re}(\omega)<0$.

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