Phase Transitions in a Vector-Spin Curie-Weiss Model

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Abstract This report explores the behaviour of a generalised Curie–Weiss model, with particular focus on the nature of its phase transitions and the fluctuations of the empirical distribution around equilibrium states.

Contents

1 Introduction: Curie-Weiss Model with Discrete Vector-Spins

The model that we are analysing in this report is a discrete vector-spin Curie-Weiss model in which spins are constrained to orthogonal axes, allowing for parallel, antiparallel and orthogonal interactions. It focuses on the asymptotic behaviour of the magnetisation density in the thermodynamic limit, identifying whether it concentrates at zero or at a spontaneous magnetisation. The key objectives include determining the dimensional dependence of critical values, conditions under which fluctuations follow Gaussian versus non-Gaussian behaviour, and how dimensionality and external fields affect continuity and fluctuation structure. More information can be referred to in [3]. The results presented in this paper are my own work.

1.1 Model Setup

Consider a system of N particles with spin configuration

$$\boldsymbol{\omega} = (\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_N)$$

Each spin $\boldsymbol{\omega}_i$ takes values in the alphabet Σ (which can be organised as seen in the matrix A),

$$\Sigma = \{\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_d\}, \qquad A = \begin{bmatrix} \mathbf{e}_1 & -\mathbf{e}_1 \\ \mathbf{e}_2 & -\mathbf{e}_2 \\ \vdots & \vdots \\ \mathbf{e}_d & -\mathbf{e}_d \end{bmatrix}$$

so that every particle chooses exactly one element of Σ. Let Ω_N denote the set of all possible spin configurations on the system.

The Vector-Spin Curie-Weiss Hamiltonian is defined by

$$\mathscr{H}_{N,\beta}(\boldsymbol{\omega}) := -\frac{\beta}{2N} \sum_{i,j=1}^{N} \boldsymbol{\omega}_i \cdot \boldsymbol{\omega}_j \tag{1.1}$$

and the corresponding Gibbs distribution on Ω_N is

$$\mu_{N,\beta}(\boldsymbol{\omega}) := \frac{e^{-\mathscr{H}_{N,\beta}(\boldsymbol{\omega})}}{Z_{N,\beta}}, \quad \text{where} \quad Z_{N,\beta} := \sum_{\boldsymbol{\omega} \in \Omega_N} e^{-\mathscr{H}_{N,\beta}(\boldsymbol{\omega})} \tag{1.2}$$

1.2 Observables

Magnetisation. The total spin of the system is

$$\boldsymbol{M}_N := \sum_{i=1}^{N} \boldsymbol{\omega}_i,$$

and the corresponding magnetisation density is

$$\boldsymbol{m}_N := \frac{\boldsymbol{M}_N}{N}, \qquad \text{define the set of magnetisation density states as } \mathscr{A}_N$$

Empirical Measure. The empirical measure of the spin configuration, which gives the fraction of spins taking each value in the alphabet a, is defined as

$$L_N^{\boldsymbol{\omega}} := \begin{bmatrix} L_N^{\boldsymbol{\omega}}(A_{1,1}) & L_N^{\boldsymbol{\omega}}(A_{1,2}) \\ L_N^{\boldsymbol{\omega}}(A_{2,1}) & L_N^{\boldsymbol{\omega}}(A_{2,2}) \\ \vdots & \vdots \\ L_N^{\boldsymbol{\omega}}(A_{d,1}) & L_N^{\boldsymbol{\omega}}(A_{d,2}) \end{bmatrix}, \quad \text{where} \quad L_N^{\boldsymbol{\omega}}(A_{i,j}) := \frac{1}{N}\sum_{k=1}^{N} \mathbf{1}_{\{\boldsymbol{\omega}_k = A_{i,j}\}}$$

Furthermore, let $\mathcal{L}_N$ be the set of all empirical measures of sequences of length N:

$$\mathcal{L}_N := \{\, \nu : \nu = L_N^{\boldsymbol{\omega}} \text{ for some } \boldsymbol{\omega} \in \Omega_N \,\}\,.$$

Hamiltonian and Gibbs Distribution in Terms of Observables. The Hamiltonian can be expressed in terms of the magnetisation density:

$$\sum_{i,j=1}^{N} \boldsymbol{\omega}_i \cdot \boldsymbol{\omega}_j = \left(\sum_{i=1}^{N} \boldsymbol{\omega}_i\right) \cdot \left(\sum_{i=1}^{N} \boldsymbol{\omega}_i\right) = \boldsymbol{M}_N \cdot \boldsymbol{M}_N = N^2\,(\boldsymbol{m}_N \cdot \boldsymbol{m}_N)$$

Hence, the Hamiltonian becomes

$$\tilde{\mathscr{H}}_{N,\beta}(\boldsymbol{m}_N) = -\frac{\beta N}{2}\,(\boldsymbol{m}_N \cdot \boldsymbol{m}_N)$$

Now we express the magnetisation in terms of the empirical measure by taking the difference of the columns of an arbitrary empirical measure $\nu \in \mathbb{R}^{d\times 2}$

$$\boldsymbol{m}_N = \nu \mathbf{x}, \quad \text{where} \quad \mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

This yields the following Hamiltonian

$$\tilde{\mathscr{H}}_{N,\beta}(\nu) = -\frac{\beta N}{2}\,(\nu \mathbf{x}) \cdot (\nu \mathbf{x})$$

2 Rate Functions and the Thermodynamic Limit

Now the goal is to obtain and minimise rate function to quantify the limiting probability distribution on the empirical measure as $N \to \infty$.

2.1 Determining the Rate Function

From [1], Sanov's Theorem states an asymptotic upper bound on the probability of getting some $\nu \in \mathcal{L}_N$ when drawing from a probability distribution μ, which is approached as $N \to \infty$.

$$\Pr_{\mu}(L_N^{\boldsymbol{\omega}} = \nu) \leq e^{-NH(\nu\|\mu)},$$

The relative entropy [2] is given as

$$H(\nu \,\|\, \mu) \coloneqq \sum_{j=1}^{2} \sum_{i=1}^{d} \left(\nu(A_{i,j}) \, \log \frac{\nu(A_{i,j})}{\mu(A_{i,j})} \right) \tag{2.1}$$

The rate function can be determined in terms of the rate function when $\beta = 0$, the Hamiltonian and a function of β.

$$I_\beta(\nu) \coloneqq I(\nu) + \frac{\tilde{\mathscr{H}}_{N,\beta}(\nu)}{N} - \varphi(\beta) \tag{2.2}$$

When $\beta = 0$, the probability distribution that is being drawn from is the uniform distribution u, giving

$$I(\nu) = H(\nu \,\|\, u) = \sum_{j=1}^{2} \sum_{i=1}^{d} \left(\nu(A_{i,j}) \, \log \frac{\nu(A_{i,j})}{u(A_{i,j})} \right)$$

The probability of getting an arbitrary element from the uniform distribution is $u(A_{i,j}) = \frac{1}{2d}$

$$I(\nu) = \sum_{j=1}^{2} \sum_{i=1}^{d} \nu(A_{i,j}) \, \log(2d \, \nu(A_{i,j})) = \log(2d) \sum_{j=1}^{2} \sum_{i=1}^{d} \nu(A_{i,j}) + \sum_{j=1}^{2} \sum_{i=1}^{d} \nu(A_{i,j}) \log(\nu(A_{i,j}))$$

But the sum of probabilities must be 1, restricting ν to the $(2d-1)$-simplex:

$$g(\nu) \coloneqq \sum_{j=1}^{2} \sum_{i=1}^{d} \nu(A_{i,j}) = 1, \qquad 0 \leq \nu(A_{i,j}) \leq 1 \tag{2.3}$$

Applying the constraint above to $I(\nu)$ and using the shorthand that $\nu_{i,j} = \nu(A_{i,j})$

$$I(\nu) = \log(2d) + \sum_{j=1}^{2} \sum_{i=1}^{d} \nu_{i,j} \log(\nu_{i,j})$$

Now substituting back into equation (2.2) yields

$$I_\beta(\nu) = \sum_{j=1}^{2} \sum_{i=1}^{d} (\nu_{i,j} \log(\nu_{i,j})) \; - \; \frac{\beta}{2} (\nu\mathbf{x}) \cdot (\nu\mathbf{x}) \; + \; \log(2d) \; - \; \varphi(\beta)$$

2.2 Generalised Extrema of the Rate Function

To find the limiting empirical vector in the thermodynamic limit, we need to find the minima of $I_\beta(\nu)$ subject to the $(2d-1)$-simplex constraint (2.3). Therefore, we will use the Lagrangian $\mathcal{L}$ with Lagrange multiplier λ.

$$\mathcal{L}_\beta(\nu, \lambda) = I_\beta(\nu) - \lambda(g(\nu) - 1)$$

The extrema can then be found using the matrix equation below

$$\nabla_{\nu,\lambda} \mathcal{L}_\beta(\nu, \lambda) = O$$

Which is equivalent to

$$\nabla_\nu I_\beta(\nu) = \lambda \nabla_\nu g(\nu) \qquad \& \qquad g(\nu) = 1$$

Looking at the component form to simplify

$$(\nabla_\nu I_\beta(\nu))_{i,j} = (\lambda \nabla_\nu g(\nu))_{i,j} \implies \frac{\partial I_\beta(\nu)}{\partial \nu_{i,j}} = \lambda \frac{\partial g(\nu)}{\partial \nu_{i,j}}$$

$$(\nabla_\nu I_\beta(\nu))_{i,j} = \frac{\partial}{\partial \nu_{i,j}} \left(\sum_{b=1}^{2} \sum_{a=1}^{d} (\nu_{a,b} \log(\nu_{a,b})) \; - \; \frac{\beta}{2} (\nu\mathbf{x}) \cdot (\nu\mathbf{x}) \; + \; \log(2d) \; - \; \varphi(\beta) \right)$$

$$(\nabla_\nu I_\beta(\nu))_{i,j} = \sum_{b=1}^{2}\sum_{a=1}^{d} \frac{\partial}{\partial \nu_{i,j}} \left(\nu_{a,b}\log(\nu_{a,b})\right) \;-\; \frac{\beta}{2}\frac{\partial}{\partial \nu_{i,j}} \left((\nu\mathbf{x})\cdot(\nu\mathbf{x})\right)$$

This partial derivative of $\nu_{a,b}\log(\nu_{a,b})$ with respect to $\nu_{i,j}$ is only non-zero when $(a,b) = (i,j)$. For the partial derivative of $(\nu\mathbf{x})\cdot(\nu\mathbf{x})$ we use vector calculus identities to get the following.

$$(\nabla_\nu I_\beta(\nu))_{i,j} = \frac{\partial}{\partial \nu_{i,j}} \left(\nu_{i,j}\log(\nu_{i,j})\right) \;-\; \beta\,(\nu\mathbf{x})_i\,\mathbf{x}_j$$

$$(\nabla_\nu I_\beta(\nu))_{i,j} = 1 + \log(\nu_{i,j}) \;-\; \beta\,(\nu\mathbf{x})_i\,\mathbf{x}_j$$

Now to separate for the cases when j=1 and j=2 (note that $\mathbf{x}_1 = 1$ and $\mathbf{x}_2 = -1$)

$$j=1: \quad (\nabla_\nu I_\beta(\nu))_{i,1} = 1 + \log(\nu_{i,1}) \;-\; \beta\,(\nu\mathbf{x})_i\,\mathbf{x}_1 = 1 + \log(\nu_{i,1}) \;-\; \beta\,(\nu_{i,1} - \nu_{i,2})$$

$$j=2: \quad (\nabla_\nu I_\beta(\nu))_{i,2} = 1 + \log(\nu_{i,2}) \;-\; \beta\,(\nu\mathbf{x})_i\,\mathbf{x}_2 = 1 + \log(\nu_{i,1}) \;+\; \beta\,(\nu_{i,1} - \nu_{i,2})$$

Now to do the same for $(\lambda\nabla_\nu g(\nu))_{i,j}$

$$(\lambda\nabla_\nu g(\nu))_{i,j} = \lambda\frac{\partial}{\partial \nu_{i,j}}\left(\sum_{b=1}^{2}\sum_{a=1}^{d}\nu_{a,b}\right) = \lambda\sum_{b=1}^{2}\sum_{a=1}^{d}\frac{\partial}{\partial \nu_{i,j}}\left(\nu_{a,b}\right)$$

This partial derivative is only nonzero when $(a,b) = (i,j)$

$$(\lambda\nabla_\nu g(\nu))_{i,j} = \lambda\frac{\partial}{\partial \nu_{i,j}}\left(\nu_{i,j}\right) = \lambda$$

Putting the equations $\nabla_\nu I_\beta(\nu) = \lambda\nabla_\nu g(\nu)$ and $g(\nu) = 1$ together for j=1 and j=2

$$\begin{cases} \log(\nu_{i,1}) - \beta(\nu_{i,1} - \nu_{i,2}) - (\lambda - 1) = 0, & j = 1 \\ \log(\nu_{i,2}) + \beta(\nu_{i,1} - \nu_{i,2}) - (\lambda - 1) = 0, & j = 2 \\ \sum_{i=1}^{d}(\nu_{i,1} + \nu_{i,2}) = 1 \end{cases}$$

The solution for any $\nu \in \mathcal{L}_N$ that satisfies equations above are the extrema. Now we must determine the global minima which will ultimately determine the limiting state as $N \to \infty$.

We can solve for $\nu_{i,1}$ and $\nu_{i,2}$ as functions of themselves

$$\begin{cases} \nu_{i,1} = e^{(\lambda-1)} \ e^{\beta(\nu_{i,1}-\nu_{i,2})}, & (1) \\ \nu_{i,2} = e^{(\lambda-1)} \ e^{-\beta(\nu_{i,1}-\nu_{i,2})}, & (2) \\ \sum_{i=1}^{d}(\nu_{i,1}+\nu_{i,2}) = 1, & (3) \end{cases}$$

Adding equations (1) and (2) and taking the sum of both sides from $i = 1$ to $i = d$ yields the identity in equation (3) which will allow us to solve for the constant $e^{\lambda-1}$

$$\sum_{i=1}^{d}(\nu_{i,1}+\nu_{i,2}) = \sum_{i=1}^{d} 2e^{(\lambda-1)} \left(\frac{e^{\beta(\nu_{i,1}-\nu_{i,2})} + e^{-\beta(\nu_{i,1}-\nu_{i,2})})}{2} \right)$$

$$e^{(\lambda-1)} = \frac{1}{2\sum_{i=1}^{d} \cosh(\beta(\nu_{i,1}-\nu_{i,2}))}$$

Now that $e^{\lambda-1}$ can be expressed in terms of ν, we can substitute that into equations (1) and (2) and subtract them.

$$(\nu_{i,1}-\nu_{i,2}) = 2e^{(\lambda-1)} \left(\frac{e^{\beta(\nu_{i,1}-\nu_{i,2})} - e^{-\beta(\nu_{i,1}-\nu_{i,2})})}{2} \right)$$

$$(\nu_{i,1}-\nu_{i,2}) = \frac{\sinh\left(\beta(\nu_{i,1}-\nu_{i,2})\right)}{\sum_{j=1}^{d} \cosh(\beta(\nu_{j,1}-\nu_{j,2}))}$$

It is not possible to simply isolate $\nu_{i,1}$ or $\nu_{i,2}$. However, by subtracting equations (1) and (2), we are able to have both sides of the equation depend on a single variable, $m_i = \nu_{i,1} - \nu_{i,2}$.

$$m_i = \frac{\sinh\left(\beta m_i\right)}{\sum_{j=1}^{d} \cosh(\beta m_j)} \tag{2.4}$$

Once β_c is determined from the above mean-field equation and the stable magnetisation vectors $\boldsymbol{m}$ are found, we can then determine $\nu_{i,1}$ and $\nu_{i,2}$ from equations (1) and (2) as

$$\nu_{i,1} = \frac{e^{\beta m_i}}{2\sum_{j=1}^{d}\cosh(\beta m_j)}, \qquad \nu_{i,2} = \frac{e^{-\beta m_i}}{2\sum_{j=1}^{d}\cosh(\beta m_j)} \tag{2.5}$$

The exponential and hyperbolic cosine functions are smooth and the denominator of the equations above are strictly positive, the maps from $\boldsymbol{m} \mapsto \nu_{i,1}$ and $\boldsymbol{m} \mapsto \nu_{i,2}$ belong to $C^{\infty}(\mathbb{R}^d)$. Consequently, the critical inverse temperature β_c at which non-trivial solutions appear is determined entirely by the mean-field equation (2.4), and is therefore identical to the critical value obtained from the empirical measure formulation.

2.3 Non-trivial Global Minima of the Rate Function

Now we will find the nature of these extrema to determine the limiting cases by using the Hessian. For notational convenience, we will use the vector form of ν denoted as χ

$$\chi = [\nu_{1,1}, \nu_{1,2}, \ldots, \nu_{d,1}, \nu_{d,2}]^T, \text{ where } \chi \in \mathbb{R}^{2d}$$

$$(\nabla_{\chi} I_{\beta}(\chi))_i = 1 + \log(\chi_i) - \beta(\chi_i - \chi_{i+(-1)^{i+1}}),$$

Then the unconstrained Hessian is symmetric and of the form

$$H = \operatorname{diag}\big(B_1, B_2, \ldots, B_d\big), \qquad B_i = \begin{bmatrix} \frac{1}{\chi_{2i-1}} - \beta & \beta \\ \beta & \frac{1}{\chi_{2i}} - \beta \end{bmatrix} \tag{2.6}$$

The eigenvalues of the Hessian can be found as the solution to: $\det(\lambda I - H) = 0$. Since $\lambda I - H$ is also block diagonal, the determinant is given as

$$\det(\lambda I - H) = \prod_{i=1}^{d} \det(\lambda I - B_i) = \prod_{i=1}^{d} \left(\left(\frac{1}{\chi_{2i-1}} - \beta - \lambda \right) \left(\frac{1}{\chi_{2i}} - \beta - \lambda \right) - \beta^2 \right)$$

Each block diagonal B_i produces two eigenvalues $\lambda_{i,+}$ and $\lambda_{i,-}$ which can be rewritten back in terms of ν. Note that $\nu_{i,1}, \nu_{i,2} > 0$ which follows directly from equation (2.5).

$$\lambda_{i,\pm} = \frac{\nu_{i,1} + \nu_{i,2} - 2\beta\nu_{i,1}\nu_{i,2} \pm \sqrt{(\nu_{i,1} - \nu_{i,2})^2 + (2\beta\nu_{i,1}\nu_{i,2})^2}}{2\nu_{i,1}\nu_{i,2}} \tag{2.7}$$

As $(\nu_{i,1} - \nu_{i,2})^2 \geq 0$, the following inequality must hold

$$\lambda_{i,+} \geq \frac{\nu_{i,1} + \nu_{i,2} - 2\beta\nu_{i,1}\nu_{i,2} + \sqrt{0 + (2\beta\nu_{i,1}\nu_{i,2})^2}}{2\nu_{i,1}\nu_{i,2}}$$

$$\lambda_{i,+} \geq \frac{\nu_{i,1} + \nu_{i,2}}{2\nu_{i,1}\nu_{i,2}}$$

$$(\nu_{i,1} + \nu_{i,2}), (2\nu_{i,1}\nu_{i,2}) > 0 \implies \lambda_{i,+} > 0$$

To find the sign of $\lambda_{i,-}$ follows a similar process

$$\lambda_{i,-} = \frac{\nu_{i,1} + \nu_{i,2} - 2\beta\nu_{i,1}\nu_{i,2} - \sqrt{(\nu_{i,1} - \nu_{i,2})^2 + (2\beta\nu_{i,1}\nu_{i,2})^2}}{2\nu_{i,1}\nu_{i,2}}$$

Since $2\nu_{i,1}\nu_{i,2} > 0$, the sign of $\lambda_{i,-}$ is given as the sign of the numerator.

$$\operatorname{sgn}(\lambda_{i,-}) = \operatorname{sgn}\left(\nu_{i,1} + \nu_{i,2} - 2\beta\nu_{i,1}\nu_{i,2} - \sqrt{(\nu_{i,1} - \nu_{i,2})^2 + (2\beta\nu_{i,1}\nu_{i,2})^2}\right)$$

If $\nu_{i,1} + \nu_{i,2} - 2\beta\nu_{i,1}\nu_{i,2} \leq 0$ then it simply follows that $\lambda_{i,-} < 0$

Now for $\nu_{i,1} + \nu_{i,2} - 2\beta\nu_{i,1}\nu_{i,2} > 0$ to have $\lambda_{i,-} < 0$ we require

$$(\nu_{i,1} + \nu_{i,2}) - (2\beta\nu_{i,1}\nu_{i,2}) < \sqrt{(\nu_{i,1} - \nu_{i,2})^2 + (2\beta\nu_{i,1}\nu_{i,2})^2}$$

Given from the assumption above both sides are positive

$$(\nu_{i,1} + \nu_{i,2})^2 - 4\beta\nu_{i,1}\nu_{i,2}(\nu_{i,1} + \nu_{i,2}) + (2\beta\nu_{i,1}\nu_{i,2})^2 < (\nu_{i,1} - \nu_{i,2})^2 + (2\beta\nu_{i,1}\nu_{i,2})^2$$

$$4\nu_{i,1}\nu_{i,2}\left(1-\beta(\nu_{i,1}+\nu_{i,2})\right)<0$$

$$1-\beta(\nu_{i,1}+\nu_{i,2})<0$$

Substituting $\nu_{i,1}$ and $\nu_{i,2}$ for the extrema from equation (2.5)

$$\nu_{i,1}=\frac{e^{\beta m_i}}{2C},\qquad \nu_{i,2}=\frac{e^{-\beta m_i}}{2C},\quad \text{where}\quad C=\sum_{j=1}^{d}\cosh(\beta m_j)$$

$$1-\frac{\beta}{C}\cosh(\beta m_i)<0$$

From equation (2.4) we get C in terms of m_i giving the following inequality

$$\beta m_i\coth(\beta m_i)>1$$

This inequality is satisfied for all $\beta m_i\in\mathbb{R}\backslash\{0\}$. Therefore $\lambda_{i,-}<0$ for all $\beta>0$ and $m_i\neq 0$

For some $\boldsymbol{m}_k$ with k magnetised directions which satisfies the mean-field equation, any of the k non-trivial components will contribute the following pair of eigenvalues

$$\lambda_{i,+}>0,\quad \lambda_{i,-}<0$$

Such a configuration $\boldsymbol{m}_k$ has at least k negative eigenvalues from each of the corresponding non-zero components. The Hessian is symmetric and therefore diagonalizable with an orthonormal eigenbasis, allowing the construction of a subspace

$$W_k=\text{span}\{\boldsymbol{w}_1,\dots,\boldsymbol{w}_k\}$$

which consists of the eigenvectors associated with these negative eigenvalues.

Furthermore, for all interior empirical vectors χ, such that $\chi_i\notin\{0,1\}$ for all i, each empirical vector can freely move in the tangent hyperplane

$$T:=\left\{\boldsymbol{u}:\sum_{i=1}^{2d}u_i=0\right\},\qquad \text{codim}(T)=1 \tag{2.8}$$

However, from equation (2.5), all components of the empirical vector for an equilibrium are strictly non-zero for finite β, meaning all equilibria are internal and can always freely move some non-zero distance in any direction of the tangent hyperplane T.

Consequently, at most one eigenvector of the Hessian may be orthogonal to T. For a configuration $\boldsymbol{m}_k$, the eigenspace W_k has dimension k. When $k = 1$, it is therefore possible that the unique negative eigenvector lies entirely in the normal direction to T, so that the negative concavity is not accessible along the constraint. In this case, the equilibrium at $k = 1$ may be a local minimum, and must be analysed separately.

For $2 \leq k \leq d$, at most one dimension of W_k can be orthogonal to T. Hence, there must exist at least one direction in W_k with a non-zero projection onto T. This implies that there is always an allowed direction of negative curvature, and therefore all equilibria with $k \geq 2$ are saddle points.

Consequently, the global minimiser of $I_\beta(\nu)$ must occur among the equilibria with $k \in \{0, 1\}$.

Now we must compare both values of k under varying β. Substituting $k = 1$ into the mean field equation (2.9) yields

$$m^* = \frac{\sinh(\beta m^*)}{\cosh(\beta m^*) + (d-1)}, \text{ where } g(m) = m \text{ and } f(m) = \frac{\sinh(\beta m)}{\cosh(\beta m) + (d-1)} \tag{2.9}$$

This equation only has the trivial solution below criticality or has three solutions $m_i \in \{-m^*, 0, m^*\}$ for some $m^* > 0$. Hence, in this critical regime, any magnetisation vector will be up to permutation of coordinates, of the form $\boldsymbol{m} = \{\pm m^* \underbrace{0, \ldots, 0}_{d-1}\}$

Now substituting the equilibria for k=1 into the rate function in terms of m_1 yields $\tilde{I}_\beta(m_1)$, which can be used to compare and determine whether the trivial equilibria or the equilibria k = 1 minimise the rate function.

I claimed that this rate function was even and concave down in the variable m. Therefore, if there exists some equilibria at $m = \pm m^*$ where $m^* \neq 0$, this equilibria will necessarily be the minimum of the rate function. Therefore, when these solutions for $m = \pm m^*$ appear is exactly when there is a phase transition.

However, it was later determined that this claim was false, and the latter parts of section (2.3) of the report rely on this claim. Therefore, it is essential to clarify that for $d > 3$ the critical value β_c is actually larger than what was calculated below, as the k=1 equilibria only become the global minimiser of the rate function as β increases after the equilibria comes into existence. For $d > 3$ the value β_c in the report correlates to the value of β at which the k=1 case have non-trivial equilibria solutions, a prerequisite for minimising the rate function internally. The rest of the section (2.3) documents my work under this false conjecture, yet it is still valuable and provides accurate results for β_c for $d \leq 3$ and a lower bound for β_c for $d > 3$

From equation (2.9), $f(m)$ has one point of inflection when $d \leq 3$, where it has positive concavity for $m < 0$ and negative concavity for $m > 0$. However, when $d > 3$, there are three points of inflection with negative concavity as $m \to 0^-$ and positive concavity as $m \to 0^+$.

For $d \leq 3$ the non-trivial solutions to the equilibria equation originate from the origin when both $f(m)$ and $g(m)$ touch. However, $m = 0$ is always a solution therefore the condition for touching is given below

$$\left.\frac{\partial f(m)}{\partial m}\right|_{m=0} = \left.\frac{\partial g(m)}{\partial m}\right|_{m=0} \qquad \frac{\beta}{d} = 1, \qquad \beta = d$$

For $d \leq 3$ the number of intersections are given as 1 for $\beta \leq d$ and 3 for $\beta > d$. The new solutions for $\beta > d$ occur at some spontaneous magnetisation density $\pm m^*$. As these solutions occur when both functions touch at the origin, we can see they appear continuously from the origin and as $\beta \to \infty, \quad m^* \to 1$. Therefore, we have $\beta_c = d, \qquad d \leq 3$. When $\beta \leq \beta_c$, the global minimum of $I_\beta(\boldsymbol{m})$ is $\boldsymbol{m} = \boldsymbol{0}$. When $\beta > \beta_c$, the global minimum of $I_\beta(\boldsymbol{m})$ is $\boldsymbol{m}_1 = (\pm m^*, \underbrace{0, \ldots, 0}_{d-1})$ for some $0 < m^* < 1$.

Now for $d > 3$, the mean-field equation (2.9) again has a point of inflection at $m = 0$ and has an additional two points of inflection on either side of the origin. Therefore, some spontaneous magnetisation $m = \pm m^*$ appears suddenly at β_c instead of continuously from $m = 0$ where $\beta_c < d$.

β_c can be found by making both functions touch at some $m = \pm m^*$ where $m^* \neq 0$. However, as the equations are transcendental, we are unable to find a closed form solution, so we will establish a lower bound by solving for when $f(m)$ and $g(m)$ touch. We will start by rewriting equation (2.9) in exponential form.

$$m^* = \frac{e^{\beta m^*} - e^{-\beta m^*}}{e^{\beta m^*} + e^{-\beta m^*} + 2(d-1)}, \text{ where } g(m) = m \text{ and } f(m) = \frac{e^{\beta m} - e^{-\beta m}}{e^{\beta m} + e^{-\beta m} + 2(d-1)}$$

Now we want to analyse the limiting behaviour as $d \to \infty$. For some $-1 < m < 1$ and some finite β, $f(m) \to 0$ which means that the only intersection between f and g is the trivial one at $m = 0$.

Therefore, for some non-trivial solution, we must have $\beta \to \infty$. For some $m^* > 0$, we can see that $e^{-\beta m^*} \to 0$ rapidly. By removing the negative exponential from the numerator and denominator we obtain $\tilde{f}(m)$ where $\tilde{f}(m) > f(m)$. This means that a non-trivial intersection for $\tilde{f}$ and g will occur for some $\beta_l < \beta_c$ giving a lower bound on the critical value.

$$m^* = \frac{e^{\beta_l m^*}}{e^{\beta_l m^*} + 2(d-1)}, \text{ where } g(m) = m \text{ and } \tilde{f}(m) = \frac{e^{\beta_l m}}{e^{\beta_l m} + 2(d-1)}$$

Taking the derivative with respect to m of $\tilde{f}$ and g and equating them.

$$1 = \frac{2(d-1)\beta_l e^{\beta_l m^*}}{(e^{\beta_l m^*} + 2(d-1))^2}, \quad \text{let} \quad x = \frac{e^{\beta_l m^*}}{2(d-1)}$$

Rewriting the conditions for touching in terms of x

$$\beta_l = \frac{(x+1)^2}{x}, \qquad m^* = \frac{x}{x+1}$$

$$\beta_l m^* = \log(2(d-1)x), \qquad \frac{(x+1)^2}{x}\frac{x}{x+1} = \log(2(d-1)x)$$

Now solving for x

$$e^{x+1} = 2(d-1)x, \qquad xe^{-(x+1)} = \frac{1}{2(d-1)}$$

As $d \to \infty, \beta \to \infty$, however since $\beta_l \sim x$ we can conclude that $x \to \infty$. Therefore when using the Lambert W function we will take the negative branch.

$$-xe^{-x} = \frac{-e}{2(d-1)}$$

Finally we have an equation for β_l and m^* in terms of d.

$$x_d = W_{-1}\left(\frac{-e}{2(d-1)}\right), \quad \text{where} \quad \beta_l = \frac{(x_d+1)^2}{x_d}, \quad \text{and} \quad m^* = \frac{x_d}{x_d+1}$$

Overall the critical value of beta is given as (note that for $d > 3$ this is a lower bound)

$$\begin{cases} \beta_c = d, & d \leq 3 \\ \beta_c \sim \frac{(x_d+1)^2}{x_d}, & d > 3 \end{cases}$$

2.4 Conclusion: Understanding Fluctuations of the Empirical Distribution around Equilibria

We can now expand a Taylor series around some χ^* that is a global minimiser of I, where $h \in T$ is a pertubation.

$$I(\chi^* + h) = I(\chi^*) + \sum_{j=1}^{\infty} \frac{1}{j!} D^j I(\chi^*)(\underbrace{h, ..., h}_{j})$$

$$I(\chi^*+h) = I(\chi^*)+\nabla I(\chi^*)\cdot h+\frac{1}{2}h^T H(\chi^*)h+\frac{1}{3!}D^3 I(\chi^*)(h,h,h)+\frac{1}{4!}D^4 I(\chi^*)(h,h,h,h)+\dots$$

For some λ we have the following by definition

$$\nabla I(\chi^*) = \lambda(\underbrace{1, \dots, 1}_{2d}), \qquad \nabla I(\chi^*) \cdot h = \lambda \sum_{i=1}^{2d} h_i = 0$$

For $\beta \neq \beta_c$ the Hessian's eigenvalues are non-zero and the fluctuations around equilibria can be described locally by the Hessian. However, when $\beta = \beta_c$, the Hessian is degenerate and the Taylor expansion must include the fourth order term in the magnetised direction, since it has zero quadratic curvature.

For $\beta \neq \beta_c \quad I(\chi^* + h) = I(\chi^*) + \frac{1}{2}h^T H(\chi^*)h + o(\|h\|^2)$

Using Sanov's theorem [1], the probability of any state approaches $e^{-NI(\chi)}$ up to sub-exponential terms as $N \to \infty$. Additionally, we let $h = t_N p$, where $t_N \in \mathbb{R}$ is the scale of the fluctuation in terms of N, and $p \in T$ which is the fluctuation vector.

$$-NI(\chi^* + t_N p) = -NI(\chi^*) - \frac{t_N^2 N}{2} p^T H(\chi^*)p + t_N^2 N o(1)$$

For the Hessian term to be non-trivial in the limit as $N \to \infty$ we require

$$-\frac{t_N^2 N}{2} = O(1) \implies t_N = O(N^{-1/2})$$

This simplifies the previous equation to the following

$$-NI(\chi^* + t_N p) = -NI(\chi^*) - \frac{1}{2} p^T H(\chi^*)p + o(1)$$

Then we obtain the random variable for fluctuations around some equilibria χ^* as the following using the Hessian constrained to the tangent hyperplane T

$$\sqrt{N}(\chi_N - \chi^*) \sim \mathcal{N}(0, H_T^{-1}(\chi^*))$$

For $\beta = \beta_c$ we have two cases, $d \leq 3$ and $d > 3$.

When $d \leq 3$, the Hessian at criticality has a one-dimensional kernel corresponding to the magnetisation direction. Decomposing the perturbations into this direction and its orthogonal components shows that quadratic fluctuations are present in all non-degenerate directions, whilst the leading order for the magnetised direction is quartic, as it can be easily shown that the odd terms vanish around the trivial solution. This means that whilst the global minimiser of the rate function is still the trivial solution, the degenerate direction has larger fluctuations.

When $d > 3$, both the trivial and k=1 magnetised states are global minima of the rate function at criticality, as a result of this we yield a mixture of states. However, around each minimiser the Hessian is non-degenerate, therefore fluctuations must be Gaussian with the covariance given by the inverse Hessian, restricted to the tangent hyperplane.

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