Dupire's Equation and SPDEs: Theory, Financial Applications, and Corollary Results

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Abstract

Dupire's equation is used model European options exhibiting generally a minimum of space and time dependence. Dupire established the equation, and there has been notable work on the issue of well-posedness of the equation as well as extension of dependence and initial data spaces. We work through a method of space dependence. The stock modelling arising from the equation is of interest due connection with optimal transport through an explicit connection of time varying measures to a martingale and derivation using Fokker-Planck, suggesting a connection with Strassen and Kellerer's theorems. A proof of the former in the case of lebesgue measures with finite second moment is given. There is a discussion of practical applications.

1 Introduction

Dupire's Equation

Dupire's equation arises in the setting of options pricing, where, generally speaking, an option is a financial instrument wherein one has the right but not the obligation to buy (or sell) an underlying asset for a fixed price (the strike price) at a fixed point in the future (the time until which is called the time to maturity). This kind of option specifically is called a European option, and is modelled based off an underlying stock S_t satisfying:

$$dS_t = \mu \ dt + \sigma \ dB_t \tag{1}$$

where B_t is the Brownian motion. The classical theory of pricing these options is due to Black and Scholes, who derived a PDE (now known as the Black-Scholes PDE) that any $C^{1,2}$ function of t, S_t must satisfy to give the option price, subject to a self-financing condition, a no-arbitrage condition, and for constant μ, ν. In particular, this may be done with a replicating portfolio and the use of Ito's formula. This last condition is the subject of our focus, and we can see that it is somewhat restrictive. It doesn't account for any fluctuation, and in fact volatility smiles, where the volatility surface has a 'smile' shape on the time axis, where observed in the market in the 80s. Dupire's equation allows q, μ, ν to have time and space dependence, stating that:

$$\partial_t C = -\frac{1}{2}\sigma^* \sigma K^2 \Delta C - \mu K \nabla C + qC \tag{2}$$

or sometimes written, especially in applications, as:

$$\sigma^2 = 2 \cdot \frac{\partial_t C + \mu K \nabla C - qC}{K^2 \Delta C} \tag{3}$$

Where C is the option price, T is the time to maturity, K is the strike price, q is the dividend rate paid by the stocks, and μ is now the interest rate with ν as before. In this case σ is often called the local volatility. This may derived somewhat similarly to Black-Scholes in broad outline but of difference is that one often makes use of the Fokker-Planck equation as in [4]. What we essentially want to know is, is this a sensible PDE? is the problem of solving it well-posed with reasonable solutions? We also desire to relate it to optimal transport theory and other stochastics, in particular Strassen and Kellerer's theorems.

1.1 C_0-semigroup theory

The theory that we wish to use to analyse Dupire's equation is that of C_0-semigroup theory, which has the advantage of being relatively well-developed, comparatively simple, and clear in motivation. The central idea is to interpret the solution as evolving in time according to some orbit, similarly to the evolution of the exponential. We shall study the simpler case of Hilbert spaces, and so we also discuss some theory of forms which is key to the main result of this section. The following definitions are from [5], although much of this can be skipped by the totally unfamiliar reader as a they do not serve to properly develop the theory on their own; primacy is given, however, to Definition 1.5, Theorem 1.1, and Theorem 1.2, which will be necessary to understand the results of this paper in this area.

Definition 1.1. A form in a Hilbert space H is a pair $(a, D(a))$, where $D(a)$ is a subspace of H, the domain of a, and $a : D(a) \times D(a) \to \mathbb{K}$ is a sesquilinear mapping.

A form a is called accretive if

$$\operatorname{Re} a(x, x) \geq 0, \qquad x \in D(a),$$

and coercive if there exists a constant $\alpha > 0$ such that

$$\operatorname{Re} a(x, x) \geq \alpha \|x\|^2, \qquad x \in D(a).$$

Definition 1.2. A family $S = (S(t))_{t\geq 0}$ of bounded operators acting on X is called a C_0-semigroup if the following three properties are satisfied:

(S1) $S(0) = I$;

(S2) (Semigroup property) $S(t)S(s) = S(t + s)$ for all $t, s \geq 0$;

(S3) (Strong continuity)

$$\lim_{t\downarrow 0} \|S(t)x - x\| = 0 \quad \text{for all } x \in X.$$

Its infinitesimal generator, or briefly the generator, is the linear operator A defined by

$$D(A) := \left\{ x \in X : \lim_{t\downarrow 0} \frac{1}{t}(S(t)x - x) \text{ exists in } X \right\},$$

$$Ax := \lim_{t\downarrow 0} \frac{1}{t}(S(t)x - x), \qquad x \in D(A).$$

Definition 1.3. A C_0-semigroup S on X is called *analytic* on Σ_ω if for all $x \in X$ the function $t \mapsto S(t)x$ extends holomorphically to Σ_ω and satisfies

$$\lim_{\substack{z\in\Sigma_\omega \\ z\to 0}} S(z)x = x.$$

We call S an analytic C_0-semigroup if it is analytic on Σ_ω for some $\omega \in (0, \pi)$.

Definition 1.4. Let $0 < \omega \leq \frac{1}{2}\pi$. A form a on H is called ω-sectorial if

$$a(v) := a(v, v) \in \Sigma_\omega \quad \text{for all } v \in D(a).$$

Theorem 1.1. *Let H be a Hilbert space and let A be the densely defined closed operator in H associated with a densely defined closed form a that is ω-sectorial for some $0 < \omega < \frac{1}{2}\pi$. Then $-A$ generates an analytic C_0-semigroup of contractions on the sector $\Sigma_{\frac{1}{2}\pi-\omega}$.*

We study the following class of differential equations:

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)), & t \in [0, T], \\ u(0) = u_0. \end{cases} \tag{SCP}$$

and within the context of the following conditions:

(i) (Strong measurability) For all $x \in X$ the function $t \mapsto f(t, x)$ is strongly measurable on $[0, T]$;

(ii) (Linear growth) There exists a constant $C \geq 0$ such that

$$\|f(t, x)\| \leq C(1 + \|x\|), \qquad t \in [0, T],\ x \in X;$$

(iii) (Lipschitz continuity) There exists a constant $L \geq 0$ such that

$$\|f(t, x) - f(t, x')\| \leq L\|x - x'\|, \qquad t \in [0, T],\ x, x' \in X.$$

We shall need a more general kind of solution than one that is strongly defined, and this is formalised as follows:

Definition 1.5. A function $u : [0, T] \to X$ is called a *mild solution* of (SCP) if it is continuous and satisfies

$$u(t) = S(t)u_0 + \int_0^t S(t - s)f(s, u(s))\, ds, \qquad t \in [0, T].$$

Theorem 1.2. *Under the assumptions (i)-(iii) above, the semilinear problem* (SCP) *admits a unique mild solution $u \in C([0, T]; X)$. This solution depends continuously, in the norm of $C([0, T]; X)$, on the initial condition $u_0 \in X$.*

Convexity and Strassen and Kellerer's Theorems

We recall that a function $f : X \to Y$ for X, Y real-valued vector spaces is convex if $f(x_1 + t(x_2 - x_1)) \leq f(x_1) + t(f(x_2) - f(x_1))$ for all $x_1, x_2 \in X$ and $t \in [0, 1]$. We can use convexity to impose an ordering upon measures by how they act upon such functions. Specifically, we are interested in measures upon $\mathbb{R}^n$. We shall not include the dimension in the notation, noting that it will always be clear from context.

Definition 1.6. We define the partial ordering $\leq_c$ on the space of measures on $\mathbb{R}^n$ such that $\mu \leq_c \nu$ if and only if:

$$\int f \, d\mu \leq \int f \, d\nu \tag{4}$$

for all convex $f \in C(\mathbb{R}^n, \mathbb{R})$, where in the integrals of (4) we admit values in the extended reals and the standard ordering therein.

We emphasise that $f \in C(\mathbb{R}^n, \mathbb{R})$ as while only an integrability condition is needed for (4), convexity fails to be well-defined if we admit, say, $f \in L^1$. One should also recall the definition of an adapted process and martingale, found in Appendix A. This allows us to state Strassen's theorem:

Theorem 1.3 (Strassen)**.** *Consider measures μ, ν on $\mathbb{R}^n$. Then there exists a martingale M_t for $t \in [0, 1]$ such that $M_0 \sim \mu$ and $M_1 \sim \nu$ if and only if $\mu \leq_c \nu$.*

The forward direction is an easy application of Jensen's inequality, and can be found in [3]. The reverse direction is considerably more difficult. There is also an extension where instead of having this martingale transporting between two measures, we instead consider a family. This notion is formalised as a 'peacock':

Definition 1.7. A family of measures $(\mu_t)_{t\in[0,1]}$ is called a *peacock* if $\mu_t \leq_c \mu_s$ for all $t < s$.

and subsequently we have the following:

Theorem 1.4 (Kellerer)**.** *Consider a family of measures $(\mu_t)_{t\in[0,1]}$ on $\mathbb{R}$. Then there exists a martingale M_t for $t \in [0, 1]$ such that $M_t \sim \mu_t$ if and only if (μ_t) is a peacock.*

Note that we have dropped from considering measures on $\mathbb{R}^n$ for arbitrary n to only the case $n = 1$. In fact the case of higher dimensions is a somewhat open problem. Strassen follows from Kellerer by simply considering the peacock given by $\mu_t = \mu + t(\nu - \mu)$.

What we should like to do is relate this back to Dupire's equation, but, as an aside, why is this a sensible approach? Well, consider that the problem of martingale construction/existence arises naturally via the martingale pricing method for financial instruments, and Dupire's equation is involved in just such pricing. Indeed, (3) gives an explicit formula for the volatility of the process underlying the call option price, and we expect the diffusive process thus arising from (2) to be a martingale. That is we re-frame the central question of Dupire's equation from one of existence of pricing no matter the volatility behaviour to deriving a volatility behaviour from a desired pricing that evolves according to an interpolation from μ to ν. Then the underlying diffusive process can be expected to give our desired martingale.

There are, as with any approach, a number of issues that we should like to briefly outline. That the martingale is generated from a diffusive process must necessarily fundamentally limit the class of measures we are considering. Of greater concern is that (3) is extremely sensitive to singular or even relatively mild non-regular behaviour. We note that there are work-arounds, for instance re-defining the volatility outside of the support of the measures as 0 is helpful. In this approach, though, an obvious first goal is to obtain L^2 behaviour of σ, which again looking at (3) can be reasonably expected to enforce harsh regularity conditions on μ and ν which are perhaps not subsequently amenable to relaxing by an approximation argument.

Instead, we will go to the related Fokker-Planck equation:

$$\partial_t \rho = -\partial_x (\mu \cdot \rho) + \frac{1}{2} \partial_{xx} \left(\sigma^2 \cdot \rho \right) \tag{5}$$

Which occurs in the sense of a stochastic process as in (1) as a condition on the evolution of its pricing kernel ρ. This works much more closely to the level of the distribution, which is what is of interest to us for Strassen and Kellerer's theorems. It is also arguably more readily adapted to the weak formulation that is desired for reasons of working with less regular σ more easily. In particular:

Definition 1.8. Let a, b be Borel maps associated to the Fokker-Planck operator $\mathcal{L}_t$ (requiring that a be symmetric and positive). A Borel curve ρ_t is said to be a (weak) solution of the Fokker-Planck equation (FPE):

$$\partial_t \rho_t = \mathcal{L}_t^* \rho_t \tag{6}$$

if and only if $a, b \in L^1_{loc}(|\nu|)$ and:

$$\int_0^T \int (\partial_t f + \mathcal{L}_t f) \, d\nu_t dt = 0 \tag{7}$$

for all $f \in C_c^{1,2}((0,T) \times \mathbb{R}^n)$[1].

Where this version is according to [8], as is much of this set-up. Our primary theorem is the Figalli-Trevisan-Ambrosio superposition principle, specifically the version given by Trevisan. This reformulates our martingale question in a weak sense, avoiding some of the issues we had with the previous approach. Spefically, this weak formulation is:

[1]The space of compactly supported functions which are differentiable at least once in time and at least twice in any spatial direction.

Definition 1.9. Let a, b be Borel maps associated to $\mathcal{L}_t$. A probability measure $\eta \in \mathcal{P}(C([0,T];\mathbb{R}^n))$[2] is said to be a solution to the martingale problem (MP) associated to $\mathcal{L}_t$ if and only if $a, b \in L^1_{loc}(\eta)$ and for every $f \in C^{1,2}((0,T) \times \mathbb{R}^n)$ the process:

$$[0,T] \ni t \mapsto f_t \circ e_t - \int_0^t (\partial_t f_s + \mathcal{L}_s f_s) \circ e_s ds \tag{8}$$

is a martingale with respect to the natural filtration on $C([0,T];\mathbb{R}^n)$[3], where e_t is the evaluation map.

Finally, a borel curve ρ_t is said to be narrowly continuous iff for every $f \in C_b(\mathbb{R}^n)$ the map $t \mapsto \int f \, d\rho_t$ is continuous. Then finally we have in [8] the following result:

Theorem 1.5 (Trevisan)**.** *If the borel curve (ρ_t) is a narrowly continuous solution of FPE then there exists a solution η of MP such that $(e_t)_\#\eta =: \eta_t = \rho_t$ almost everywhere (in L^1).*

This is almost exactly what we want, and essentially reduces our method of proof to a technical problem. This was somewhat surprising to the author, and upon further literature review it turns out that proofs of Strassen and Kellerer's theorem via the Fokker-Planck equation are a relatively well-studied area. In particular, there is a proof by Hirsch and Roynette [3] that uses what is very loosely a blend of the two methods so far outlined to obtain a proof of Kellerer's theorem in its full generality. It does not, however, make direct use of the superposition principle, which in what follows greatly simplifies matters.

1.2 Statement of Authorship

The results Proposition 2.1 and Theorem 3.1 are my own, and the rest of the paper is largely based on [5] and [8].

2 Well-posedness of Dupire's Equation

To apply this result, we now wish to write Dupire's equation in the form of (SCP), so we write:

$$\partial_t c = -Ac + q(T)c \tag{9}$$

[2]The space of probability measures on continuous functions from $[0,T]$ to $\mathbb{R}^n$.

[3]The natural filtration being the one generated by the evaluation maps e_t upon evaluation of a path γ.

where we have lowered the case for clarity. Now take:

$$A = -\frac{1}{2}\sigma^*\sigma K^2\Delta - \mu K\nabla \tag{10}$$

Then under the mild condition of $q \in L^\infty$ gives[4] satisfaction of conditions (i-iii). It then suffices to show that $-A$ generates a C_0-semigroup. This relates back to Theorem 1.1, and so we should wish to prove that A is ω-sectorial. But at least a priori there is no especially good reason to expect that it is, and indeed what we shall do is change the Hilbert space H it is operating on by endowing $\mathbb{R}^n$ with a modified inner product.

Proposition 2.1. *Within the space $H = L^2(\mathbb{R}, \langle\rangle_g)$ endowed with the inner product:*

$$\langle f_1, f_2\rangle_g = \int (f_1 \cdot f_2) \cdot g \; dx \tag{11}$$

where:

$$g(x) = e^{H(x)} \tag{12}$$

$$H(x) = -\int_1^x \left(\frac{2}{t} + \frac{2\mu}{t \cdot \sigma\sigma^*} + \frac{div(\sigma^*\sigma)}{\sigma^*\sigma}\right) dt \tag{13}$$

taken in a weak sense, the operator A is ω-sectorial (in fact positive).

Proof. This is essentially a computational verification. We write:

$$\begin{aligned}
\langle Af, f\rangle_g &= \left\langle \frac{1}{2}\sigma^*\sigma K^2(-\Delta), f\right\rangle_g + \langle \mu K\nabla f, f\rangle_g \\
&= \frac{1}{2}\left\langle \operatorname{div}\left(\sigma^*\sigma(-\nabla)f\right), K^2 f\right\rangle_g + \frac{1}{2}\left\langle K^2 \operatorname{div}\left(\sigma^*\sigma\right)\nabla f, f\right\rangle_g + \langle \nabla f, \mu K f\rangle_g \\
&= \frac{1}{2}\langle \sigma K\nabla f, \sigma K\nabla f\rangle_g + \left\langle \nabla f, \sigma^*\sigma\left(K^2\frac{\nabla g}{2g} + K\right) f\right\rangle_g \\
&+ \left\langle \nabla f, \mu K + \frac{1}{2}K^2 \operatorname{div}\left(\sigma^*\sigma\right) f\right\rangle_g \\
&= \frac{1}{2}\langle \sigma K\nabla f, \sigma K\nabla f\rangle_g + \left\langle \nabla f, \left(\mu + \sigma^*\sigma\right) K f + \frac{1}{2}\left(\sigma^*\sigma\frac{\nabla g}{g} + \operatorname{div}\left(\sigma^*\sigma\right)\right) K^2 f\right\rangle_g
\end{aligned}$$

[4]In practical terms all we are asking is that the dividend rate will not blow up to infinity, which should be true in a regulated market with limited funds.

where in the second line the new second term arises to deal with the extra term arising from application of the chain rule in the first term after the manipulation. In the third line we have applied the adjoint of divergence, taking note that this also affects the g and K^2 terms for the addition of the new term (having aggregated the other relevant terms). It now suffices to show that this additional term is in fact zero. But this follows easily from this definitions, as:

$$\frac{\nabla g}{g} = \nabla H(x) = -\left(\frac{2}{K} + \frac{2\mu}{K \cdot \sigma\sigma^*} + \frac{\mathrm{div}\,(\sigma^*\sigma)}{\sigma^*\sigma}\right)$$

so that in particular:

$$\frac{1}{2}K^2\sigma^*\sigma\frac{\nabla g}{g} = -\frac{1}{2}K^2\mathrm{div}\,(\sigma)^* - K\sigma^*\sigma - K\mu$$

which exactly cancels the other terms. Thus:

$$\begin{aligned}\langle Af, f\rangle &= \frac{1}{2}\langle \sigma K\nabla f, \sigma K\nabla f\rangle_g \\ &\geq 0\end{aligned}$$

for all f, proving positivity of the operator. □

An obvious point of contention in the above is the definition of H; we require sufficient regularity conditions on $\mu, \sigma^*\sigma$ for this to well-defined and behave sensibly. Nonetheless, the result is sufficient to have:

Corollary 2.1. *For any $f \in H$ we have there exists a mild solution to* (2) *with initial data f.*

Proof. This is an immediate corollary of Theorem 1.1 and 1.5 when considering Proposition 2.1 and our prior exposition. □

3 Strassen via the Superposition Principle

Theorem 3.1. *Strassen's theorem holds for $n = 1$ with $d\mu = u(x)dx, d\nu = v(x)dx$, where $u, v \in L^1_{loc}$ and μ, ν have finite second moment.*

Proof. We define, taking $\kappa = \mu - \nu, k(x) = u(x) - v(x)$:

$$F(x) := \int_0^x k(y)\ dy$$
$$G(x) := \int_0^x F(u)du$$

Where we note that $\partial_{xx}G(x) = k(x)$. Write $p_t = |u|+|v|+(u+t\cdot k)$, and $\rho_t = p_t dx$. We now define:

$$a_t(x) := \begin{cases} \frac{G(x)}{p_t(x)} & \text{if } p_t(x) \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

This then defines the operator, and we essentially need only verify that this is well-defined, and that ρ_t gives a narrowly continuous weak solution. For the operator arising from a_t to be well-defined in the context of the problem, we note that it is trivially symmetric in 1 dimension and that we require that a_t is non-negative. For this it suffices to verify that $G(x)$ is non-negative since $p_t(x)$ is already due to the addition of the absolute value factors. But this follows easily from re-writing $G(x)$ and using convex ordering directly:

$$\begin{aligned} G(x) &= \int_0^x \int_0^u k(y)dy\ du \\ &= \int \int 1_{[0,x]}(u) 1_{[0,u]}(y) k(y) dy\ du \\ &= \int \int 1_{[0,x]}(u) 1_{[0,u]}(y) k(y) du\ dy \end{aligned}$$

by an application of Tonelli, which is justified by L^1_{loc} conditions. We note that $1_{[0,x]}(u)1_{[0,u]}(y) = 1_{[0,x]}(u)1_{[0,y]}(u) = 1_{[x,y]}(u)$, and so we write:

$$\begin{aligned} G(x) &= \int k(y) \int 1_{[x,y]}(u) du\ dy \\ &= \int k(y) \, |x-y| \, dy \\ &\geq 0 \end{aligned}$$

Since $|x-y|$ is convex in y, and applying the definition of convex ordering. We proceed to verification that ρ_t is a weak solution. That $a_t \in L^1_{loc}(|\rho_t|)$ may be done by imposing the additional assertion that μ, ν have finite second moment. For (7) we shall proceed with integration by parts:

$$\begin{aligned}\int_0^T \int (\partial_t f + L_t f) d\rho_t dt &= \int_0^T \int (\partial_t f + L_t f) p_t dx dt \\ &= \int_0^T \int p_t \partial_t f \ dx \ dt + \int_0^T \int p_t L_t f \ dx \ dt \\ &= \int_0^T \int p_t \partial_t f \ dt \ dx + \int_0^T \int G(x) \partial_{xx} f \ dx \ dt\end{aligned}$$

where we have used Fubini. The fact that we only require f to be of compact support allows us to disregard the additional term introduced by integrating by parts, so:

$$\begin{aligned}\int_0^T \int (\partial_t f + L_t f) d\rho_t dt &= \int_0^T \int (-\partial_t p_t + \partial_{xx} G(x)) f \ dx \ dt \\ &= \int_0^T \int (-k(x) + k(x)) f \ dx \ dt \\ &= 0\end{aligned}$$

Since our integrand is 0. As such, we have that ρ_t is a solution to the FPE. To invoke the superposition principle, we only now require that the curve ρ_t is narrowly continuous. But this is an easy consequence of its definition via interpolation, as:

$$\begin{aligned}\int f d\rho_t &= \int f p_t \ dx \\ &= \int f(x) u(x) \ dx + t \int f(x) k(x) \ dx\end{aligned}$$

which is clearly continuous in time as an affine map. As a result, we have the existence of η which solves the martingale problem for the associated operator $\mathcal{L}_t$. In particular, if we substitute $f \equiv 1$ into (8) we see that the evaluation map $e_t : \gamma \mapsto \gamma_t$ is a martingale under the natural filtration when $C([0, T]; \mathbb{R})$ is endowed with η as a measure. In particular, $M_t = e_t$ has marginals $M_t = (e_t)_{\#}\eta = \eta_t$, which by the result of the principle is ρ_t. Thus, we almost have our martingale transport from μ to ν[5]. We require an additional constant martingale to remove the absolute value terms that were added, but this is easily done.

[5] Technically this is only L^1 almost everywhere, but as in what follows this only requires introducing a constant martingale to rectify the resulting issue

Consider a second probability space Ω defined such that $Y : \Omega \to \mathbb{R}$ is a random variable with distribution $|\mu| + |\nu|$[6]. Taking $\Omega' = \Omega \times [0, T]$ we endow it with the σ-algebra $\mathcal{F}' = \{F \times [0, T] : F \in \mathcal{F}\}$ and the trivial filtration $\mathcal{F}'_t = \mathcal{F}'$ for all t. Now consider $Y_t : \Omega' \to \mathbb{R}$ the process, which is trivially a martingale, that is the projection of the Ω coordinate through Y. We then form a new space $C([0, T]; \mathbb{R}) \times \Omega'$. Then extending both M_t, Y_t in the natural way through pointwise projection and endowing the space with the cross product σ-algebra and filtration the process $M_t - Y_t$ is a martingale transport from μ to ν.

□

A key drawback of the superposition principle is that it is not readily amenable to approximation arguments or broader classes of measures. In terms of approximation we are limited in that the space is what is being changed to generate η, and so the approximation would need to pass through in this way. We also require some kind of integration by parts, which while easily handled in the case of measures that are absolutely continuous with respect to the Lebesgue is not so easily handled in broader generality. Still, it is perhaps possible that this result implies the full one as is the case for other seemingly restrictive versions such as in [7]; perhaps a messy approximation of spaces is avoidable. Some investigation was done as part of this project into extending this to a version of Kellerer, but many of the technical points that are easily put aside when linearly interpolating in time are somewhat thorny for more complicated time dependent behaviour.

4 Discussion, Applications and Conclusion

Recall that our study of Dupire's equation was itself motivated by the fact that the Black-Scholes model did not account for reasonable conditions of the stock model, and subsequent to its adoption such conditions were indeed observed in the market. In practical terms, the well-posedness actually guarantees the ability to price an option as long as the market behaves in a much broader sense than in Black-Scholes. While the broader theory of time dependence and general measure conditions is beyond our scope, we note that it can be done [2]. What is something of a bigger issue is that our model does not account for any randomness within the market, which is where the titular Stochastic PDEs (SPDEs) come in. Namely, we allow $\sigma(K, T, \omega)$ for $\omega \in \Omega$ some probability space. This, however, adds to the already significant complexity of our limited space and time dependence, and is therefore

[6] Which necessarily exists.

unfortunately beyond the scope of this report. We might also wish for an L^p theory to account for other market conditions, but this is in fact an area of active research and thus also unfortunately beyond our scope.

In terms of the work on Strassen and Kellerer, the limitations have been outlined previously. In terms of applications, what is of greatest interest is the application of these ideas to pricing and in particular portfolio replication. Namely, we have some ability to transport a distribution at current time to a desired distribution, and have at least outlined some ideas for how the relevant martingale might be constructed, if perhaps in a mathematically loose sense. We note for the reader that while martingale pricing of options is comparatively easy, portfolio replication is a harder problem. Consequently, more careful analysis related to the Fokker-Planck and Dupire equations may yield explicit representations of practical use.

With regards to algorithmic computation of implicit volatility, there is often an issue of interpolation in real-world use cases wherein only a finite number of strike price/time price pairs are known and trying to fill in the remainder of the surface can cause singular behaviour that subsequently causes failure of the theory. The use of optimal transport theory such as further in Strassen/Kellerer has been used to create an interpolation method without such a problem [1], though this method is potentially not computationally viable for applications. A clear direction is to pair this more closely with functional analytic nature of this report and seek a computationally simpler method that serves as intermediary between the practical and the theoretically advantageous, perhaps through machine learning methods for quadratic PDEs such as in [6].

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Appendix A: Probability Theory

Definition 4.1. A *probability space* is a triple

$$(\Omega, \mathcal{F}, \mathbb{P}),$$

where $\mathcal{F}$ is a σ-algebra of subsets of Ω and $\mathbb{P} : \mathcal{F} \to [0, 1]$ is a probability measure with $\mathbb{P}(\Omega) = 1$.

We shall refer to the whole triple only by Ω where no confusion shall arise.

Definition 4.2. A *filtration* is a family of sub-σ-algebras

$$\{\mathcal{F}_t\}_{t \in T}$$

indexed by a totally ordered set such that

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$$

whenever $s \leq t$.

It is common to interpret $\mathcal{F}_t$ as the 'information' available up to time t within the context of the probability space.

Definition 4.3. For Ω a probability space, a stochastic process is a collection of random variables

$$X = \{X_t : \Omega \to \mathbb{R}\}_{t \in T}$$

where a random variable is the probability theory terminology for a measurable function.

The process X is said to be *adapted* to the filtration $\{\mathcal{F}_t\}$ if for every t, the random variable X_t is $\mathcal{F}_t$-measurable.

Definition 4.4. Let X be an integrable random variable: $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$

Then the *expectation* of X is

$$\mathbb{E}[X] = \int_\Omega X(\omega)\, \mathbb{P}(d\omega).$$

Definition 4.5. Given a sub-σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, the *conditional expectation* of X with respect to $\mathcal{G}$, denoted $\mathbb{E}[X \mid \mathcal{G}]$, is defined as a $\mathcal{G}$-measurable random variable Y satisfying

$$\int_G Y(\omega)\, d\mathbb{P} = \int_G X(\omega)\, d\mathbb{P}$$

for every $G \in \mathcal{G}$.

which one can interpret as the probability theory version of projection onto a subspace. Existence and uniqueness can be established in the L^2 case by an application of the Riesz representation theorem.

Definition 4.6. Let Ω be a probability space and $\mathcal{F}_t$ a filtration on Ω. We say a stochastic process $\{M_t\}_{t \in T}$ is a *martingale* with respect to the filtration $\{\mathcal{F}_t\}$ and measure $\mathbb{P}$ if M_t is adapted, M_t is integrable for every t, and the *martingale property* is satisfied:

$$\mathbb{E}[\, M_t \mid \mathcal{F}_s \,] = M_s$$

for all $s \leq t$.

This may be interpreted as being a process that is well-behaved, where one has no future knowledge, and where current knowledge will - loosely - not alter the overarching behaviour of the process.

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