AMSI SUMMER RESEARCH SCHOLARSHIPS 2025–26
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The Hopf Fibration

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Abstract

The Hopf fibration is an example of what is known as a nontrivial fibre bundle in pure mathematics. This report provides the necessary background to both define the Hopf fibration and to write it in three different representations. We also prove some of the fibration's key properties, including how the fibres of the map are circles and the linking of these fibre circles. Finally, the report introduces some basic quantum mechanics in order to illustrate how the Hopf fibration appears naturally in a two-state quantum system.

1 Introduction

The Hopf fibration is a construction in mathematics with several interesting properties, making it an object of study for pure mathematicians. The Hopf fibration has several different representations, including spheres in real space, the quaternions, and complex projective space. Not only that, but the fibration appears at least seven times across physics (Urbantke 2003), making it a naturally occurring phenomenon and of interest to physicists.

This report aims to make the Hopf map accessible to undergraduate students by explaining the background knowledge on spheres, quaternions, and complex projective space to describe the Hopf map in each of these representations. Then, the report will investigate some of the properties of the Hopf map, with a focus on its fibres and their arrangement in space. Finally, the connection with quantum mechanics is investigated by analysing what is known as the two-state system, with little prior knowledge in physics assumed.

1.1 Statement of Authorship

The work by Lyons 2022 and Treisman 2009 formed the basis of Section 2, with some results from Berger 1994. The author summarises the key ideas and definitions, presenting them in an easy-to-follow manner. The author also fills in detail left out of these works for proofs of the properties described. Section 3 relies on work by David J. Griffiths 2018 to introduce quantum mechanics in 3.1. The results on qubits in 3.2 are based on the work done in Urbantke 2003 and Urbantke 1991.

2 The Hopf Map

The Hopf map is a particular map $h : S^3 \to S^2$ with S^1 fibres. This section aims to make that statement understandable by providing the required background mathematics and tools necessary to define the map and investigate some of its key features.

2.1 Background Mathematics

2.1.1 Spheres and Stereographic Projection

To begin, we give a definition of a sphere that generalises to all dimensions.

Definition 1. An n-dimensional sphere, or n-sphere, for $n \geq 0$ is the set S^n of unit vectors in $\mathbb{R}^{n+1}$:

$$S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}$$

The circle in the xy-plane S^1 and the 'regular' sphere S^2 in $\mathbb{R}^3$ are two familiar examples of n-spheres and essential to the Hopf map. The final sphere in the Hopf map is the 3-sphere S^3, which is harder to conceptualise, but is just the unit ball in $\mathbb{R}^4$. A key fact about S^3 is that it is equivalent to a subset of $\mathbb{C}^2$. Much like how we can treat $\mathbb{R}^2$ as $\mathbb{C}$ and hence identify S^1 with a subset of $\mathbb{C}$, we identify $\mathbb{R}^4$ with $\mathbb{C}^2$ in the following way:

$$(x_1, y_1, x_2, y_2) \leftrightarrow (x_1 + iy_1, x_2 + iy_2).$$

Then S^3 is the subset of $\mathbb{C}^2$ given by

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}.$$

This way of associating S^3 with a subset of $\mathbb{C}^2$ becomes very useful for one representation of the Hopf fibration and is also related to how the Hopf fibration will later appear in quantum mechanics. From now on, S^3 will be interchangeably treated as a subset of $\mathbb{R}^4$ or $\mathbb{C}^2$, depending on which is most beneficial in the moment.

Another tool we will need to fully investigate the Hopf map is *stereographic projection*. Stereographic projection is a homeomorphism $s_n : S^n \setminus \{N\} \to \mathbb{R}^n$, where $N = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$ is the 'north pole', that exists for all dimensions (Lyons 2022). It being a homeomorphism allows us to investigate properties of a set that resides in $\mathbb{R}^{n+1}$ instead in $\mathbb{R}^n$. The point N we choose to project from is arbitrary—a stereographic projection of S^n exists for all $P \in S^n$. We will now explicitly define the two stereographic projection maps used in this research:

Definition 2. (Lyons 2022) Let $N = (0, 0, 1)$ be the north pole of S^2. Then the map $s_2 : S^2 \setminus \{N\} \to \mathbb{R}^2$ defined by

$$s_2(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

is the stereographic projection of S^2.

Definition 3. (Lyons 2022) Let $N = (0, 0, 0, 1) \in S^3$. Then the map $s_3 : S^3 \setminus \{N\} \to \mathbb{R}^3$ defined by

$$s_3(x, y, z, w) = \left(\frac{x}{1-w}, \frac{y}{1-w}, \frac{z}{1-w}\right)$$

is the stereographic projection of S^3.

An important property of stereographic projection is that it preserves circles. More precisely, circles on S^n that do not pass through the projection point P are mapped to circles in $\mathbb{R}^n$ and circles on S^n that do pass through the projection point P are mapped to lines in $\mathbb{R}^n$ (Lyons 2022; Berger 1994; Treisman 2009).

Finally, we can use stereographic projection to establish an important homeomorphism of the 2-sphere. Since s_2 is a homeomorphism $S^2 \setminus \{N\} \to \mathbb{R}^2$, it is not defined at N. But if we extend the definition of s_2 to N by defining $s_2(N) = \infty$, we get a homeomorphism $S^2 \to \mathbb{R}^2 \cup \{\infty\}$. Furthermore, we can easily connect $\mathbb{R}^2$ to $\mathbb{C}$ by the correspondence $(x, y) \leftrightarrow x + iy$. Therefore, we can show that S^2 is homeomorphic to $\mathbb{C} \cup \{\infty\}$.

2.1.2 Complex Projective Line

A very useful representation of the Hopf map is in terms of its connection to the *complex projective line* $\mathbb{CP}^1$.

Definition 4. (Treisman 2009; Berger 1994) The complex projective line $\mathbb{CP}^1$ is the set of one-dimensional complex linear subspaces of $\mathbb{C}^2$:

$$\mathbb{CP}^1 = \{\text{span}_{\mathbb{C}}(z_1, z_2) : (z_1, z_2) \in \mathbb{C}^2 \setminus \{0\}\}$$

This is equivalent to the quotient $(\mathbb{C}^2 \setminus \{0\})/\sim$ by the equivalence relation given by $x \sim y$ if and only if $x = \lambda y$ for some $\lambda \in \mathbb{C}$. We write elements of $\mathbb{CP}^1$ as ratios $[z_1 : z_2]$.

Therefore, points $[z_1 : z_2] \in \mathbb{CP}^1$ have the property that $[\lambda z_1 : \lambda z_2] = [z_1 : z_2]$ for all nonzero $\lambda \in \mathbb{C}$. This also allows us to rescale our point $[z_1 : z_2]$ in $\mathbb{CP}^1$ to the equivalent point $[1 : z_2]$, so long as $z_1 \neq 0$. The case of $z_1 = 0$ is special and the ratio $[0 : 1]$ is called the 'point at infinity'.

The description of $\mathbb{CP}^1$ as a set equivalence classes gives a canonical projection to the associated equivalence class.

Definition 5. (Berger 1994) The canonical projection $\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}^1$ is given by

$$\pi((z_1, z_2)) = [z_1 : z_2] \tag{1}$$

This projection is particularly relevant to the Hopf map, as will be seen when we define the map and investigate its properties.

Lastly, we will find that $\mathbb{CP}^1$ is homeomorphic to $\mathbb{C} \cup \{\infty\}$. Take a ratio $[z_1 : z_2] \in \mathbb{CP}^1$. Then recall we can rescale to $[z_1 : z_2] = [1 : z]$ as long as $z_1 \neq 0$. Then it is quite easy to see that we can map $[1 : z]$ to the complex number z. Different choices of z give different elements of $\mathbb{CP}^1$, and hence there is a correspondence $[1 : z] \leftrightarrow z$. If we then establish $[0 : 1]$ as the point at infinity, we have a homeomorphism between $\mathbb{CP}^1$ and $\mathbb{C} \cup \{\infty\}$.

2.1.3 Quaternions

To end our section on background mathematics, we will talk about the *quaternions*.

Definition 6. (Lyons 2022; Treisman 2009) The quaternions $\mathbb{H}$ are given by

$$\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$$

where i, j, k satisfy $i^2 = j^2 = k^2 = -1$ and $ij = k$, $jk = i$, $ki = j$.

The quaternions do not form a field, since they are not commutative but anti-commutative. All other field axioms are satisfied, however. We can easily see from their definition that they are equivalent to $\mathbb{R}^4$: we have $a + bi + cj + dk \leftrightarrow (a, b, c, d)$. Similarly to $\mathbb{C}$, we define a conjugate:

Definition 7. (Lyons 2022; Treisman 2009) Let $r = a + bi + cj + dk \in \mathbb{H}$.

- We call a the real part of r and b, c, d the imaginary parts.
- We call a quaternion with real part $a = 0$ a purely imaginary quaternion. We will define $\text{Im}(\mathbb{H}) \subseteq \mathbb{H}$ as the set of all purely imaginary quaternions.
- The conjugate $\bar{r}$ of r is defined as $\bar{r} = a - bi - cj - dk$.

- The magnitude of r is defined as $|r| = \sqrt{r\bar{r}} = \sqrt{a^2 + b^2 + c^2 + d^2}$.
- If $r \neq 0$, then the inverse of r is $r^{-1} = \bar{r}/|r|^2$.

Quaternions r with $|r| = 1$, when viewed as a subset of $\mathbb{R}^4$, is easily seen to be S^3. We can view purely imaginary quaternions as a copy of $\mathbb{R}^3$ sitting inside $\mathbb{R}^4$. Hence, we can also identify unit purely imaginary quaternions with a copy of S^2 inside $\mathbb{R}^4$. The quaternions are very practical for their ability to describe rotations and later we will see how this leads to the Hopf map. If $r \in \mathbb{H}$, $p \in \mathrm{Im}(\mathbb{H})$, then $rpr^{-1} \in \mathrm{Im}(\mathbb{H})$ (Lyons 2022). Hence, we can define a map $R_r : \mathrm{Im}(\mathbb{H}) \to \mathrm{Im}(\mathbb{H})$ by $R_r(p) = rpr^{-1}$. Identifying purely imaginary quaternions with $\mathbb{R}^3$, this is a linear map $\mathbb{R}^3 \to \mathbb{R}^3$ and defines a rotation in three dimensions (Lyons 2022; Treisman 2009). We also can easily see that if $k \in \mathbb{R}$ and $k \neq 0$, then $R_{kr}(p) = (kr)p(kr)^{-1} = kk^{-1}rpr^{-1} = rpr^{-1} = R_r(p)$. Hence, we can rescale our quaternions to only ever consider rotations by unit quaternions and write $R_r(p) = rp\bar{r}$.

2.2 Defining the Hopf Map

Now that we've set up the requisite mathematics, we can actually write down the Hopf map. The Hopf map has several different representations. The three included in this report are in terms of quaternions, real vectors, and complex projective space.

2.2.1 As a Map to purely imaginary quaternions

The way that quaternions describe a rotation of three-space lets us define the Hopf map. Fix a pure unit quaternion $p_0 \in S^2$. Let $r = a + bi + cj + dk$ be the unit quaternion corresponding to $(a, b, c, d) \in S^3$. Then the Hopf fibration is $h : S^3 \to \mathrm{Im}(\mathbb{H})$ defined by $h(r) = R_r(p_o) = rp_0\bar{r}$. For our purposes, we will pick $p_0 = (0, 1, 0, 0) = i$, and so the Hopf fibration is given by

$$h(r) = ri\bar{r}.$$

Since $r, i, \bar{r}$ all have unit length, we have that $h(r)$ is always a unit pure quaternion, and so $h(S^3)$ is a copy of S^2 when $\mathrm{Im}(\mathbb{H})$ is thought of as $\mathbb{R}^3$. We see then that this map of quaternions can be considered a map $S^3 \to S^2$.

2.2.2 As a Map to Real Space

If we write out the Hopf fibration in terms of the components of the quaternions, we can determine an exact map $h(a, b, c, d)$ in terms of real numbers. Let $r = a + bi + cj + dk$. Then the Hopf map is

$$\begin{aligned} h(a, b, c, d) &= (a + bi + cj + dk)i(\overline{a + bi + cj + dk}) \\ &= (a + bi + cj + dk)i(a - bi - cj - dk) \\ &= (a + bi + cj + dk)(ai + b - ck - dj) \\ &= (a^2 + b^2 - c^2 - d^2)i + 2(ad + bc)j + 2(bd - ac)k. \end{aligned}$$

Written in terms of real vectors, we have $h : S^3 \to \mathbb{R}^3$ given by

$$h(a, b, c, d) = \big(a^2 + b^2 - c^2 - d^2, 2(ad + bc), 2(bd - ac)\big)$$

Like discussed with the quaternionic representation, the image of h is always unit length, and so this is a map $S^3 \to S^2$.

2.2.3 As a Map to Complex Projective Space

We can very easily define the Hopf map h in terms of complex projective space. Treating S^3 as a subset of $\mathbb{C}^2$, we define the Hopf map as follows:

Definition 8. (Treisman 2009) The Hopf map $h : S^3 \to \mathbb{C} \cup \{\infty\}$ is defined by

$$h(z_1, z_2) = \frac{z_2}{z_1},$$

where we define $h(0, z_2) = \infty$.

Since we know that $\mathbb{C} \cup \{\infty\}$ is homeomorphic to $\mathbb{CP}^1$, where $z_2/z_1 \leftrightarrow [z_1 : z_2]$, we see that this is a map $S^3 \to \mathbb{CP}^1$. In fact, it is just the restriction of π to S^3. We also know that $\mathbb{C} \cup \{\infty\}$ is homeomorphic to S^2, meaning we do indeed have a map between spheres.

2.3 Properties of the Hopf Map

In the previous section, we gave several equivalent ways of defining the Hopf map. The main variant of the Hopf map that will be used going forward will be the description as a map $S^3 \to \mathbb{CP}^1$. Many of the properties of the map we can observe by looking at the 'fibres' of the map. The fibres of the map are simply preimage sets $h^{-1}(p)$ where $p \in S^2$. The particular reason for calling these fibres will become apparent when we see how the preimage sets relate to each other.

2.3.1 Topologically Circular Fibres

We know from complex projective space that if $[z_1 : z_2] \in \mathbb{CP}^1$, then $[\lambda z_1 : \lambda z_2] = [z_1 : z_2]$ for all $\lambda \in \mathbb{C}$. Since the Hopf map h is the restriction of the map $\pi(z_1, z_2) = [z_1 : z_2]$ to S^3, it has this same property: if $(z_1, z_2) \in S^3$ and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, then $h(\lambda(z_1, z_2)) = h((z_1, z_2))$. The requirement that $|\lambda| = 1$ comes from the domain of the Hopf map—we only considering $\lambda(z_1, z_2) \in S^3$. It follows, then, that the fibres of h are these unit complex multiples $\lambda z \in S^3$. Let $[z_1 : z_2] \in \mathbb{CP}^1$. Then we have

$$h^{-1}([z_1 : z_2]) = \{e^{i\theta}(z_1, z_2) \, : \, \theta \in \mathbb{R}\}.$$

We can define our set in terms of the single parameter θ, and this parameter causes the set to 'wrap back around' on itself, just like a circle. In fact, it is quite easy to show that this fibre is homeomorphic to a circle. We use the point $(z_1, z_2) \in S^3$ as a representative for the equivalence class $[z_1 : z_2]$. Hence, we can write each point in the fibre $h^{-1}([z_1 : z_2])$ as $e^{i\theta}(z_1, z_2)$, like above. Then, if we let $f : h^{-1}([z_1 : z_2]) \to \mathbb{C}$ be the map $e^{i\theta}(z_1, z_2) \mapsto e^{i\theta}$ and we let $g : \mathbb{C} \to \mathbb{R}^2$ be the map $e^{i\theta} \mapsto (\cos\theta, \sin\theta)$, we can take our homeomorphism to be the composition $g \circ f$. It it easy to see that $(g \circ f)(h^{-1}([z_1 : z_2])) = S^1$. Our fibres are all circles, topologically speaking. This is an astonishing fact that makes our map $S^3 \to S^2$ much more interesting—our map is an example of what is known as a nontrivial fibre bundle. We will not give a technical definition of a fibre bundle in this report but one is given in Cohen 1998. The fact that the Hopf fibration is a fibre bundle is greatly important in geometry and algebraic topology (Cohen 1998).

2.3.2 Geometrically Circular Fibres

We've determined that our fibres are actually all copies of S^1, that is, they are homeomorphic to S^1. However, our space sits inside a bigger one—one for which we can measure distance and angles—and being homeomorphic to S^1 does not ensure that these fibres are geometrically circles. It could instead be the case that these fibres are deformed circles sitting on the surface of the sphere. A geometric circle on S^3 would be a subset $U \subseteq S^3$ such that $U \simeq S^1$ and there is a point $x \in \mathbb{R}^4$ such that x is equidistant to all $y \in U$. Examples of geometric circles on S^3 are what is known as the great circles of S^3.

Definition 9. (Berger 1994) A great circle of S^n, where $n \geq 2$, is a subset of S^n of the form $S^n \cap U$, where U is a two-dimensional subspace of $\mathbb{R}^{n+1}$.

It becomes easy for $n = 2$ to see that great circles are geometric circles. A great circle in S^2 is simply the curve along the sphere lying on some plane through the origin. The centre of the great circle is the origin and, by definition of S^2, all the points on the curve must have unit distance from the origin. This generalises to higher dimensions, in particular S^3, and we will use this to show that fibre circles are geometric circles.

First, fix a point $[z_1 : z_2]$ in $\mathbb{CP}^1$. In $\mathbb{CP}^1$, so long as $z_1 \neq 0$, we can rescale this point to be of the form $[1 : z]$, where $z = z_2/z_1$. Recall the map π defined in (1). The preimage of $[1 : z]$ under this map is

$$\pi^{-1}([1 : z]) = \{\lambda(1, z) \,:\, \lambda \in \mathbb{C}, \lambda \neq 0\}.$$

We can explicitly write $\lambda = a + bi$ and $z = x + iy$ for some $a, b, x, y \in \mathbb{R}$ with a, b not simultaneously 0. If we do so, we get the following:

$$\begin{aligned}
\pi^{-1}([1 : z]) &= \{\lambda(1, z) \,:\, \lambda \in \mathbb{C}, \lambda \neq 0\} \\
&= \{(a + bi)(1, x + iy) \,:\, a, b \in \mathbb{R}, a + bi \neq 0\} \\
&= \{(a + bi, (a + bi)(x + iy)) \,:\, a, b \in \mathbb{R}, a + bi \neq 0\} \\
&= \{(a + bi, ax + bxi + ayi - by)) \,:\, a, b \in \mathbb{R}, a + bi \neq 0\} \\
&= \{(a + bi, (ax - by) + i(ay + bx)) \,:\, a, b \in \mathbb{R}, a + bi \neq 0\}
\end{aligned}$$

Now, we identify this with the corresponding subset of $\mathbb{R}^4$ to get

$$\begin{aligned}
\pi^{-1}([1 : z]) &= \{(a, b, ax - by, ay + bx) \,:\, a, b \in \mathbb{R}, a + bi \neq 0\} \\
&= \{a(1, 0, x, y) + b(0, 1, -y, x) \,:\, a, b \in \mathbb{R}, a + bi \neq 0\}.
\end{aligned}$$

This is nearly a subspace of $\mathbb{R}^4$, except it does not contain the zero vector. If we let

$$U = \{a(1, 0, x, y) + b(0, 1, -y, x) \,:\, a, b \in \mathbb{R}\},$$

that is, the subspace of $\mathbb{R}^4$ spanned by $(1, 0, x, y)$ and $(0, 1, -y, x)$, we can write the preimage under π as $\pi^{-1}([1 : z]) = U \setminus \{0\}$. Since the Hopf map is just the restriction of π to S^3, we have that the preimage of $[1 : z]$ under h is simply $h^{-1}([1 : z]) = (U \setminus \{0\}) \cap S^3$. But $0 \notin S^3$, so $h^{-1}([1 : z]) = U \cap S^3$, and the fibres of the Hopf map are great circles of S^3. Thus, we see that our fibres are not just topological circles, but actual circles on the surface of S^3, and we can build S^3 out of an S^2 worth of circles on its surface. However, it is not true that every great circle is a fibre of the Hopf map. Take, for example, the great circle given by

$$K = \{(\cos\alpha, 0, \sin\alpha, 0) \,:\, \alpha \in [0, 2\pi]\}.$$

Now consider the fibre given by

$$h^{-1}([1:0]) = \{(\cos\alpha, \sin\alpha, 0, 0) \,:\, \alpha \in [0, 2\pi]\}.$$

Note that $(1, 0, 0, 0)$ is an element of K and $h^{-1}([1:0])$. Since fibres are disjoint sets and $K \cap h^{-1}([1:0]) \neq \emptyset$, either $K = h^{-1}([1:0])$ or K is not a fibre of the Hopf map. It is fairly easy to see that K cannot be $h^{-1}([1:0])$, so K is not a fibre of the Hopf map.

2.3.3 Linkedness of Fibre Circles

A truly remarkable property of the Hopf map is that all the fibre circles are linked in $\mathbb{R}^4$. Not only is this linking property interesting, but all of the fibres are necessary for the nontriviality of the Hopf fibration—taking only part of S^3 trivialises the fibration (Cohen 1998).

The process for showing all fibre circles are linked with each other is a two-step process, described by both Lyons 2022 and Treisman 2009 but with the details filled out here. First, we will show that each fibre circle is linked to a particular fibre in the xy-plane under stereographic projection. Next, we take a pair of fibre circles and demonstrate a rotation of $\mathbb{R}^4$ which takes one of the pair to the fibre in the xy-plane and the other to a different circular fibre. This proves that the two are linked after the rotation and therefore must be linked before applying the rotation.

We start by taking a point $[1:z] \in \mathbb{CP}^1$ and describing the preimage under h. Similarly to 2.3.1, we have

$$h^{-1}([1:z]) = \left\{ \frac{e^{i\theta}}{\sqrt{1+|z|^2}}(1, z) \,:\, \theta \in [0, 2\pi] \right\},$$

where we've picked up a factor to ensure our points are unit length. Then, taking our first fibre to be $h^{-1}([1:0])$, we have

$$h^{-1}([1:0]) = \left\{ e^{i\theta}(1, 0) \,:\, \theta \in [0, 2\pi] \right\}$$

As a subset of $\mathbb{R}^4$, this is

$$h^{-1}([1:0]) = \{(\cos\theta, \sin\theta, 0, 0) \,:\, \theta \in [0, 2\pi]\}.$$

Using the stereographic projection s_3, we then have that

$$s \circ h^{-1}([1:0]) = \{(\cos\theta, \sin\theta, 0) \,:\, \theta \in [0, 2\pi]\},$$

meaning this fibre is the unit circle in the xy-plane.

Now, let $z = re^{i\phi} \in \mathbb{C}$ with $r > 0$. Then $h^{-1}([1:z])$ is a different fibre to $h^{-1}([1:0])$. We will show it is linked to the $h^{-1}([1:z])$ after stereographic projection by finding the intersection points of $s \circ h^{-1}([1:0])$ with the xy-plane. We have

$$\begin{aligned} h^{-1}([1:z]) &= \left\{ \frac{e^{i\theta}}{\sqrt{1+r^2}}(1, re^{i\phi}) \,:\, \theta \in [0, 2\pi] \right\} \\ &= \left\{ \frac{1}{\sqrt{1+r^2}}(e^{i\theta}, re^{i(\theta+\phi)}) \,:\, \theta \in [0, 2\pi] \right\} \end{aligned}$$

Writing this as a real subset, we have

$$h^{-1}([1:z]) = \left\{ \left(\frac{\cos\theta}{\sqrt{r^2+1}}, \frac{\sin\theta}{\sqrt{r^2+1}}, \frac{r\cos(\theta+\phi)}{\sqrt{r^2+1}}, \frac{r\sin(\theta+\phi)}{\sqrt{r^2+1}} \right) \,:\, \theta \in [0, 2\pi] \right\},$$

and stereographically projecting gives

$$s \circ h^{-1}([1:z]) = \left\{ \left(\frac{\cos\theta}{\sqrt{r^2+1} - r\sin(\theta+\phi)}, \frac{\sin\theta}{\sqrt{r^2+1} - r\sin(\theta+\phi)}, \frac{r\cos(\theta+\phi)}{\sqrt{r^2+1} - r\sin(\theta+\phi)} \right) : \theta \in [0, 2\pi] \right\}$$

We want to find solutions in the xy-plane, that is, we solve

$$\frac{r\cos(\theta+\phi)}{\sqrt{r^2+1} - r\sin(\theta+\phi)} = 0$$

for $\theta \in [0, 2\pi]$. It is easy to see that this occurs only when $\theta + \phi = \pi/2$ or $\theta + \phi = 3\pi/2$. Let $\theta_1 = \frac{\pi}{2} - \phi$ and $\theta_2 = \frac{3\pi}{2} - \phi$. Then $\cos\theta_1 = \sin\phi$, $\sin\theta_1 = \cos\phi$, $\cos\theta_2 = -\sin\phi$, $\sin\theta_2 = -\cos\phi$, $\sin(\theta_1 + \phi) = 1$, and $\sin(\theta_2 + \phi) = -1$. Now, we substitute θ_1, θ_2 into the above expression to get

$$\begin{aligned} u_1 &= \left(\frac{\cos\theta_1}{\sqrt{r^2+1} - r\sin(\theta_1+\phi)}, \frac{\sin\theta_1}{\sqrt{r^2+1} - r\sin(\theta_1+\phi)}, \frac{r\cos(\theta_1+\phi)}{\sqrt{r^2+1} - r\sin(\theta_1+\phi)} \right) \\ &= \left(\frac{\sin\phi}{\sqrt{r^2+1} - r}, \frac{\cos\phi}{\sqrt{r^2+1} - r}, 0 \right) \end{aligned}$$

and

$$\begin{aligned} u_2 &= \left(\frac{\cos\theta_2}{\sqrt{r^2+1} - r\sin(\theta_2+\phi)}, \frac{\sin\theta_2}{\sqrt{r^2+1} - r\sin(\theta_2+\phi)}, \frac{r\cos(\theta_1+\phi)}{\sqrt{r^2+1} - r\sin(\theta_1+\phi)} \right) \\ &= \left(-\frac{\sin\phi}{\sqrt{r^2+1} + r}, -\frac{\cos\phi}{\sqrt{r^2+1} + r}, 0 \right) \end{aligned}$$

The distance from the origin to u_2 is $\|u_2\| = \frac{1}{\sqrt{r^2+1}+r}$. Since $r > 0$, we have $r^2 + 1 > 1$, so $\sqrt{r^2+1} > 1$ and $\sqrt{r^2+1} + r > 1$. Thus, $\|u_2\| < 1$ and u_2 is inside the unit circle $s \circ h^{-1}([1:0])$. Similarly, the distance from the origin to u_1 is $\|u_1\| = \frac{1}{\sqrt{r^2+1}-r}$. Performing some algebraic manipulation, we find

$$\begin{aligned} \|u_1\| &= \frac{1}{\sqrt{r^2+1} - r} \\ &= \frac{\sqrt{r^2+1} + r}{(\sqrt{r^2+1} - r)(\sqrt{r^2+1} + r)} \\ &= \frac{\sqrt{r^2+1} + r}{r^2 + 1 - r^2} \\ &= \frac{\sqrt{r^2+1} + r}{1} \\ &= \frac{1}{\|u_2\|}. \end{aligned}$$

Since $\|u_2\| < 1$, we must have $\|u_1\| > 1$, and so u_1 is outside the unit circle $s \circ h^{-1}([1:0])$. Since there are always two intersection points, one inside and one outside $s \circ h^{-1}([1:0])$, the projected circles are linked. But stereographic projection is a homeomorphism, meaning the circles were linked before the projection too. Finally, consider the fibre $h^{-1}([0:1])$ given by

$$h^{-1}([0:1]) = \left\{ e^{i\theta}(0,1) : \theta \in [0, 2\pi] \right\}.$$

In real coordinates,

$$h^{-1}([0:1]) = \{(0, 0, \cos\theta, \sin\theta) : \theta \in [0, 2\pi]\}.$$

We see that this fibre passes through our s_3 projection point $(0,0,0,1)$. To avoid the projection point, we can shift the interval of θ and exclude the end points:

$$h^{-1}([0:1]) \setminus \{N\} = \{(0,0,\cos\theta,\sin\theta) \,:\, \theta \in (-3\pi/2, \pi/2)\}\,.$$

The stereographic projection gives

$$s_3(h^{-1}([0:1]) \setminus \{N\}) = \left\{ \left(0, 0, \frac{\cos\theta}{1-\sin\theta}\right) \,:\, \theta \in (-3\pi/2, \pi/2) \right\}.$$

As θ approaches $-3\pi/2$, the fraction diverges to negative infinity, while as θ approaches $\pi/2$, the fraction diverges to positive infinity. Hence, this is simply the z-axis:

$$s_3(h^{-1}([0:1]) \setminus \{N\}) = \{(0,0,z) \,:\, z \in \mathbb{R}\}\,.$$

This intersects the unit circle $s \circ h^{-1}([1:0])$ at $(0,0,0)$. As our circle passes through the projection point, its stereographic projection becomes a circle of infinite radius, and so we consider passing through $s \circ h^{-1}([1:0])$ as being linked in $\mathbb{R}^3$. Hence, we see that all fibres of the Hopf map are linked with the fibre $h^{-1}([1:0])$ in $\mathbb{R}^4$.

To show that all fibres are linked with each other, not just $h^{-1}([1:0])$, we start by taking the fibre $h^{-1}([z_1:z_2])$ where $(z_1,z_2) \in S^3$. Now, the claim is this: there is $A \in SU(2)$ such that

$$A \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

where $SU(2)$ are the *special unitary matrices* and have the form (Hall 2015)

$$SU(2) = \left\{ \begin{bmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{bmatrix} \,:\, \alpha, \beta \in \mathbb{C},\ |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

Let $A \in SU(2)$. Then $\det A = \alpha\bar{\alpha} + \beta\bar{\beta} = |\alpha|^2 + |\beta|^2 = 1$ and A is invertible. Hence, we can invert A to get the equation

$$A^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

Taking the inverse, we get

$$\begin{aligned} A^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\ \begin{bmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\ \begin{bmatrix} \bar{\alpha} & \bar{\beta} \\ -\beta & \alpha \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \end{aligned}$$

giving $\alpha = \bar{z}_1$ and $\beta = -z_2$ as solutions. Hence, the matrix

$$A = \begin{bmatrix} \bar{z}_1 & \bar{z}_2 \\ -z_2 & z_1 \end{bmatrix}$$

rotates (z_1, z_2) onto $(1, 0)$. Since matrix multiplication is linear, we have that A rotates $e^{i\theta}(z_1, z_2)$ onto $e^{i\theta}(1, 0)$, meaning it rotates the entire fibre $h^{-1}([z_1 : z_2])$ onto the fibre $h^{-1}([1 : 0])$. Now, take a fibre $h^{-1}([w_1 : w_2])$ for $(w_1, w_2) \in S^3$. We show that A rotates this fibre onto a different fibre. For this to be true, we must show

$$A \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_1' \\ w_2' \end{bmatrix}$$

where $(w_1', w_2') \in S^3$. We have

$$\begin{aligned} A \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} &= \begin{bmatrix} \bar{z}_1 & \bar{z}_2 \\ -z_2 & z_1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\ &= \begin{bmatrix} \bar{z}_1 w_1 + \bar{z}_2 w_2 \\ z_1 w_2 - z_2 w_1 \end{bmatrix} \\ &= \begin{bmatrix} w_1' \\ w_2' \end{bmatrix} \end{aligned}$$

Then we have

$$\begin{aligned} |w_1'|^2 &= (\bar{z}_1 w_1 + \bar{z}_2 w_2)(\overline{\bar{z}_1 w_1 + \bar{z}_2 w_2}) \\ &= (\bar{z}_1 w_1 + \bar{z}_2 w_2)(z_1 \bar{w}_1 + z_2 \bar{w}_2) \\ &= z_1 \bar{z}_1 w_1 \bar{w}_1 + z_2 \bar{z}_2 w_2 \bar{w}_2 + z_1 \bar{z}_2 \bar{w}_1 w_2 + \bar{z}_1 z_2 w_1 \bar{w}_2 \end{aligned}$$

and

$$\begin{aligned} |w_2'|^2 &= (z_1 w_2 - z_2 w_1)(\overline{z_1 w_2 - z_2 w_1}) \\ &= (z_1 w_2 - z_2 w_1)(\bar{z}_1 \bar{w}_2 - \bar{z}_2 \bar{w}_1) \\ &= z_1 \bar{z}_1 w_2 \bar{w}_2 + z_2 \bar{z}_2 w_1 \bar{w}_1 - z_1 \bar{z}_2 \bar{w}_1 w_2 - \bar{z}_1 z_2 w_1 \bar{w}_2, \end{aligned}$$

hence

$$\begin{aligned} |w_1'|^2 + |w_2'|^2 &= z_1 \bar{z}_1 w_1 \bar{w}_1 + z_2 \bar{z}_2 w_2 \bar{w}_2 + z_1 \bar{z}_1 w_2 \bar{w}_2 + z_2 \bar{z}_2 w_1 \bar{w}_1 \\ &= z_1 \bar{z}_1 (w_1 \bar{w}_1 + w_2 \bar{w}_2) + z_2 \bar{z}_2 (w_1 \bar{w}_1 + w_2 \bar{w}_2) \\ &= (z_1 \bar{z}_1 + z_2 \bar{z}_2)(w_1 \bar{w}_1 + w_2 \bar{w}_2) \\ &= 1 \cdot 1 = 1, \end{aligned}$$

proving $(w_1', w_2') \in S^3$. Due to the linearity of A, the point $e^{i\theta}(w_1, w_2)$ is sent to the point $e^{i\theta}(w_1', w_2')$, meaning A rotates the fibre $h^{-1}([w_1 : w_2])$ onto the fibre $h^{-1}([w_1' : w_2'])$. Thus, A is an invertible transformation rotating $h^{-1}([z_1 : z_2])$ onto $h^{-1}([1 : 0])$ and $h^{-1}([w_1 : w_2])$ onto a different fibre $h^{-1}([w_1' : w_2'])$. Since $h^{-1}([1 : 0])$ and $h^{-1}([w_1' : w_2'])$ are linked (from earlier argument), the fibres $h^{-1}([z_1 : z_2])$ and $h^{-1}([w_1 : w_2])$ must have been linked. This proves that all fibre circles in S^3 are linked with each other—a remarkable property of the Hopf fibration.

3 The Hopf Map in Quantum Physics

The Hopf map can be seen to have very interesting mathematical properties, including being an example of a nontrivial fibre bundle, making it an object of study for pure mathematics. But, as it turns out, the Hopf fibration appears at least seven times throughout physics (Urbantke 2003), making it somehow a naturally occurring phenomenon. The example we will look at in this section is how the Hopf map arises in two-state systems, called 'qubits'.

3.1 Quantum Systems

In order to describe how the Hopf map appears in a two-state system, we must establish some basic quantum mechanics.

3.1.1 States and State Space

In quantum mechanics, the states of the system reside in a complex Hilbert space H equipped with inner product $\langle \cdot, \cdot \rangle$, where the basis is the possible states a particle of the system could be in. We typically label these states ψ_i, with the Hilbert basis being

$$\{\psi_1, \psi_2, \psi_3, \dots\}.$$

These states often arise from solutions to state equations, e.g. the Schrödinger equation. A classical particle of the system will be in exactly one of these states at a given time. That is, we know which state we will observe at a given time. A quantum particle, however, may be in a *superposition of states*.

Definition 10. (David J. Griffiths 2018) A superposition of states is a complex-linear combination of the basis states

$$\Psi = \sum_{i=1}^{\infty} c_i \psi_i$$

where $c_i \in \mathbb{C}$.

The quantum mechanical interpretation, when Ψ is a wavefunction, is that $|\Psi|^2$ represents a probability density function, and so we may observe our particle in any state where $c_i \neq 0$, with probability of observing state ψ_i proportional to the magnitude $|c_i|^2$ (David J. Griffiths 2018).

This also leads, naturally, to the idea of normalised states. A superposition Ψ is normalised if and only if $\langle \Psi, \Psi \rangle = 1$. Since our Hilbert basis is orthonormal,

$$\langle \psi_i, \psi_j \rangle = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases},$$

we have that Ψ is normalised if and only if $\sum_i \langle c_i, c_i \rangle = 1$.

3.1.2 Observables

A quantity that we can measure about our system is called an 'observable'. Since our quantum particle may be in any number of different states with varying probability of observing that state, we often care more about the expected value, or expectation, of an observable particle. Observables are measured using hermitian operators in the following way:

Definition 11. (David J. Griffiths 2018) Let $\hat{Q}$ be the Hermitian operator for observable Q and $\langle \cdot, \cdot \rangle$ the inner product for our state space. Then the expectation of Q for the state Ψ is $\langle Q \rangle = \langle \Psi, \hat{Q}\Psi \rangle$.

Some common observables and their operators are position, $\hat{x} = x$; momentum, $\hat{p} = -i\hbar\frac{\partial}{\partial x}$; and energy, $\hat{E} = i\hbar\frac{\partial}{\partial t}$.

3.2 Qubits

In general, a Hilbert space may be infinite-dimensional, and our superpositions are infinite complex linear combinations. As it turns out, however, interesting quantum physics can appear even in the simplest of systems, so we now focus our attention to the simplest, nontrivial quantum mechanical system. A system with only one state is trivial, since we can form no superpositions of states. A system with two states can have quantum superpositions, and so we look at a two-state system, or a 'qubit' system.

3.2.1 States of a Qubit System

Consider calling the two states of the system $|0\rangle$ and $|1\rangle$. In classical mechanics, a two-state particle can be called a 'bit', and it is either in the $|0\rangle$ state or in the $|1\rangle$ state. If we now allow our particle to be in a superposition of the $|0\rangle$ and $|1\rangle$ states, we have what we call a 'qubit' (Urbantke 2003). A superposition of this system can then be written as $\Psi = c_0 |0\rangle + c_1 |1\rangle$ for $c_i \in \mathbb{C}$. The basis for our state space is simply $\{|0\rangle, |1\rangle\}$. Hence, we see that our system is simply $\mathbb{C}^2$, which we equip with the hermitian inner product $\langle z, w\rangle = z^\dagger w$, where $z^\dagger$ is the hermitian conjugate of z (Urbantke 1991). To simplify notation, we can use the familiar mathematics notation for $\mathbb{C}^2$, and we write a superposition of states as $z = z_1(1,0) + z_2(0,1) = (z_1, z_2)$ for $z_1, z_2 \in \mathbb{C}$.

For this system, we see that the normalisation condition $\langle z, z\rangle = 1$ is equivalent to $|z_1|^2 + |z_2|^2 = 1$. That is, a state $z \in \mathbb{C}^2$ is normalised if and only if $z \in S^3 \subseteq \mathbb{C}^2$. Furthermore, if we take a unit complex number $\lambda \in \mathbb{C}$, we have that

$$\begin{aligned}\langle \lambda z, \lambda z\rangle &= (\lambda z)^\dagger(\lambda z)\\ &= \bar{\lambda} z^\dagger \lambda z\\ &= \bar{\lambda}\lambda z^\dagger z\\ &= z^\dagger z\\ &= \langle z, z\rangle.\end{aligned}$$

Thus, multiplying our normalised state by a unit complex number produces a normalised state. We call λ the 'phase factor' and can write it $\lambda = e^{i\alpha}$ (Urbantke 1991). Furthermore, we see that the expectation behaves similarly. If $\hat{Q}$ is a hermitian operator, then $\langle \lambda z, \hat{Q}(\lambda z)\rangle = \bar{\lambda} z^\dagger \hat{Q} \lambda z = z^\dagger \hat{Q} z = \langle z, \hat{Q} z\rangle$. Hence, our expectation ignores any phase factor multiplying our state z. For this reason, we define what we call a 'pure state' or sometimes 'unitary ray'.

Definition 12. (Urbantke 1991) Let $z \in \mathbb{C}^2$ be a normalised state. Then the pure state associated with z is the set of all phase factor multiples of z

$$\{\lambda z : \lambda \in \mathbb{C}, |\lambda| = 1\}$$

Therefore, the expectation depends only on the chosen pure state. Pure states can be easily seen to be equivalent to elements of $\mathbb{CP}^1$ using π: A representative $z = (z_1, z_2)$ of a pure state corresponds to $[z_1 : z_2]$ in $\mathbb{CP}^1$.

3.2.2 Expectation of Spin Angular Momentum

Of particular importance in quantum mechanics are the *Pauli spin matrices*—the operators for spin angular momentum for spin-1/2 particles.

Definition 13. (Urbantke 2003) The Pauli spin matrices $\sigma_1, \sigma_2, \sigma_3$ are given by

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The importance of these to physics goes beyond the scope of this report, so it will be taken for granted that we care about the quantities these measure. We will determine what the possible expectation values for our qubits are under these operators. To do so, we will define

$$R(z) = (\langle z, \sigma_1 z\rangle, \langle z, \sigma_2 z\rangle, \langle z, \sigma_3 z\rangle) = \left(z^\dagger \sigma_1 z,\ z^\dagger \sigma_2 z,\ z^\dagger \sigma_3 z\right)$$

to be the 3-tuple of expectation values. Let's determine expressions in terms of z_1, z_2 for each component of R:

$$\begin{aligned} \langle z, \sigma_1 z\rangle &= z^\dagger \sigma_1 z \\ &= \begin{bmatrix} \bar{z}_1 & \bar{z}_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\ &= \begin{bmatrix} \bar{z}_1 & \bar{z}_2 \end{bmatrix} \begin{bmatrix} z_2 \\ z_1 \end{bmatrix} \\ &= z_1 \bar{z}_2 + z_2 \bar{z}_1 \end{aligned}$$

$$\begin{aligned} \langle z, \sigma_2 z\rangle &= z^\dagger \sigma_2 z \\ &= \begin{bmatrix} \bar{z}_1 & \bar{z}_2 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\ &= \begin{bmatrix} \bar{z}_1 & \bar{z}_2 \end{bmatrix} \begin{bmatrix} iz_2 \\ -iz_1 \end{bmatrix} \\ &= i(z_1 \bar{z}_2 - z_2 \bar{z}_1) \end{aligned}$$

$$\begin{aligned} \langle z, \sigma_3 z\rangle &= z^\dagger \sigma_3 z \\ &= \begin{bmatrix} \bar{z}_1 & \bar{z}_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\ &= \begin{bmatrix} \bar{z}_1 & \bar{z}_2 \end{bmatrix} \begin{bmatrix} z_1 \\ -z_2 \end{bmatrix} \\ &= z_1 \bar{z}_1 - z_2 \bar{z}_2 \end{aligned}$$

We can write the final expression instead as $z_1\bar{z}_1 - z_2\bar{z}_2 = |z_1|^2 - |z_2|^2$, meaning that the third component is always real. As it turns out, we can show that all three components are always real. Let $z_1 = b + ai$ and $z_2 = c + di$. Then $z_1\bar{z}_2 = (b+ai)(c-di) = bc + aci - ibd + ad$ and $z_2\bar{z}_1 = (c+di)(b-ai) = bc + bdi - aci + ad$, so we have $z_1\bar{z}_2 + z_2\bar{z}_1 = 2(ad+bc) \in \mathbb{R}$ and $i(z_1\bar{z}_2 - z_2\bar{z}_1) = 2i(aci - bdi) \in \mathbb{R}$. Hence, we find that $R(z) \in \mathbb{R}^3$ for all z.

The expectation vector R being a real vector is not necessarily surprising. If the Pauli spin matrices corresponded to an observable quantity, we would expect to see values we could measure in the real world. However, what we will show next is that this vector is always unit length for a pure state. To do so, we will introduce the *density matrix* of a normalised state.

Definition 14. (Urbantke 1991) The density matrix ρ of a state z is given by

$$\rho = \begin{bmatrix} z_1\bar{z}_1 & z_1\bar{z}_2 \\ z_2\bar{z}_1 & z_2\bar{z}_2 \end{bmatrix}$$

We see easily from the definition that if $\lambda \in \mathbb{C}$ and $|\lambda| = 1$ that the density matrix associated with λz is the same as the one for z. Hence, density matrices are related to the pure state of a normalised state, rather than the state itself. We introduce the density matrix for the following reason: If $R = (R_1, R_2, R_3)$ is the expectation under the spin angular momentum operators, then

$$\rho = \frac{1}{2}\begin{bmatrix} 1+R_3 & R_1 - iR_2 \\ R_1 + iR_2 & 1 - R_3 \end{bmatrix}.$$

Above, we have expressions for R_1, R_2, R_3 in terms of z_1, z_2 and we can use those to show this is true:

$$\begin{aligned}
1 + R_3 &= 1 + (z_1\bar{z}_1 - z_2\bar{z}_2) = z_1\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_1 - z_2\bar{z}_2 = 2z_1\bar{z}_1 \\
1 - R_3 &= 1 - (z_1\bar{z}_1 - z_2\bar{z}_2) = z_1\bar{z}_1 + z_2\bar{z}_2 - z_1\bar{z}_1 + z_2\bar{z}_2 = 2z_2\bar{z}_2 \\
R_1 + iR_2 &= z_1\bar{z}_2 + z_2\bar{z}_1 + (z_1\bar{z}_2 - z_2\bar{z}_1) = 2z_1\bar{z}_2 \\
R_1 - iR_2 &= z_1\bar{z}_2 + z_2\bar{z}_1 - (z_1\bar{z}_2 - z_2\bar{z}_1) = 2z_2\bar{z}_1
\end{aligned}$$

$$\frac{1}{2}\begin{bmatrix} 1+R_3 & R_1 - iR_2 \\ R_1 + iR_2 & 1 - R_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2z_1\bar{z}_1 & 2z_1\bar{z}_2 \\ 2z_2\bar{z}_1 & 2z_2\bar{z}_2 \end{bmatrix} = \rho$$

Taking the determinant of both sides, we have

$$\begin{aligned}
\det\begin{bmatrix} z_1\bar{z}_1 & z_1\bar{z}_2 \\ z_2\bar{z}_1 & z_2\bar{z}_2 \end{bmatrix} &= \frac{1}{4}\det\begin{bmatrix} 1+R_3 & R_1 - iR_2 \\ R_1 + iR_2 & 1 - R_3 \end{bmatrix} \\
z_1\bar{z}_1 z_2\bar{z}_2 - z_1\bar{z}_2 z_2\bar{z}_1 &= \frac{1}{4}(1+R_3)(1-R_3) - \frac{1}{4}(R_1 - iR_2)(R_1 + iR_2) \\
0 &= \frac{1}{4}(1 - R_3^2) - \frac{1}{4}(R_1^2 + R_2^2) \\
0 &= 1 - (R_1^2 + R_2^2 + R_3^2) = 1 - \|R\|^2,
\end{aligned}$$

so we find that $\|R(z)\| = 1$ for all pure states z. This means that not only is $R(z)$ real for all $z \in S^3$, but it is also unit length, that is, $R(z) \in S^2$.

Consider the mapping $S^3 \to S^2$ given by $z \mapsto R(z)$. Let's explicitly write R_1, R_2, R_3 in terms of $z_1 = b + ai$ and $z_2 = c + di$. From earlier, where we confirmed R_1, R_2, R_3 were real, we had

$$R(z) = (R_1, R_2, R_3) = \big(2(ad+bc), 2i(aci - bdi), |z_1|^2 - |z_2|^2\big)$$

Writing it completely in terms of a, b and simplifying, we have

$$R(z) = \left(2(ad+bc), 2(bd-ac), a^2+b^2-c^2-d^2\right),$$

and by permuting the coordinates, we see we have the real form of the Hopf map from Section 2!

The appearance of the Hopf map here is extraordinary for a few reasons. We were able to derive the Hopf map entirely from quantum mechanics concepts, giving a very beautiful connection between an area of study in pure mathematics and physics. This allows us to infer facts about one from the other. For example, asking which states of our system map to the same spin angular momentum expectation is equivalent to asking what the fibres of the Hopf map are. Not only this, but our study of the Hopf map tells us exactly what our fibres are and how they are arranged in space. We know that our fibres are circular, hence moving along a fibre of the Hopf map is equivalent to changing the phase of a normalised state. When studying the time evolution of qubit state z, we may have a non-constant phase $\lambda(t)$ which is not observed when we measure spin angular momentum, for example. These are just a few of the ways that we can apply knowledge from one field to glean information in the other field.

4 Conclusion

We've discussed how the Hopf fibration is a map $S^3 \to S^2$ which has fibres that are topologically circles S^1. We then saw that these fibres are actually great circles of S^3, making them geometrically circles, too. We also investigated how these fibres are all linked together with each other. Next, we looked at the two-state system from quantum mechanics and discovered that the expectation of normalised states under the Pauli spin matrices gives the Hopf fibration.

Further areas of research may include investigating how the Hopf fibration naturally occurs in other areas of physics—in particular, the six other occurrences pointed out by Urbantke 2003—or to continue the study of qubits by looking at the solutions to the Schrödinger equation for the system, as done in Urbantke 1991. The Hopf fibration also has a higher dimensional analogue which relates to a system of entangled qubits, which is discussed in Remy Mosseri 2001, and is therefore a natural extension to the research done on qubits in this project.

5 Acknowledgements

I would like to thank my supervisors, Michael Albanese and Raymond Vozzo, who took valuable time out of preparing for the merger of the University of Adelaide and the University of South Australia into Adelaide University to supervise summer research projects, ensuring the opportunity to complete an AMSI Summer Research Scholarship was still available during this transition process.

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