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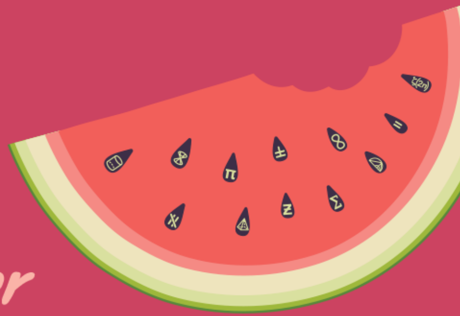

Minimal Spectrahedral Representations for Facial Dimension Realisations

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Abstract

We study the problem of determining, for a prescribed facial dimension signature, the minimal-size spectrahedral representation that exhibits it. We characterise affine subspaces whose intersections with the three-dimensional cone of positive semidefinite matrices produce spectrahedra with prescribed facial dimensions. We also obtain minimal spectrahedral representations for a particular family of signatures, and we recast existing upper bounds in a spectrahedral framework. Ultimately, we present some open questions and discuss possible extensions.

MSC: 52A20

1 Statement of authorship

This report is the work of the author. Where the work of others has been used, it has been acknowledged and referenced according to proper procedure. Any assistance received in the writing of this report has been appropriately disclosed.

Contents

2 Introduction

A set $S \subseteq \mathbb{R}^n$ is called a *spectrahedron* if there exist m-by-m real symmetric matrices $A_0, \dots, A_n$ such that

$$S = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : A_0 + A_1 x_1 + \cdots + A_n x_n \succeq 0\},$$

where, for equally sized symmetric matrices, $M_1 \succeq M_2$ means that $(M_1 - M_2)$ is positive semidefinite (all eigenvalues are greater than or equal to zero).

Spectrahedra are sets that arise as feasible regions in semidefinite programming. Semidefinite programming is a generalisation of linear programming and is notable for its applications in control theory and finance. In the field of combinatorial optimisation, semidefinite programming can provide reasonable bounds on solutions to NP-hard problems [3]. A key geometric feature of spectrahedra is their boundary structure, which influences feasibility, uniqueness, and the algorithmic difficulty of optimisation problems. A simpler spectrahedral representation of a particular facial dimension signature can reduce computational cost by lowering the size and complexity associated with semidefinite constraints.

A convex subset F of a convex set C is a *face* of C, denoted by $F \trianglelefteq C$, if for any $x, y \in C$ with $(x, y) \cap F \neq \emptyset$, we have $x, y \in F$. Further, the facial dimension signature of a convex set C is the set

$$\sigma(C) = \{\dim F : F \trianglelefteq C, F \neq \emptyset\},$$

where the dimension of a face, $\dim F$, is the dimension of the smallest affine subspace containing F [1].

Spectrahedra can also be thought of as affine slices of the space of real positive semidefinite matrices. The space of m-by-m real positive semidefinite matrices is denoted $\mathbb{S}^m_+$. The space $\mathbb{S}^m_+$ forms a closed convex cone inside the real vector space $\mathbb{S}^m$ of m-by-m real symmetric matrices [2]. We say that a set K is a *cone* if it is closed under multiplication by non-negative scalars; that is, if $x \in K$ then $\lambda x \in K$ for any real scalar $\lambda \geq 0$.

This report begins by developing necessary theorems in convex analysis. We characterise when the intersection of a closed convex cone with an affine line, $\{c\} + \operatorname{span}\{v\}$ for some $c, v \in \mathbb{R}^n$ where $v \neq 0$, is empty, a point, a line segment, a ray, or the whole line. We then look at affine slices of closed convex cones in $\mathbb{R}^3$ and develop Theorem 4.5 which can be used to characterise the dimension of one-dimensional slices of the cone of positive semidefinite matrices. This is then applied to characterise one-dimensional affine slices of the cone $\mathbb{S}^m_+$ in Theorem 5.1. In Theorem 6.8 we find the smallest m for which there exists an affine slice of the cone $\mathbb{S}^m_+$ whose resulting spectrahedron can yield a facial dimension signature corresponding to a special family of finite integer sequences. Additionally, we reformulate an existing method for constructing a spectrahedron with prescribed facial dimension signature to obtain an explicit upper bound on the required matrix size.

3 Background in convex analysis

Before we observe affine slices of closed convex cones, we first develop the required background in convex analysis by introducing some definitions and proving lemmas that will be used in the main results of this report.

Definition 3.1. The Minkowski sum of two sets $A, B \subseteq \mathbb{R}^n$ is defined as

$$A + B = \{a + b : a \in A, b \in B\}.$$

Definition 3.2. Let V be an inner product space with $U \subseteq V$. The orthogonal complement of U is defined as

$$U^\perp = \{v \in V : \langle v, u\rangle = 0 \text{ for all } u \in U\}.$$

Definition 3.3. The negative of a set $C \subset \mathbb{R}^n$ is the set $-C = \{-c : c \in C\}$.

Definition 3.4. For any set $C \subseteq \mathbb{R}^n$, the *dual cone* is the set

$$C^* = \{y : \langle x, y\rangle \geq 0 \text{ for all } x \in C\}.$$

Lemma 3.5 is a standard lemma from [2], which will be useful for building intuition about duality. This observation will be useful when we interpret dual variables as defining supporting halfspaces in later arguments.

Lemma 3.5. *For any $C \subseteq \mathbb{R}^n$, the dual of C is always a convex cone.*

Lemma 3.6. *For a closed convex cone $K \subset \mathbb{R}^n$ and a linear subspace $L \subset \mathbb{R}^n$, $(K + L)^* = K^* \cap L^\perp$.*

Proof: For the first inclusion, let $y \in (K + L)^*$. Then, $\langle x, y\rangle \geq 0$ for all $x \in K + L$. It follows that $\langle x, y\rangle \geq 0$ for all $x \in K$ and $\langle x, y\rangle \geq 0$ for all $x \in L$. Consequently, $y \in K^*$, and using the fact that L is a linear subspace, $y \in L^* = L^\perp$. Therefore $y \in K^* \cap L^\perp$ and $(K + L)^* \subseteq K^* \cap L^\perp$.

For the second inclusion, let $y \in K^* \cap L^\perp$. Moreover, $\langle y, p\rangle \geq 0$ for all $p \in K$. Likewise, $\langle y, q\rangle \geq 0$ for all $q \in L = (L^\perp)^*$. Then $\langle y, p + q\rangle \geq 0$ for all $p \in K$ and $q \in L$. Therefore, $\langle y, x\rangle \geq 0$ for all $x = (p + q) \in K + L$. This implies that $y \in (K + L)^*$. Thus, $(K + L)^* = K^* \cap L^\perp$. □

Theorem 3.7 is a well-established result contained in [1]. This theorem will prove useful in the justification of the lemmas required for the main result.

Theorem 3.7. *Strict separating hyperplane theorem: Let A and B be two disjoint, non-empty convex subsets of $\mathbb{R}^n$. If A is closed and B compact, then there exists a non-zero vector v and a real number α such that,*

$$\langle y, v\rangle < \alpha \leq \langle x, v\rangle,$$

for all $x \in A$ and $y \in B$.

4 Characterising affine slices of closed convex cones

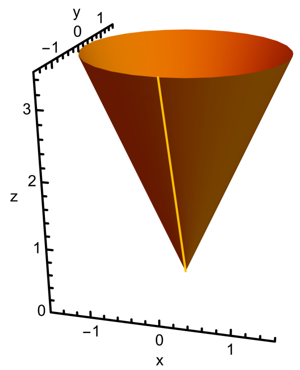
Let $K \subset \mathbb{R}^n$ be a closed convex cone, and let $\{c\} + L \subseteq \mathbb{R}^n$ be an affine subspace for some $c \in \mathbb{R}^n$. Define the linear map,

$$\mathscr{L}(a) := a_1 v_1 + a_2 v_2 + \cdots + a_k v_k,$$

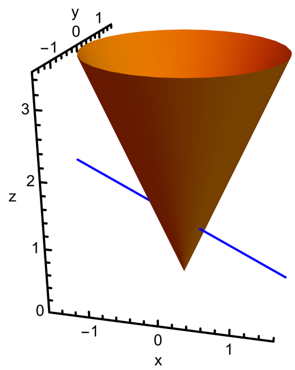
where $v_1, \dots, v_k \in \mathbb{R}^n$ and $a = (a_1, \dots, a_k) \in \mathbb{R}^k$, with image

$$L := \operatorname{im} \mathscr{L} = \left\{ \sum_{i=1}^{k} a_i v_i : a \in \mathbb{R}^k \right\}.$$

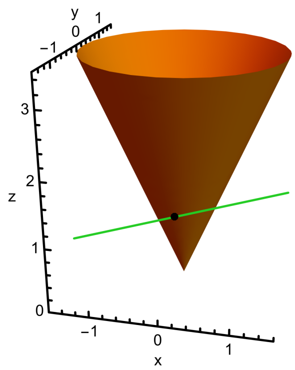
For now, we assume $\ker \mathscr{L} = \{0\}$. That is $\mathscr{L}$ is injective or equivalently $v_1, \dots, v_k$ are linearly independent. We now introduce some lemmas and theorems that are useful for making inferences about the facial structure of the intersection $K \cap (\{c\} + L)$.



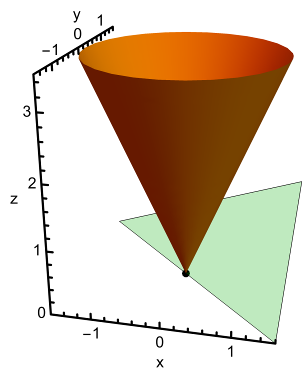
(a) One-dimensional affine subspace with one-dimensional intersection with the cone.



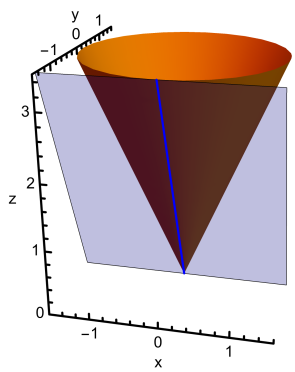
(b) One-dimensional affine subspace with one-dimensional intersection with the cone.



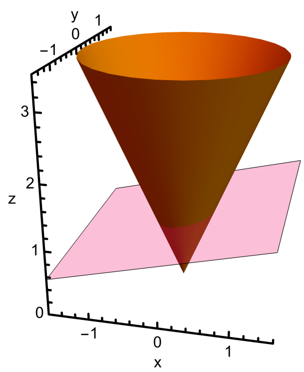
(c) One-dimensional affine subspace with zero-dimensional intersection with the cone.



(d) Two-dimensional affine subspace with zero-dimensional intersection with the cone.



(e) Two-dimensional affine subspace with one-dimensional intersection with the cone.



(f) Two-dimensional affine subspace with two-dimensional intersection with the cone.

Figure 1: Affine slices of the circular cone in $\mathbb{R}^3$.

Example 4.1. Figure 1 depicts different affine subspaces that form zero-, one-, and two-dimensional intersections with the circular cone in $\mathbb{R}^3$, where the translate of the linear subspace is an element of the Minkowski sum of the cone and that linear subspace (that is, the intersection is non-empty). Recall that spectrahedra arise as affine slices of cones of positive semidefinite matrices [4]. Since $\mathbb{S}^m$ is a finite-dimensional vector space, it can be identified isometrically with $\mathbb{R}^N$, with $N = \dim(\mathbb{S}^m) = \frac{m(m+1)}{2}$. Therefore, any results obtained for slices of cones in $\mathbb{R}^N$ also extend to slices of the cone $\mathbb{S}^m_+$. This perspective will allow us to make statements about the facial dimensions of spectrahedra.

Lemma 4.2. *Let $K \subseteq \mathbb{R}^n$ and let $\{c\} + L$ be an affine subspace of $\mathbb{R}^n$. Then, $K \cap (\{c\} + L) \neq \emptyset$ if and only if $c \in K + L$.*

Proof. For the forward implication, if $K \cap (\{c\} + L) \neq \emptyset$, then there exists an x such that $x \in K \cap (\{c\} + L)$. So $x = c + l_1$, for some $l_1 \in L$. Since $x \in K$ and $c = x - l_1$, where $-l_1 \in L$, clearly $c \in K + L$. On the other hand, if $c \in K + L$ then $c = k + l_2$, for some $k \in K$ and $l_2 \in L$. Since $k = c - l_2$ is an element of $\{c\} + L$, then $k \in K \cap (\{c\} + L) \neq \emptyset$. Therefore, $K \cap (\{c\} + L) \neq \emptyset$ if and only if $c \in K + L$. □

Lemma 4.3. *If $K \subset \mathbb{R}^n$ is a closed convex cone, then*

$$K = \{x \in \mathbb{R}^n : \langle x, y \rangle \geq 0 \text{ for all } y \in K^*\}.$$

Proof. Firstly, let $x \in K$. By definition for any $y \in K^*$ we have that $\langle x, y \rangle \geq 0$. Since the real inner product is symmetric, $\langle y, x \rangle \geq 0$ which implies that $x \in K^{**}$. So $K \subseteq K^{**}$.

For the second inclusion, we prove the contrapositive. That is, we show that if $x \notin K$ then $x \notin K^{**}$. Since K is closed and the point x is compact, Theorem 3.7 implies that there exists an $\alpha \in \mathbb{R}$ and a non-zero $y \in \mathbb{R}^n$ such that

$$\langle x, y \rangle < \alpha \leq \langle k, y \rangle, \text{ for all } k \in K. \tag{1}$$

Since $0 \in K$, we substitute $k = 0$ into the expression above to obtain,

$$\langle x, y \rangle < \alpha \leq 0. \tag{2}$$

Now, using the fact that K is a cone, for any $k \in K$ we have that $\lambda k \in K$ for all scalars $\lambda \geq 0$. This implies that $\langle \lambda k, y \rangle = \lambda \langle k, y \rangle \geq \alpha$ for all $\lambda \geq 0$. This requires $\langle k, y \rangle \geq \frac{\alpha}{\lambda}$ for all $\lambda > 0$ and $k \in K$. Consequently, $\langle k, y \rangle \geq 0$ and this means $y \in K^*$. Since

$$K^{**} = \{x \in \mathbb{R}^n : \langle x, y \rangle \geq 0 \text{ for all } y \in K^*\},$$

and $\langle x, y \rangle < 0$, it is clear that $x \notin K^{**}$. Hence, if $x \in K$ then $x \in K^{**}$ so $K \subseteq K^{**}$ and $K = K^{**} = \{x \in \mathbb{R}^n : \langle x, y \rangle \geq 0 \text{ for all } y \in K^*\}$. □

Lemma 4.4. *Let $K \subset \mathbb{R}^n$ be a closed convex cone and let $L = \operatorname{span}\{v\}$ with $v \in \mathbb{R}^n \setminus \{0\}$. Define*

$$T := \{t \in \mathbb{R} : c + tv \in K\},$$

where $c \in \mathbb{R}^n$. Then,

$$K \cap (\{c\} + L) = \{c + tv : t \in T\}.$$

Proof. Take $x \in K \cap (\{c\} + L)$. Then $x = c + \ell$ for some $\ell \in L$. Since $L = \operatorname{span}\{v\}$, there exists $t_0 \in \mathbb{R}$ such that $\ell = t_0 v$ which implies that $x = c + t_0 v$. By assumption, $x \in K$, so $c + t_0 v \in K$. Further, by the definition of T, $t_0 \in T$ and $x = c + t_0 v$ implying $x \in \{c + tv : t \in T\}$. Thus, $K \cap (\{c\} + L) \subseteq \{c + tv : t \in T\}$. On the other hand, let $y \in \{c + tv : t \in T\}$. We then have that $y = c + t_1 v$ for some $t_1 \in T$. Considering that $t_1 \in T$, then $y = c + t_1 v \in K$. Moreover, $t_1 v \in L$ hence $y = c + tv \in \{c\} + L$. Finally, $\{c + tv : t \in T\} \subseteq K \cap (\{c\} + L)$ and we can indeed conclude that $K \cap (\{c\} + L) = \{c + tv : t \in T\}$. □

Theorem 4.5. *Let $K \subset \mathbb{R}^n$ be a closed convex cone and let $L = \operatorname{span}\{v\}$ with $v \in \mathbb{R}^n \setminus \{0\}$. Write the affine line as $\{c\} + L = \{c + tv : t \in \mathbb{R}\}$ for $c \in \mathbb{R}^n$. Firstly, $K \cap (\{c\} + L) = \emptyset$ if and only if $c \notin K + L$. Otherwise, if $c \in K + L$ let $\alpha, \beta \in \mathbb{R} \cup \{\pm\infty\}$ such that*

$$\alpha := \sup_{\substack{y \in K^* \\ \langle v, y\rangle > 0}} \left(-\frac{\langle c, y\rangle}{\langle v, y\rangle} \right) \quad \text{and} \quad \beta := \inf_{\substack{y \in K^* \\ \langle v, y\rangle < 0}} \left(-\frac{\langle c, y\rangle}{\langle v, y\rangle} \right),$$

with the convention that if the supremum is over an empty set $\alpha = -\infty$ and if the infimum is over an empty set $\beta = \infty$. Then,

1. *$K \cap (\{c\} + L)$ is zero-dimensional if and only if $c \in K + L$ and $\alpha = \beta$;*
2. *$K \cap (\{c\} + L)$ is one-dimensional if and only if $c \in K + L$ and $\alpha < \beta$.*

Proof: To show that $K \cap (\{c\} + L) = \emptyset$ if and only if $c \notin K + L$, we simply apply the contrapositive of the forward and reverse implications of Lemma 4.2.

On the other hand, if $c \in K + L$, we define $T := \{t \in \mathbb{R} : c + tv \in K\}$ and take α and β as defined above. Assume now that $c \in K + L$ and define,

$$T := \{t \in \mathbb{R} : c + tv \in K\}.$$

Then, by Lemma 4.4 $K \cap (\{c\} + L) = \{c + tv : t \in T\}$. Since K is convex, T is an interval. We claim that $T = [\alpha, \beta]$.

By Lemma 4.3, for a closed convex cone K we have that $x \in K$ if and only if $\langle x, y\rangle \geq 0$ for all $y \in K^*$. Applying this to $x = c + tv$ gives that $t \in T$ if and only if,

$$\langle c + tv, y\rangle = \langle c, y\rangle + t\langle v, y\rangle \geq 0 \text{ for all } y \in K^*.$$

Now fix $y \in K^*$. If $\langle v, y\rangle > 0$, then

$$t \geq -\frac{\langle c, y\rangle}{\langle v, y\rangle},$$

and if $\langle v, y\rangle < 0$ it is equivalent to

$$t \leq -\frac{\langle c, y\rangle}{\langle v, y\rangle}.$$

Alternatively, if $\langle v, y\rangle = 0$, the inequality becomes $\langle c, y\rangle \geq 0$. In this case, we employ Lemma 3.6 which states that $(K + L)^* = K^* \cap L^\perp$. In particular, note that $L^\perp = \{y \in \mathbb{R}^n : \langle v, y\rangle = 0\}$. Thus whenever $y \in K^*$ satisfies $\langle v, y\rangle = 0$, we have that $y \in (K + L)^*$. Since $c \in K + L$ and $y \in (K + L)^*$, it follows that $\langle c, y\rangle \geq 0$. This implies that $\langle v, y\rangle = 0$ imposes no further restrictions on t. Consequently, the conditions $c + tv \in K$ are exactly the collection of lower bounds coming from $y \in K^*$ with $\langle v, y\rangle > 0$ and upper bounds coming from $y \in K^*$ with $\langle v, y\rangle < 0$. Taking the tightest such bounds yields for all $t \in T$,

$$t \geq \sup_{\substack{y\in K^* \\ \langle v,y\rangle > 0}} \left(-\frac{\langle c, y\rangle}{\langle v, y\rangle}\right) = \alpha \quad \text{and} \quad t \leq \inf_{\substack{y\in K^* \\ \langle v,y\rangle < 0}} \left(-\frac{\langle c, y\rangle}{\langle v, y\rangle}\right) = \beta.$$

Therefore $T \subseteq [\alpha, \beta]$. Conversely, let $t \in [\alpha, \beta]$ and let $y \in K^*$ be arbitrary. If $\langle v, y\rangle > 0$, then by definition of α,

$$-\frac{\langle c, y\rangle}{\langle v, y\rangle} \leq \alpha \leq t,$$

which implies that $\langle c, y\rangle + t\langle v, y\rangle \geq 0$. Similarly, if $\langle v, y\rangle < 0$, then by definition of β,

$$t \leq \beta \leq -\frac{\langle c, y\rangle}{\langle v, y\rangle},$$

which also implies that $\langle c, y\rangle + t\langle v, y\rangle \geq 0$. Finally, as already established, if $\langle v, y\rangle = 0$, then $\langle c, y\rangle \geq 0$, hence $\langle c, y\rangle + t\langle v, y\rangle = \langle c, y\rangle \geq 0$. Therefore, $\langle c + tv, y\rangle \geq 0$ for all $y \in K^*$. So $c + tv \in K$ and $t \in T$. Hence, $[\alpha, \beta] \subseteq T$ and then $T = [\alpha, \beta]$.

The dimension statements now follow immediately. Firstly, if $\alpha = \beta$, then $K \cap (\{c\} + L) = \{c + \alpha v\}$, so the intersection is a single point and thus zero-dimensional. If $\alpha < \beta$, then $K \cap (\{c\} + L) = \{c + tv : \alpha \leq t \leq \beta\}$ is a nondegenerate interval in the affine line (possibly bounded or unbounded), and is therefore one-dimensional. □

5 Applications to spectrahedra

Leveraging the general characterisation of affine slices of closed convex cones, we can make statements about the facial dimensions present within spectrahedra in the aforementioned special cases. As mentioned earlier, the set of symmetric matrices $\mathbb{S}^m$ forms a vector space. For our purposes, we introduce the standard matrix inner product over this vector space as

$$\langle A, B\rangle = \operatorname{tr}(AB) = \sum_{1\leq i,j\leq m} A_{ij}B_{ij},$$

where $A, B \in \mathbb{S}^m$ [2]. With respect to this inner product, we now introduce a key theorem.

Theorem 5.1. *Let $L = \operatorname{span}\{A_1\}$ with $A_1 \in \mathbb{S}^m \setminus \{0\}$. Write the affine line as $\{A_0\} + L = \{A_0 + tA_1 : t \in \mathbb{R}\}$ for $A_0 \in \mathbb{S}^m$. Firstly, $\mathbb{S}^m_+ \cap (\{A_0\} + L) = \emptyset$ if and only if $A_0 \notin \mathbb{S}^m_+ + L$. Otherwise, if $A_0 \in \mathbb{S}^m_+ + L$ let $\alpha, \beta \in \mathbb{R} \cup \{\pm\infty\}$ such that*

$$\alpha := \sup_{\substack{Y\in \mathbb{S}^m_+ \\ \langle A_1,Y\rangle > 0}} \left(-\frac{\langle A_0, Y\rangle}{\langle A_1, Y\rangle}\right) \quad \text{and} \quad \beta := \inf_{\substack{Y\in \mathbb{S}^m_+ \\ \langle A_1,Y\rangle < 0}} \left(-\frac{\langle A_0, Y\rangle}{\langle A_1, Y\rangle}\right),$$

with the convention that if the supremum is over an empty set $\alpha = -\infty$ and if the infimum is over an empty set $\beta = \infty$. Then,

1. *$\mathbb{S}^m_+ \cap (\{A_0\} + L)$ is zero-dimensional if and only if $A_0 \in \mathbb{S}^m_+ + L$ and $\alpha = \beta$;*

2. $\mathbb{S}_+^m \cap (\{A_0\} + L)$ *is one-dimensional if and only if* $A_0 \in \mathbb{S}_+^m + L$ *and* $\alpha < \beta$.

Proof: Theorem 4.5 holds in any finite-dimensional inner product space. We apply it to the real vector space $\mathbb{S}^m$ equipped with the standard matrix inner product

$$\langle A, B\rangle = \mathrm{tr}(AB).$$

The cone $\mathbb{S}_+^m$ is a closed convex cone in $\mathbb{S}^m$, and it is self-dual with respect to this inner product; that is, $(\mathbb{S}_+^m)^* = \mathbb{S}_+^m$. Now take

$$K = \mathbb{S}_+^m, \qquad c = A_0, \qquad L = \mathrm{span}\{A_1\}.$$

Then the affine line $\{A_0\} + L$ is precisely

$$\{A_0 + tA_1 : t \in \mathbb{R}\}.$$

Applying Theorem 4.5 with these choices, and using the self-duality of $\mathbb{S}_+^m$, we obtain that

$$\mathbb{S}_+^m \cap (\{A_0\} + L) = \emptyset \quad \text{if and only if} \quad A_0 \notin \mathbb{S}_+^m + L.$$

Otherwise, if $A_0 \in \mathbb{S}_+^m + L$, then

$$\alpha := \sup_{\substack{Y \in \mathbb{S}_+^m \\ \langle A_1, Y\rangle > 0}} \left(-\frac{\langle A_0, Y\rangle}{\langle A_1, Y\rangle}\right), \qquad \beta := \inf_{\substack{Y \in \mathbb{S}_+^m \\ \langle A_1, Y\rangle < 0}} \left(-\frac{\langle A_0, Y\rangle}{\langle A_1, Y\rangle}\right),$$

with the same conventions on empty index sets, and

$$\mathbb{S}_+^m \cap (\{A_0\} + L)$$

is zero-dimensional if and only if $\alpha = \beta$, and one-dimensional if and only if $\alpha < \beta$. This is exactly the claimed result. □

6 Minimal spectrahedral representations

Let $I = \{d_0, d_1, \ldots, d_N\}$ be a finite integer sequence with $\min_{i \in \mathbb{N}}\{d_i\} = 0$ and $\max_{i \in \mathbb{N}}\{d_i\} = d_{max}$. We seek to find the smallest $m \in \mathbb{N}$ for which there exists real symmetric m-by-m matrices $A_0, A_1, \ldots, A_n$ such that the spectrahedron

$$S = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : A_0 + A_1x_1 + \cdots + A_nx_n \succeq 0\},$$

satisfies $\sigma(S) = I$. For a particular family of integer sequences, we determine this minimal possible m. We also reinterpret existing upper bounds on m in terms of spectrahedra. To develop intuition, we will begin with illustrative examples and develop necessary supporting theorems.

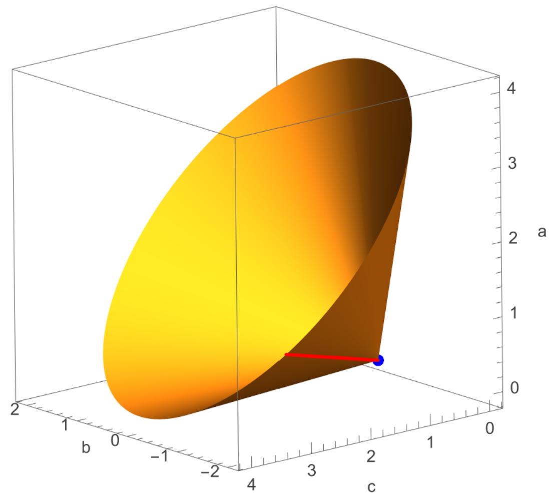


Figure 2: The boundary of $\mathbb{S}^2_+$.

Example 6.1. Figure 2 depicts the boundary of the cone $\mathbb{S}^2_+$. This cone has zero-dimensional faces (blue) and one-dimensional faces (red). Additionally, since

$$\mathbb{S}^2_+ = \left\{ X = \begin{pmatrix} a & b \\ b & c \end{pmatrix} : X \succeq 0 \right\}$$

is three-dimensional. Therefore, $\mathbb{S}^2_+$ also has a three-dimensional face. This demonstrates that the corresponding facial dimension signature is $\sigma\left(\mathbb{S}^2_+\right) = \{0, 1, 3\}$.

Lemma 6.2. *Non-empty faces of $\mathbb{S}^m_+$ are cones.*

Proof: Let $F \neq \emptyset$ be a face and choose some $M \in F$. We can write

$$M = \frac{1}{2}(2M) + \frac{1}{2}(0),$$

where $0, 2M \in \mathbb{S}^m_+$. Since F is a face, this implies that $0 \in F$. Now fix $\lambda > 0$. If $0 \le \lambda \le 1$, then $\lambda M = \lambda M + (1-\lambda)(0)$ lies in F by convexity of F. On the other hand, if $\lambda > 1$, then

$$M = \frac{1}{\lambda}(\lambda M) + \left(1 - \frac{1}{\lambda}\right)(0),$$

which is a convex combination of λM and 0, which are both elements of $\mathbb{S}^m_+$. Since $M \in F$ and F is a face, this forces $\lambda M \in F$. Therefore, F is a cone. □

Lemma 6.3. *Let P be an m-by-m positive definite matrix. Then, for every $A \succeq 0$, there exists $\lambda > 0$ such that $\lambda P \succeq A$.*

Proof: Define the continuous function $f : \{x : x^T x = 1\} \to \mathbb{R}$ such that $f(x) = x^T P x$. Since P is positive definite, $f(x) > 0$ for all $x \in \{x : x^T x = 1\}$. Therefore,

$$k = \min_{\|x\|=1} x^T P x,$$

exists and satisfies $k > 0$. Hence, $x^T P x \geq k\|x\|^2$ for all $x \in \mathbb{R}^m$. It follows that, for all symmetric matrices $A \succeq 0$, we apply Cauchy-Schwarz to obtain

$$x^T A x = \langle A, xx^T \rangle \leq \|A\|_F \|xx^T\|_F = \|A\|_F \|x\|^2.$$

Here we use $\|\cdot\|_F$ to distinguish between the Frobenius norm and the standard Euclidean norm. Combining these two inequalities for all x gives

$$x^T A x \leq \frac{\|A\|_F}{k} x^T P x = x^T \left(\frac{\|A\|_F}{k} P \right) x,$$

or equivalently,

$$\lambda P \succeq A \text{ with } \lambda = \frac{\|A\|_F}{k}.$$

□

Lemma 6.4. *Let $G \subseteq \mathbb{S}_+^m$ be a face of $\mathbb{S}_+^m$. If there is a $P \in G$ such that $P \succ 0$, then $G = \mathbb{S}_+^m$.*

Proof: Fix $A \succeq 0$. Next, as per Lemma 6.3, select a $\lambda > 0$ such that $A \preceq \lambda P$, so $\lambda P - A \succeq 0$. As established in Lemma 6.2, G is a cone so if $P \in G$, then $\lambda P \in G$. It follows that,

$$\lambda P = \frac{1}{2}(2A) + \frac{1}{2}(2(\lambda P - A))$$

is a convex combination of elements of $\mathbb{S}_+^m$. Since $\lambda P \in G$ and G is a face, this representation forces each endpoint of the convex combination to lie in G. Therefore, $2A \in G$ and because G is a cone $A = \frac{1}{2}(2A) \in G$. Therefore, every $A \succeq 0$ lies in G and $\mathbb{S}_+^m \subseteq G$. By assumption $G \subseteq \mathbb{S}_+^m$, so we conclude $G = \mathbb{S}_+^m$. □

Lemma 6.5. *Let $A, B \in \mathbb{S}_+^m$. Then*

$$\ker(A + B) = \ker(A) \cap \ker(B).$$

Proof: First, let $v \in \ker(A + B)$. By definition, $(A + B)v = 0$, so

$$0 = v^T (A + B)v = v^T Av + v^T Bv.$$

Using the fact that $A, B \succeq 0$, we have that $v^T Av \geq 0$ and $v^T Bv \geq 0$. Further, $v^T Av = 0$ and $v^T Bv = 0$. Since both A and B are positive semidefinite, they have unique positive semidefinite square roots $A^{1/2}$ and $B^{1/2}$ respectively. This implies that

$$v^T Av = \|A^{1/2} v\|^2 = 0$$

and $A^{1/2} v = 0$. Similarly $B^{1/2} v = 0$. This indicates that $A^{1/2}(A^{1/2} v) = Av = 0$ and in a similar manner it can be shown that $Bv = 0$. Thus, $v \in \ker(A) \cap \ker(B)$ and $\ker(A + B) \subseteq \ker(A) \cap \ker(B)$.

On the other hand, if $v \in \ker(A) \cap \ker(B)$, then $Av = Bv = 0$, hence $(A + B)v = 0$ and $v \in \ker(A + B)$. Then $\ker(A + B) = \ker(A) \cap \ker(B)$. □

Lemma 6.6. *Every face $F \neq \emptyset$ of the cone of positive semidefinite matrices is exactly the set of positive semidefinite matrices that vanish on some fixed subspace $V \subset \mathbb{R}^m$. That is*

$$F = \{X \succeq 0 : V \subseteq \ker(X)\}.$$

Proof: Fix a subspace $V \subset \mathbb{R}^m$ and $F_V = \{X \succeq 0 : V \subseteq \ker(X)\}$. We now demonstrate that F_V is a face of $\mathbb{S}^m_+$. First, take $X \in F_V$ and write $X = tA + (1-t)B$ with $A, B \succeq 0$ for $t \in (0,1)$. Let $v \in V$. Then,

$$v^T X v = t v^T A v + (1-t) v^T B v = 0.$$

Since $v^T A v \geq 0$ and $v^T B v \geq 0$, it must be the case that $v^T A v = 0$ and $v^T B v = 0$. Since $A, B \succeq 0$, they have unique positive semidefinite square roots $A^{1/2}$ and $B^{1/2}$ respectively. Therefore, $v^T A v = \|A^{1/2} v\|^2 = 0$ implies $Av = 0$ and likewise $Bv = 0$ for all $v \in V$. Then $V \subseteq \ker(A)$ and $V \subseteq \ker(B)$ which implies that $A, B \in F_V$. This demonstrates that F_V is a face of $\mathbb{S}^m_+$.

Now we choose a particular $X_0 \in F$ such that X_0 has the maximum rank amongst all the elements of F. Set $V = \ker(X_0)$. Since $\operatorname{rank}(X) \in \{0, 1, \dots, m\}$, there exists such an $X_0 \in F$.

We now show that $F \subseteq F_V$. Since F is a face, it is convex; and by Lemma 6.2 we know that F is a cone. Using the fact that F is a convex cone, $X_0 + X \in F$. As established in Lemma 6.5, if $A, B \succeq 0$ then $\ker(A+B) = \ker(A) \cap \ker(B)$. When $A = X_0$ and $B = X$,

$$\ker(X_0 + X) = \ker(X_0) \cap \ker(X) = V \cap \ker(X).$$

If $V \not\subseteq \ker(X)$, clearly $V \cap \ker(X) \subset V$ and $\dim(\ker(X_0 + X)) < \dim(V)$. This implies that

$$\operatorname{rank}(X_0 + X) > \operatorname{rank}(X_0),$$

which is a contradiction. Therefore, $V \subseteq \ker(X)$ for all $X \in F$. Thus, $F \subseteq F_V$.

Next, we check that $F_V \subseteq F$. Let $V^\perp$ be the complementary subspace to V; that is choose an orthonormal basis adapted to $\mathbb{R}^m = V^\perp \oplus V$. With respect to this decomposition, any $Y \in F_V$ can be written as,

$$Y = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \text{ with } A \in \mathbb{S}^{m-k}_+ \text{ on } V^\perp,$$

where $k = \dim(V)$. Any matrix of this form is in F_V. Moreover, since $\ker(X_0) = V$, we have that

$$X_0 = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix},$$

with $A_0 \succ 0$ on $V^\perp$. Now define a subset of the cone $\mathbb{S}^{m-k}_+$ by the set

$$F' = \left\{ A \in \mathbb{S}^{m-k}_+ : \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in F \right\}.$$

Let $A \in F'$ and suppose that $A = tB + (1-t)C$ for $t \in (0,1)$ with $B, C \in \mathbb{S}^{m-k}_+$. Consider the linear map

$$\phi : \mathbb{S}^{m-k}_+ \to \mathbb{S}^m_+ \quad \text{where} \quad \phi(A) = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $\phi(A) = t\phi(B) + (1-t)\phi(C)$, with $\phi(B), \phi(C) \in \mathbb{S}^m_+$. By definition, $A \in F'$ implies that $\phi(A) \in F$. Additionally, because F is a face, $\phi(B), \phi(C) \in F$. Hence, $B, C \in F'$. Thus F' is a face of $\mathbb{S}^{m-k}_+$. Since A_0 is positive definite and $A_0 \in F'$, Lemma 6.4 indicates that $F' = \mathbb{S}^{m-k}_+$. Finally, if $Y \in F_V$. Then $Y = \phi(N)$ for some $N \in \mathbb{S}^{m-k}_+$. Since $F' = \mathbb{S}^{m-k}_+$, we have that $N \in F'$, so $\phi(N) = Y \in F$. Then, $F_V \subseteq F$. This shows that $F_V = F$. □

Example 6.7. The cone $\mathbb{S}^3_+$ has a facial dimension signature $\{0, 1, 3, 6\}$. Hence, signatures like

$$\{0, 6\}, \quad \{0, 1, 6\}, \quad \text{and} \quad \{0, 2, 3, 6\},$$

are not attainable as affine sections of $\mathbb{S}^3_+$. More generally, among signatures whose maximum is six, the only one attainable from $\mathbb{S}^3_+$ is $\{0, 1, 3, 6\}$ (the entire cone). Indeed, for any other signature with six as the maximal element, we require an affine section of $\mathbb{S}^m_+$ for some $m > 3$.

Theorem 6.8. *The smallest $m \in \mathbb{N}$ for which there exists a spectrahedron S representable by an m-by-m linear matrix inequality with*

$$\sigma(S) = \left\{ \frac{j(j+1)}{2} \ : \ j = 0, 1, \dots, n \right\}$$

is $m = n$.

Proof: First, we demonstrate that there exists a spectrahedron with the relevant signature that is representable by an n-by-n linear matrix inequality. Let $N = \dim(\mathbb{S}^n) = \frac{n(n+1)}{2}$ and choose a basis $E_1, E_2, \dots, E_N$ of $\mathbb{S}^n$. Define

$$S := \left\{ x \in \mathbb{R}^N : \sum_{i=1}^{N} x_i E_i \succeq 0 \right\}.$$

Then, the linear map $x \mapsto \sum_{i=1}^{N} x_i E_i$ is a vector space isomorphism between $\mathbb{R}^N$ and $\mathbb{S}^n$. It follows that S is linearly isomorphic to $\mathbb{S}^n_+$. Therefore, S has the same facial structure as $\mathbb{S}^n_+$. This implies that

$$\sigma(S) = \sigma(\mathbb{S}^n_+) = \left\{ \frac{j(j+1)}{2} \ : \ j = 0, 1, \dots, n \right\},$$

so the required facial dimension signature is achievable when using an n-by-n linear matrix inequality.

Next, we show that the smallest such linear matrix inequality has size $m = n$. The highest-dimensional face in the signature $\sigma(S)$ is of dimension $\frac{n(n+1)}{2}$. Hence $\dim(S) \geq \frac{n(n+1)}{2}$. On the other hand, if

$$S = \left\{ x \in \mathbb{R}^k : \ A_0 + \sum_{i=1}^{k} x_i A_i \succeq 0 \right\},$$

where without loss of generality we assume the $A_1, \dots, A_k \in \mathbb{S}^m$ are linearly independent, then the affine map

$$\mathcal{A} : \mathbb{R}^k \to \mathbb{S}^m, \qquad \mathcal{A}(x) = A_0 + \sum_{i=1}^{k} x_i A_i,$$

is injective so $k \leq \dim(\mathbb{S}^m) = \frac{m(m+1)}{2}$. It follows that since $S \subseteq \mathbb{R}^k$, we have that $\dim(S) \leq k$. By combining inequalities, we obtain

$$\frac{n(n+1)}{2} \leq \dim(S) \leq \frac{m(m+1)}{2}.$$

This implies that $m \geq n$ and no spectrahedron with the required signature can be expressed with $m < n$. Therefore, this demonstrates that n is the smallest value of m that yields this signature. □

Corollary 6.8.1. *Let S be a spectrahedron with $\max \sigma(S) = \frac{n(n+1)}{2}$. If $\sigma(S) \neq \sigma(\mathbb{S}^n_+)$, then S admits no $n \times n$ linear matrix inequality representation. Equivalently, any linear matrix inequality representation of S has size $m > n$.*

In addition to the minimal representation for signatures of the aforementioned form, we find an explicit upper bound on spectrahedral representation size for general facial dimension signatures using the results of [5]. In order to obtain a facial dimension signature $\Sigma \subseteq \{0, 1, \ldots, n\}$, where $0 = \min\{\Sigma\} < \max\{\Sigma\} = n$, [5] starts with a Euclidean ball in $\mathbb{R}^n$, which has facial dimension signature $\{0, n\}$, and takes careful intersections with cylinders to obtain faces of other dimensions. They define the parameters $c, r \in \mathbb{R}$ such that

$$1 + c > r > \sqrt{c^2 + \sqrt{2}c + 1},$$

to ensure the cylinder cuts the unit ball in a clean way that creates faces of the desired signature. We denote the Euclidean unit ball centred at the origin B^n. Additionally, they define

$$C_i^n := \left\{x \in \mathbb{R}^n : (x_{i+1} + c)^2 + x_{i+2}^2 + \cdots + x_n^2 \leq r^2\right\} \tag{3}$$

$$= \left\{x \in \mathbb{R}^n : rI_{n+1} + \left(e_{i+1}e_{n+1}^T + e_{n+1}e_{i+1}^T\right)c + \sum_{\ell=i+1}^{n} \left(e_\ell e_{n+1}^T + e_{n+1}e_\ell^T\right)x_\ell \succeq 0\right\}, \tag{4}$$

where I_{n+1} is the $(n+1)$-by-$(n+1)$ identity matrix and $e_i \in \mathbb{R}^{n+1}$ is the i^{th} standard basis vector [5]. If $S_0 = B^n$, then for all $i \in \{1, \ldots n-1\}$, they define

$$S_i := \begin{cases} S_{i-1} & \text{if } i \notin \Sigma, \\ S_{i-1} \cap C_i^n & \text{if } i \in \Sigma. \end{cases} \tag{5}$$

Finally, they let $S = S_{n-1}$.

We also construct a spectrahedron with this signature; however, we find its explicit representation and size as a linear matrix inequality. We begin by rewriting B^n as a $(n+1)$-by-$(n+1)$ linear matrix inequality. In doing so, we obtain

$$x \in B^n \quad \text{if and only if} \quad \begin{pmatrix} 1 & x^T \\ x & I_n \end{pmatrix} \succeq 0.$$

Additionally, if we let $d_i := n - i$, we write

$$y_i(x) := \begin{pmatrix} x_{i+1} + c \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^{d_i}.$$

Furthermore, if we seek to represent (3) as a linear matrix inequality, $(x_{i+1} + c)^2 + x_{i+2}^2 + \cdots + x_n^2 \leq r^2$ is $\|y_i(x)\| \leq r$ with respect to the standard inner product, or equivalently

$$\begin{pmatrix} r & y_i(x)^T \\ y_i(x) & rI_{d_i} \end{pmatrix} \succeq 0. \tag{6}$$

This block is of size $(d_i + 1)$-by-$(d_i + 1)$. Let $J := \Sigma \cap \{1, \ldots, n-1\}$ and let

$$M_0(x) = \begin{pmatrix} 1 & x^T \\ x & I_n \end{pmatrix} \in \mathbb{S}^{n+1} \quad \text{and} \quad M_j(x) = \begin{pmatrix} r & y_j(x)^T \\ y_j(x) & rI_{d_j} \end{pmatrix} \in \mathbb{S}^{n-j+1},$$

for $j \in J$. Then the required S can be represented as

$$S = \{x \in \mathbb{R}^n : \mathcal{M}(x) \succeq 0\},$$

where $\mathcal{M}(x) = \mathrm{diag}(M_0(x), (M_j(x))_{j\in J})$. It can be shown that this matrix is of size

$$m = n + 1 + \sum_{j\in J}(n - j + 1).$$

Thus, it is possible to obtain a spectrahedron with a given facial dimension signature represented as an m-by-m linear matrix inequality for some

$$m \leq n + 1 + \sum_{j\in J}(n - j + 1).$$

7 Discussion and conclusion

We developed a complete intersection criterion for affine subspaces with the positive semidefinite cone and characterised one-dimensional affine slices of closed convex cones in Theorem 4.5. We further proved a tight bound on the minimal size m of matrices required in spectrahedral representations that realise prescribed facial-dimension signatures for certain families of finite integer sequences in Theorem 6.8. In addition, leveraging [5], we obtained an explicit upper bound on spectrahedral representation size for facial dimension realisations.

It should be possible to sharpen the current upper bounds on spectrahedral representation size. Moreover, for several natural families of facial-dimension signatures, the optimal representation size remains unknown. A particularly interesting example is the finite integer sequence composed of zero and the first n prime numbers. Evidently, determining tight matrix bounds for signatures of this type appears to require new ideas beyond the constructions and estimates developed here.

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