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Superspaces, Superalgebras and Supersymmetry!

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Abstract

In this paper, we introduce the definitions of Lie algebras and Lie superalgebras, and the notion of a representation. We explore the representation theory of a particular Lie superalgebra known as $\mathfrak{gl}(1|1)$ and apply its consequences to a theory of quantum mechanics with a particular property known as supersymmetry.

Contents

Introduction

The universe has many symmetries within it; the laws of physics are unchanged from one position to another and from one moment to the next. These symmetries allow us to gain insight into the fundamentals of our universe. There is, however, a particular dichotomy that seems as if there should be a symmetry there, where none has been observed. This is the types of particles, the bosons and the fermions. Supersymmetry has been proposed as a possible property that the theory of quantum field theory may have, where there exists a symmetry that connects the fermions, the matter particles, and the bosons, the force mediators. As with all quantum physics, this property can be described mathematically, in this case with a structure called a Lie superalgebra. The aim of this paper is to introduce the reader to the concepts of Lie algebras and then Lie superalgebras, and to explore what is known as the representation theory of Lie superalgebras. The representation theory allows us to realise how the actions of particular operators behave, and thus can be applied to supersymmetric quantum mechanics where these operators act on physical states, allowing us to infer properties that a supersymmetric theory of quantum mechanics may possess.

Statement of Authorship

All results in this paper have already been discovered and proved by other authors, however, the derivations, calculations and proofs in this paper have been redone independently of previous literature.

1 Lie Algebras

1.1 Definition of a Lie Algebras

A Lie algebra over a field $\mathcal{F}$ is a vector space V with a bilinear map $V \times V \to V$, $(x, y) \mapsto [x, y]$ where $[\cdot, \cdot]$ is called the Lie bracket, satisfying the following properties:

1. $[x, x] = 0, \ \forall x \in V$
2. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \ \forall x, y, z \in V$ (Jacobi Identity)

Additionally, property 1. implies that $[x, y] = -[y, x], \ \forall x, y \in V$.

1.2 Some Examples of Lie Algebras

- $\mathbb{R}^3$ along with the cross product is a Lie algebra.
- Any vector space V with Lie bracket defined by $[x, y] = 0, \ \forall x, y \in V$, known as the abelian Lie algebra on V.
- The vector space of all $n \times n$ matrices over $\mathcal{F}$ with Lie bracket $[x, y] = xy - yx$ is known as $\mathfrak{gl}(n, \mathcal{F})$, the general linear Lie algebra. Also denoted as $\mathfrak{gl}(V)$ when considering instead the linear maps from V to itself, which is isomorphic to $\mathfrak{gl}(n, \mathcal{F})$. (Note: For convenience, whenever $\mathcal{F} = \mathbb{C}$, the field will be omitted from the notation $\mathfrak{gl}(n, \mathcal{F})$, becoming simply $\mathfrak{gl}(n)$).
- The matrices with trace equal to zero is a subalgebra of $\mathfrak{gl}(n, \mathcal{F})$ known as the special linear Lie algebra. It is denoted by $\mathfrak{sl}(n, \mathcal{F})$.

1.3 Subalgebras, Ideals and Homomorphisms

A Lie subalgebra of a Lie algebra V is a subspace W of V such that $[x, y] \in W, \ \forall x, y \in W$, i.e. It is a subspace of V which is closed under the Lie bracket.

An ideal of a Lie algebra V is a subspace I of V such that $[x,y] \in I,\ \forall x \in V,\, y \in I$. An ideal is necessarily a subalgebra, specifically one that is 'absorbing'; taking the Lie bracket of an element in the ideal with any other element yields another element of the ideal. This mirrors the concept of a normal subgroup.

$\mathfrak{sl}(n)$ is an ideal of $\mathfrak{gl}(n)$, since if $A \in \mathfrak{sl}(n)$ and $B \in \mathfrak{gl}(n)$, $\mathrm{tr}[A,B] = \mathrm{tr}(AB - BA) = \mathrm{tr}(AB) - \mathrm{tr}(BA) = 0$, hence the result of the Lie bracket is also traceless, indicating that $\mathfrak{sl}(n)$ is an ideal.

If U and V are Lie algebras, then the map $\varphi : U \to V$ is a homomorphism if φ is linear and $\varphi([x,y]) = [\varphi(x), \varphi(y)],\ \forall x, y \in U$.

A typical example of a homomorphism is the adjoint homomorphism, $\mathrm{ad} : V \to \mathfrak{gl}(n)$, defined by $(\mathrm{ad}\, x)(y) = [x,y],\ \forall x, y \in V$, which one can prove is a homomorphism using the Jacobi identity.

1.4 Representations of Algebras

A representation of a Lie algebra A is a homomorphism $\varphi : A \to \mathfrak{gl}(V)$ determined by the action $\cdot$ of elements of A on a vector space V, such that $a \cdot v = \varphi(a)v,\ \forall a \in A, v \in V$.

In order to find representations of a Lie algebra A, we consider an A-module where elements of A act onto the vectors of V, and write down the corresponding linear transformation, which in turn gives us the map φ. When a basis is chosen, this yields a representation matrix for the particular element a. Both the map and the module (the vector space V equipped with the action of elements in A) are referred to as a representation.

1.5 Representations of $\mathfrak{sl}(2)$

The matrix definition of $\mathfrak{sl}(2)$ is the traceless 2×2 matrices, i.e.:

$$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, a + d = 0 \right\} \implies a = -d$$

A frequent basis of $\mathfrak{sl}(2)$ is $\{e, f, h\}$, where $e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. The map from $\{e, f, h\}$ onto their respective matrices is called the defining representation of $\mathfrak{sl}(2)$ since it is the representation based on its definition.

The commutation relations between these elements are as follows:

$$[h, e] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = 2e$$

$$[h, f] = hf - fh = h^T e^T - e^T f^T = -2f$$

$$[e, f] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = h$$

Another representation is the adjoint representation, which is found when considering the following action:

$$x \cdot y = [x, y] = (\operatorname{ad} x)(y)$$

The action of an element x on each basis element gives the adjoint representation:

$$\operatorname{ad} e = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \operatorname{ad} f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{bmatrix} \quad \operatorname{ad} h = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

There is also the trivial representation, where each of e, f and h are mapped to $[0]$.

1.6 Terminology for Modules

Some definitions:

- A module is called irreducible if it contains exactly 2 submodules, itself and $\{0\}$.
- A module is called indecomposable if it cannot be written as a non-trivial direct sum, i.e. if $M = N_1 \oplus N_2 \implies N_1 = 0$ or $N_2 = 0$ (Direct sum $V = V_1 \oplus V_2$ refers to the vector space with elements of the form $v = v_1 + v_2$ where $v_1 \in V_1$, $v_2 \in V_2$ and $V_1 \cap V_2 = \{0\}$, where the action is given by $a \cdot v = a \cdot v_1 + a \cdot v_2$).

1.7 General Representations of $\mathfrak{sl}(2)$

Let V be an $\mathfrak{sl}(2)$-module, i.e. there exists a representation $\pi : \mathfrak{sl}(2) \to \mathfrak{gl}(V)$.
Assume $\dim V < \infty$ and V is complex and irreducible. Then the fundamental theorem of algebra states that h has an eigenvalue, which implies the existence of an eigenvalue λ with maximal real part. Let v_λ be the eigenvector. In order to find the representation of h, we consider the action of h on ev_λ and fv_λ:

(hev_λ) From the commutation relations, we know that $[h, e] = 2e = he - eh$

$$\implies 2ev_\lambda = hev_\lambda - ehv_\lambda \implies hev_\lambda = 2ev_\lambda + \lambda ev_\lambda = (\lambda + 2)v_\lambda$$

Thus it seems that ev_λ is an eigenvector of h with eigenvalue $\lambda + 2$, but since λ has maximal real part, this cannot be the case. Thus $ev_\lambda = 0$.

(hfv_λ) From the commutation relations, we know that $[h, f] = -2f = hf - fh$

$$\implies -2fv_\lambda = hfv_\lambda - fhv_\lambda \implies hfv_\lambda = -2fv_\lambda + \lambda fv_\lambda = (\lambda - 2)fv_\lambda$$

Hence fv_λ is an eigenvector of h with eigenvalue $\lambda - 2$.

In general, $hf^n v_\lambda = (\lambda - 2n)f^n v_\lambda$, $n \in \mathbb{N}$ and since each of the eigenvectors of h have distinct eigenvalues, they are linearly independent and form a basis for V. Thus, since V is finite dimensional, $\exists m \in \mathbb{N}$ such that $f^m v_\lambda \neq 0$ and $f^{m+1} v_\lambda = 0$.

Now we consider $ef^n v_\lambda$, utilising the fact that $[e,f] = ef - fe = h$. Computing $ef^n v_\lambda$ for the first few n yields:
$ef^0 v_\lambda = 0, \quad ef^1 v_\lambda = (hv_\lambda - fev_\lambda) = \lambda v_\lambda,$
$ef^2 v_\lambda = (ef)(fv_\lambda) = (h + fe)(fv_\lambda) = hfv_\lambda + fefv_\lambda = (\lambda - 2)fv_\lambda + \lambda fv_\lambda = (2\lambda - 2)fv_\lambda.$
From these computations, it can be supposed that $ef^n v_\lambda = (n\lambda - n(n-1))f^{n-1}v_\lambda$, which will be proven by induction.

Assume that for some $k > 0$, we have that $ef^k = (k\lambda - k(k-1))v_\lambda$ We now must show, given the expression is true for $n = k$, it is true for $n = k+1$. Thus, we want to show that $ef^{k+1}v_\lambda = ((k+1)\lambda - k(k+1))v_\lambda$.

$\text{LHS} = ef^{k+1}v_\lambda = (ef)(f^k v_\lambda) = (h + fe)(f^k v_\lambda) = hf^k v_\lambda + fef^k v_\lambda$
$= (\lambda - 2k)f^k v_\lambda + f(k\lambda - k(k-1))f^{k-1}v_\lambda = (\lambda - 2k + k\lambda - k^2 + k)f^k v_\lambda$
$= ((k+1)\lambda - k(k+1))f^k v_\lambda = \text{RHS}. \quad \square$
Thus $ef^n v_\lambda = (n\lambda - n(n-1))f^{n-1}v_\lambda := c_{n,\lambda}f^{n-1}v_\lambda$

Hence the general representation of $\mathfrak{sl}(2)$ is as follows:

$$e \mapsto \begin{bmatrix} 0 & c_{\lambda,1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & c_{\lambda,2} & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & c_{\lambda,m} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad f \mapsto \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad h \mapsto \begin{bmatrix} \lambda & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda-2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda-4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda+2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\lambda \end{bmatrix}$$

2 Superspaces and Lie Superalgebras

2.1 Vector Superspaces

A vector superspace, which is also known as a vector space with $\mathbb{Z}_2$-grading, is a vector space V with a decomposition $V = V_{\bar{0}} \oplus V_{\bar{1}}$, where $V_{\bar{0}}$ is known as the even space and $V_{\bar{1}}$ is known as the odd space. Elements of each of the subspaces are called even and odd respectively, and their parity is denoted with a bar, i.e. $\bar{v}$ takes values 0 or 1 ($\bar{v} \in \mathbb{Z}_2 = \{0, 1\}$).

2.2 Superspace Homomorphisms

One can consider homomorphisms between vector superspaces, which are linear maps that preserve the decomposition. If $V = V_{\bar{0}} \oplus V_{\bar{1}}$ and $W = W_{\bar{0}} \oplus W_{\bar{1}}$, then a superspace homomorphism $\varphi : V \mapsto W$ would map $V_{\bar{0}}$ to $W_{\bar{0}}$ and map $V_{\bar{1}}$ to $W_{\bar{1}}$, which is called parity-preserving. This is quite constraining however, so we may allow maps to take a linear combination of the two following forms:

$$\left[\begin{array}{c|c} \star & 0 \\ \hline 0 & \star \end{array}\right]$$

or

$$\left[\begin{array}{c|c} 0 & \star \\ \hline \star & 0 \end{array}\right]$$

A superspace homomorphism of the first kind maps the even space of V onto the even space of W, and the odd space of V onto the odd space of W and is called parity-preserving. The second kind maps even onto odd and odd onto even and is called parity-reversing. Linear combinations of these may not necessarily preserve the $\mathbb{Z}_2$-grading.

2.3 Lie Superalgebras

A Lie superalgebra is a vector superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ with a bilinear map called Lie superbracket $[\cdot, \cdot]$, which satisfies two properties similar to those of the Lie bracket:

1. $[x, y] = -(-1)^{\bar{x}\bar{y}}[y, x], \quad \forall x, y \in V$
2. $(-1)^{\bar{x}\bar{z}}[x, [y, z]] + (-1)^{\bar{x}\bar{y}}[y, [z, x]] + (-1)^{\bar{y}\bar{z}}[z, [x, y]] = 0$ (Super Jacobi identity)

$[\cdot, \cdot]$ also satsfies $\forall v_i \in V_i$ and $v_j \in V_j$, $[v_i, v_j] \in V_{i+j}$. What this means is that the Lie superbracket of two elements of the same parity will yield an even element, and the Lie superbracket of two elements of the opposite parity will yield an odd element.

Common examples of Lie superalgebras are:

- $\mathfrak{gl}(n|m)$, the general linear Lie superalgebra which consists of all $(n + m) \times (n + m)$ supermatrices.
- $\mathfrak{sl}(n|m)$, the special linear Lie superalgebra, consisting of all elements of $\mathfrak{gl}(n|m)$ with zero supertrace, where the supertrace is given by:

$$\mathbf{str}\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \mathbf{tr}(A) - \mathbf{tr}(D)$$

- $\mathfrak{osp}(m|2n)$, the orthosymplectic superalgebra.

Each of the Lie superalgebras above utilise the same superbracket, known as the supercommutator which is given by the following expression:

$$[A, B] = AB - (-1)^{\bar{A}\bar{B}}BA$$

For a bracket involving an even element, $\bar{A}\bar{B} = 0$, and hence this expression reduces into the usual commutator. However, when both A and B are odd, $\bar{A}\bar{B} = 1$, meaning that instead the supercommutator is given by the anticommutator $AB + BA$ which is denoted by $\{A, B\}$.

2.4 $\mathfrak{osp}(1|2)$

$\mathfrak{osp}(m|2n)$ is known as the orthosymplectic Lie superalgebra, and is comprised of elements which preserve a supersymmetric bilinear form on $\mathbb{R}^{m|2n}$. This definition is equivalent to $g^{\mathrm{sT}}\,\mathrm{J}_{m|2n} = -\,\mathrm{J}_{m|2n}\,g$, where $\mathbf{sT}$ denotes the supertranspose, i.e.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{\mathbf{sT}} = \begin{bmatrix} A^{\mathbf{T}} & C^{\mathbf{T}} \\ -B^{\mathbf{T}} & D^{\mathbf{T}} \end{bmatrix}$$

In the case of $\mathfrak{osp}(1|2)$, $\mathrm{J}_{1|2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{I} & 0 \\ \hline 0 & \mathbf{J} \end{array}\right]$

Let $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where A is 1×1, B is 1×2, C is 2×1, D is 2×2.

From the definition of $\mathfrak{osp}(1|2)$:

$$\left[\begin{array}{c|c} A^{\mathbf{T}} & C^{\mathbf{T}} \\ \hline -B^{\mathbf{T}} & D^{\mathbf{T}} \end{array}\right] \left[\begin{array}{c|c} \mathbf{I} & 0 \\ \hline 0 & \mathbf{J} \end{array}\right] = \left[\begin{array}{c|c} \mathbf{I} & 0 \\ \hline 0 & \mathbf{J} \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

$$\implies A^{\mathbf{T}} + A = 0 \implies A = [0] \in \mathfrak{o}(1), \quad D^{\mathbf{T}}J = -JD \implies \begin{bmatrix} d_{11} & d_{21} \\ d_{12} & d_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$$

$$\implies \begin{bmatrix} -d_{21} & d_{11} \\ -d_{22} & d_{12} \end{bmatrix} = \begin{bmatrix} -d_{21} & -d_{22} \\ d_{11} & d_{12} \end{bmatrix} \implies d_{11} = -d_{22}$$

Therefore:

$$D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & -d_{11} \end{bmatrix} \in \mathfrak{sl}(2) \simeq \mathfrak{sp}(2)$$

So far:

$$g = \begin{bmatrix} 0 & b_{11} & b_{12} \\ c_{11} & d_{11} & d_{12} \\ c_{21} & d_{21} & -d_{11} \end{bmatrix} \implies \begin{bmatrix} 0 & c_{11} & c_{21} \\ -b_{11} & d_{11} & d_{12} \\ -b_{12} & d_{21} & -d_{11} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & b_{11} & b_{12} \\ c_{11} & d_{11} & d_{12} \\ c_{21} & d_{21} & -d_{11} \end{bmatrix}$$

Hence, a natural basis for $\mathfrak{osp}(1|2)$ is:

$$\left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}$$

The first 3 elements are labelled e, f and h and are called the even elements, since they are block diagonal. The latter 2 elements are labelled x and y and are called the odd elements, being block antidiagonal. The even part of $\mathfrak{osp}(1|2)$ is isomorphic to $\mathfrak{o}(1) \oplus \mathfrak{sp}(2) \simeq \mathfrak{sl}(2)$.

2.5 Representations of Lie Superalgebras

Representation of Lie superalgebras are similar to representations of Lie algebras, being parity-preserving linear maps $\varphi : A \mapsto \mathfrak{gl}(V)$ where $V = V_{\bar{0}} \oplus V_{\bar{1}}$, determined by considering the module produced when we equip the vector superspace V with the action of elements from the Lie superalgebra. The representation is found in a similar way to that described in 1.4, and similarly, the modules are also referred to as representations along with the map.

2.6 Representations of $\mathfrak{gl}(1|1)$

$\mathfrak{gl}(1|1)$ consists of all 2×2 supermatrices. A natural basis is:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

These are called N, E, ψ^+ and ψ^- respectively and they have the following commutation relations:

$$\{\psi^\pm, \psi^\pm\} = 0, \quad [N, \psi^\pm] = \pm 2\psi^\pm, \quad \{\psi^+, \psi^-\} = E$$

In order to find representations of $\mathfrak{gl}(1|1)$, we consider a $\mathfrak{gl}(1|1)$-module which contains a highest weight vector $v_{n,e}$ such that:

$$N v_{n,e} = n v_{n,e} \text{ and } E v_{n,e} = e v_{n,e}$$

The vector $v_{n,e}$ being highest weight means that it is annihilated by the raising operator ψ^+. This is possible to find since according to the fundamental theorem of algebra, N must have an eigenvalue and thus an eigenspace, and we consider the restriction of E to that eigenspace. Since E commutes with N, $E(Nv) = N(Ev)$ which means that E restricted to the eigenspace of N is an endomorphism, and thus we can consider an eigenspace of E giving us the simultaneous eigenvector $v_{n,e}$ above. If this vector is not annihilated by ψ^+, we may redefine it as $\psi^+ v_{n,e}$ which remains an eigenvector of N and E and is annihilated by ψ^+ since $(\psi^+)^2 = 0$.
Additionally, since $(\psi^-)^2 = 0$, $(\psi^-)^2 v_{n,e} = 0$, resulting in the following Verma module:

$$\mathcal{V}_{n,e} = \textbf{span}\{v_{n,e}, \psi^- v_{n,e}\}$$

Now we act each of N, E, ψ^+ and ψ^- on the basis vectors of $\mathcal{V}_{n,e}$:
$N v_{n,e} = n v_{n,e}$, $N\psi^- v_{n,e} = ([N, \psi^-] + \psi^- N) v_{n,e} = (n-2) v_{n,e}$
$E v_{n,e} = e v_{n,e}$, $E\psi^- v_{n,e} = ([E, \psi^-] + \psi^- E) v_{n,e} = e\psi^- v_{n,e}$
$\psi^+ \psi^- v_{n,e} = (\{\psi^+, \psi^-\} - \psi^- \psi^+) v_{n,e} = E v_{n,e} = e v_{n,e}$
Hence:

$$N \mapsto \begin{bmatrix} n & 0 \\ 0 & n-2 \end{bmatrix}, E \mapsto \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}, \psi^+ \mapsto \begin{bmatrix} 0 & e \\ 0 & 0 \end{bmatrix}, \psi^- \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

When $e \neq 0$, the module along with the above map from $\mathfrak{gl}(1|1)$ to $\textbf{End}(\mathbb{C}^{1|1})$ is known as the typical representation, denoted by $\mathcal{T}_{n,e}$. This module is two dimensional as its basis is $\{v_{n,e}, \psi^- v_{n,e}\}$. It is also irreducible as it has no proper submodules. When $e = 0$, both E and ψ^+ vanish, meaning that $\psi^+\psi^- v_{n,0} = ev_{n,0} = 0$, implying $\mathbf{span}(\{\psi^- v_{n,0}\})$ is a proper submodule of $\mathcal{V}_{n,0}$, labelled $\mathcal{S}_{n,0}$. Since we want to consider irreducible modules, we may define the atypical representation as

$$\mathcal{A}_n = \mathcal{V}_{n,0}/\mathcal{S}_{n,0} = \mathbf{span}(\{[v_{n,0}]\}$$

where $[v_{n,0}] = \{v_{n,0} + \alpha\psi^- v_{n,0} : \alpha \in \mathbb{C}\}$. This is a one dimensional irreducible representation, with the action defined as $A \cdot [v] = [A \cdot v]$.

If V and W are representations, then $V \otimes W$ is also a representation, the action of an element a given by:

$$a \cdot (v \otimes w) = (av) \otimes w + (-1)^{\bar{a}\bar{v}} v \otimes (aw)$$

(For the following calculations the parity of elements of the module are taken to be even.)

Taking the tensor of two different atypical representations $\mathcal{A}_n$ and $\mathcal{A}_{n'}$ results in:

$N([v_{n,0}] \otimes [v_{n',0}]) = N[v_{n,0}] \otimes [v_{n',0}] + [v_{n,0}] \otimes N[v_{n',0}] = (n + n')([v_{n,0}] \otimes [v_{n',0}])$

$\implies \mathcal{A}_n \otimes \mathcal{A}_{n'} \simeq \mathcal{A}_{n+n'}$.

Taking the tensor of an atypical module $\mathcal{A}_n$ and a typical module $\mathcal{T}_{n',e'}$ gives us two basis vectors; $[v_{n,0}] \otimes v_{n',e'}$ and $[v_{n,0}] \otimes \psi^- v_{n',e'}$. Acting with N, E, ψ^+ and ψ^- on these yields:

$$N([v_{n,0}] \otimes v_{n',e'}) = (n + n')([v_{n,0}] \otimes v_{n',e'})$$

$$N([v_{n,0}] \otimes \psi^- v_{n',e'}) = (n + n' - 2)([v_{n,0}] \otimes \psi^- v_{n',e'})$$

$$E([v_{n,0}] \otimes v_{n',e'}) = e'([v_{n,0}] \otimes v_{n',e'}), \quad E([v_{n,0}] \otimes \psi^- v_{n',e'}) = e'([v_{n,0}] \otimes \psi^- v_{n',e'})$$

$$\psi^+([v_{n,0}] \otimes v_{n',e'}) = 0, \quad \psi^+([v_{n,0}] \otimes \psi^- v_{n',e'}) = e'([v_{n,0}] \otimes \psi^- v_{n',e'})$$

$$\psi^-([v_{n,0}] \otimes v_{n',e'}) = [v_{n,0}] \otimes \psi^- v_{n',e'}, \quad \psi^-([v_{n,0}] \otimes \psi^- v_{n',e'}) = 0$$

This gives us the following map:

$$N \mapsto \begin{bmatrix} n+n' & 0 \\ 0 & n+n'-2 \end{bmatrix}, \quad E \mapsto \begin{bmatrix} e' & 0 \\ 0 & e' \end{bmatrix}, \quad \psi^+ \mapsto \begin{bmatrix} 0 & e' \\ 0 & 0 \end{bmatrix}, \quad \psi^- \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Which is isomorphic to the representation $\mathcal{T}_{n+n',e'}$.

For the tensor product of two different typical representations, a similar technique may be used to show that $\mathcal{T}_{n,e} \otimes \mathcal{T}_{n',e'} \simeq \mathcal{T}_{n+n',e+e'} \oplus \mathcal{T}_{n+n'-2,e+e'}$ given that $e, e', e + e' \neq 0$.

3 Supersymmetric Quantum Mechanics

3.1 The Algebra of SUSY QM

Supersymmetric refers to the supersymmetry algebra, which contains a supercharge Q, which is an odd symmetry operator between fermions and bosons. Supersymmetric quantummechanics describes Hamiltonians of the form

$H = \frac{1}{2}\{Q, Q^\dagger\}$ where $Q^2 = 0$.

Additionally, we have that $[H, Q] = [H, Q^\dagger] = 0$.

Proof:

$[H,Q] = \frac{1}{2}[\{Q, Q^\dagger\}, Q = \frac{1}{2}[Q, \{Q^\dagger, Q\}] + \frac{1}{2}[Q^\dagger, \{Q, Q\}]$

$= -\frac{1}{2}[\{Q^\dagger, Q\}, Q] = -\frac{1}{2}[\{Q, Q^\dagger\}, Q] = -[H, Q] \implies [H, Q] = 0$

Similarly $[H, Q^\dagger] = 0$. $\square$

The Lie superalgebra spanned by $\{H, Q, Q^\dagger\}$ is isomorphic to $\mathfrak{sl}(1|1)$, the subalgebra of $\mathfrak{gl}(1|1)$ spanned by $\{E, \psi^+, \psi^-\}$, where $H \simeq E$, and $Q \simeq \psi^+$ and $Q^\dagger \simeq \psi^-$. The representation theory of $\mathfrak{sl}(1|1)$ is similar to that of $\mathfrak{gl}(1|1)$ but since $N \notin \mathfrak{sl}(1|1)$, the representations no longer depend on n.

Since $H|\psi\rangle = e|\psi\rangle$ (here $e \in \mathbb{R}$ denotes the energy eigenvalue), we have that $\{Q, Q^\dagger\}|\psi\rangle = 2H|\psi\rangle = 2e|\psi\rangle$. If we take the expectation value of H for some state $|\psi\rangle = \psi$, we get:

$$2\langle H\rangle = \langle \psi, 2H\psi\rangle = \langle \psi, (QQ^\dagger + Q^\dagger Q)\psi\rangle = \langle \psi, QQ^\dagger\psi\rangle + \langle \psi, Q^\dagger Q\psi\rangle = \langle Q^\dagger\psi, Q^\dagger\psi\rangle + \langle Q\psi, Q\psi\rangle = \|Q^\dagger\psi\|^2 + \|Q\psi\|^2 \geq 0$$

This means that the energy of a state is always positive unless it is annihilated by both Q and $Q^\dagger$, in which case it is zero. We can now apply the representation theory to investigate what happens when $e \neq 0$ versus when $e = 0$. For $e \neq 0$: We start by taking a highest weight vector ψ (i.e. $Q\psi = 0$). Then $2e\psi = 2H\psi = \{Q, Q^\dagger\}\psi = QQ^\dagger\psi + Q^\dagger Q\psi = QQ^\dagger\psi$. Hence for $e \neq 0$, $Q^\dagger\psi \neq 0$, giving a 2D representation spanned by $\psi := |0\rangle$ and $Q^\dagger\psi := |1\rangle$. This algebra is similar to the typical representation of $\mathfrak{gl}(1|1)$, however now we are restricted to $\mathfrak{sl}(1|1)$. Since it is two dimensional, all non-zero energy states must come in pairs for each energy e (although there may be more degeneracy with several pairs).

When $e = 0$, we set $Q^\dagger\psi = 0$ by taking the quotient $\mathcal{V}/\mathcal{S}$, where $\mathcal{V} = \mathbf{span}(\{\psi, Q^\dagger\psi\})$ and $\mathcal{S} = \mathbf{span}(\{Q^\dagger\psi\})$. This gives our state the form of the equivalence class $[\psi]$ which is annihilated by both Q and $Q^\dagger$, agreeing with the result above regarding the expectation value of H for when $e = 0$.

We may define the fermion number operator $F := \frac{1}{2e}QQ^\dagger$. This has commutator relations $[F, Q] = -Q$, $[F, Q^\dagger] = Q^\dagger$ and $[F, H] = 0$. Its action on the basis $\{|1\rangle, |0\rangle\}$ is given by $F|0\rangle = \frac{1}{2e}QQ^\dagger|0\rangle = 0$ and $F|1\rangle = \frac{1}{2e}QQ^\dagger|1\rangle = |1\rangle$, which gives F the matrix representation $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ which corresponds to $\frac{1}{2}N$ with $n = 2$ from $\mathfrak{gl}(1|1)$. The fermion number operator thus allows us to reduce the Hilbert space of states accordingly into "bosonic states" with $F = 0$ and "fermionic states" with $F = 1$, i.e. $\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F$ which is a $\mathbb{Z}_2$-grading. For pairs of states with $e \neq 0$, 1 state is in $\mathcal{H}_B$ and the other is in $\mathcal{H}_F$.

3.2 Solving for the Ground States using SUSY QM

We now consider a particle moving along a line in a potential. The Hilbert space here is given by $\mathcal{H} = L^2(\mathbb{R})\otimes\mathbb{C}^2$. Hence $\mathcal{H} = L^2(\mathbb{R})|0\rangle \oplus L^2(\mathbb{R})|1\rangle = \mathcal{H}_B \oplus \mathcal{H}_F$

The supercharge Q is given by $Q = (p-ih'(x))\otimes \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and its adjoint is given by $Q^\dagger = (p+ih'(x))\otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, where $p = -i\frac{\partial}{\partial x}$ is the momentum operator and $h(x)$ is a real function. Consequently, $Q^2 = (Q^\dagger)^2 = 0$.

We may now find an expression for the Hamiltonian H. Since we are dealing with supersymmetric quantum mechanics, $H = \frac{1}{2}\{Q, Q^\dagger\}$, thus:

$$H = \frac{1}{2}QQ^\dagger + \frac{1}{2}Q^\dagger Q$$

$$= \frac{1}{2}\left((p-ih')\otimes \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)\left((p+ih')\otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) + \frac{1}{2}\left((p+ih')\otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)\left((p-ih')\otimes \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)$$

$$= \frac{1}{2}(p-ih')(p+ih')\otimes \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2}(p+ih')(p-ih')\otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{2}(p^2+h'^2)\otimes \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2}(p^2+h'^2)\otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{2}(pih'-ih'p)\otimes \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2}(-pih'+ih'p)\otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$pih' - ih'p = [p, ih']$, thus we compute the bracket of p and ih':

$[p, ih']\psi = (-i\frac{\partial}{\partial x}ih' - (ih')(-i)\frac{\partial}{\partial x})\psi$

$= \frac{\partial}{\partial x}h'\psi - h'\frac{\partial}{\partial x}\psi = h''\psi + h'\psi = h'\psi = h''\psi$

Hence $[p, ih'] = h''$.

Therefore $H = \frac{1}{2}(p^2+h'^2)\otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{2}h''\otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

The first term is the usual Hamiltonian for a particle moving along a line with a potential given by $V(x) = \frac{1}{2}(h')^2$ and involves a matrix the corresponding to E from $\mathfrak{sl}(1|1)$, whilst the second term involves the matrix known as the Pauli matrix whic corresponds to N from $\mathfrak{gl}(1|1)$, and this matrix distinguishes between bosonic and fermionic states due to the minus sign in the second column. We thus consider states that look like:

$$\psi(x)|0\rangle = \psi \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathcal{H}_B, \quad \psi(x)|1\rangle = \psi \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathcal{H}_F$$

which are called bosonic states and fermionic states respectively. We can now solve for a general ground state $\Psi(x) = \begin{bmatrix} \psi(x) \\ \varphi(x) \end{bmatrix}$. According to the previous consequences of supersymmetry, a ground state must be annihilated by both Q and $Q^\dagger$. Hence $Q\Psi = Q^\dagger\Psi = 0$, i.e.

$$\left((p-ih')\otimes\begin{bmatrix}0&0\\1&0\end{bmatrix}\right)\begin{bmatrix}\psi\\ \varphi\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \text{ and } \left((p+ih')\otimes\begin{bmatrix}0&1\\0&0\end{bmatrix}\right)\begin{bmatrix}\psi\\ \varphi\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\implies (p-ih')\psi=0 \quad\text{and}\quad (p+ih')\varphi=0$$
$$\implies (-i\frac{\partial}{\partial x}-ih')\psi=0 \quad\text{and}\quad (-i\frac{\partial}{\partial x}+ih')\varphi=0$$
$$\implies \psi'=-h'\psi \quad\text{and}\quad \varphi'=h'\varphi$$
$$\implies \psi=e^{-h} \quad\text{and}\quad \varphi=e^{h}.$$

We must have at least one of ψ and φ be normalisable, which is only possible if, when $|x|\to\infty$, $h\to\infty$ or $h\to-\infty$. If the former holds, then ψ can be normalised and we have a fermionic ground state. If the latter holds, then φ can be normalised and we have a bosonic ground state. If neither hold, then there is no $E=0$ ground state, and we say that the supersymmetry is broken. This means that the ground state will have $E\neq 0$ and is hence degenerate.

3.3 The Witten Index

Supersymmetry allows for simple calculation of the Witten index, which is a partition function given by:

$\mathcal{I}=\mathbf{str}_{\mathcal{H}}q^H=\mathbf{tr}_{\mathcal{H}}(-1)^F q^H$, where $(-1)^F=\begin{cases}1 & \text{if} \quad F=0\\ -1 & \text{if} \quad F=1\end{cases}$

Since for all $\mathcal{H}_B$ and $\mathcal{H}_F$ with $E>0$ the basis states are paired, the contributions from each cancel with each other and leave only the contributions from the zero energy states, hence:

$\mathcal{I}=\mathbf{dim}(\mathcal{H}_{0,B})-\mathbf{dim}(\mathcal{H}_{0,F})$, where $\mathcal{H}_{0,B}$ represents the $e=0$ bosonic state, and $\mathcal{H}_{0,F}$ represents the $e=0$ fermionic state.

Discussion and Conclusion

This paper has described the representation theory of the Lie superalgebra $\mathfrak{gl}(1|1)$, defining both the one-dimensional atypical and two-dimensional typical representations. When these two representations are applied to supersymmetric quantum mechanics, it leads to results such as a simpler approach for solving for the ground states of a system, and an easier calculation of the Witten index. An interesting course of further study would be to look at supersymmetric quantum mechanics for higher dimensions and/or with more than one supercharge.

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