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Multiplicative Dependence in Linear Recurrence Sequences

David Nachuan Chen

Supervised by Associate Professor Alina Ostafe

University of New South Wales

Contents

Abstract

For a linear recurrence sequence $(u(n))_{n=1}^{\infty}$, we bound $M_s^*(N)$ the number of s-tuples among $u(1), u(4), \ldots, u(N^2)$ that are multiplicatively dependent of maximal rank, lowering from the trivial bound N^s. We find that $M_s^*(N) \ll N^{s(1-1/(d+1)+2((d+1)!-d)/(1+2(d+1)(d+1)!-2d(d+1))}$.

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2 Introduction

2.1 Statement of Authorship

The main result of this paper, **Theorem 3.6**, is proven by the author along with supporting **Theorem 3.2** and **Lemma 3.4**. Meanwhile, **Lemma 3.3** and **Lemma 3.5** are attributed to [5] and [2] respectively. For **Remark 3.1**, a widely known result in number theory, the version of the proof presented in the appendix is the author's own work.

2.2 Motivation

Although linear recurrences are predictable, it is difficult to predict the arithmetic properties of the terms themselves. One of these challenges is understanding the multiplicative structure of the elements in linear recurrences. This motivated the study of multiplicative dependence in binary sequences with bounded exponents [3], of k-Fibonacci and k-Lucas sequences [4], and more recently, of any linear recurrence sequence [2]. In these works, the goal is to obtain good upper bounds of the number of multiplicatively dependent tuples in linear recurrences, obtaining a power-saving over the trivial bound.

This paper explores a variation of this problem, seeking to bound the number of multiplicative dependent s-tuples from linear recurrences at quadratic terms. Due to the nature of the growth of linear recurrences at quadratic terms, this problem requires different results from the previously mentioned papers, which usually work with a range of N consecutive terms. We follow similar strategies as presented in previous works, however, we have to overcome several technical difficulties and obtain similar mathematical tools in our setting.

3 Preliminaries

3.1 Definitions and Setup

We define $\mathbf{u} = (u(n))_{n=1}^{\infty}$ as an *integer linear recurrence sequence* of order $d \geqslant 1$ if its terms are recursively defined through some relation

$$u(n+d) = c_{d-1}u(n+d-1) + \ldots + c_0u(n), \qquad n = 1, 2, \ldots,$$

for some integers $c_0, \ldots, c_{d-1}$ and where $c_0 \neq 0$[1]. Moreover, the characteristic polynomial of $\mathbf{u}$ is defined as

$$f(x) = x^d - c_{d-1}x^{d-1} - \ldots - c_0 \in \mathbb{Z}[x].$$

Faithful to the original paper [2], we define $\mathbf{u}$ to be *simple* if there are no repeating roots in f and non-degenerate if there are no roots of unity among the ratios of the distinct roots of f. A root Λ is said to be *dominant* among roots $\lambda_1, \ldots, \lambda_d$ of a polynomial f if $|\Lambda| \geqslant |\lambda_i|$ for $1 \leqslant i \leqslant d$ and Λ is one of the roots.

[1] If $c_0 = 0$, then the relation could instead be defined as $u(n+d) = c_{d-1}u(n+d-1) + \ldots + c_1u(n+1)$, effectively making $u(n)$ redundant.

Remark 3.1. *Let $\lambda_1, \ldots, \lambda_d$ be the roots f. If $\boldsymbol{u}$ is simple, then the closed-form solution to $u(n)$ is a linear combination of $\lambda_1^n, \ldots, \lambda_d^n$. That is, we can write*

$$u(n) = a_1\lambda_1^n + \ldots + a_n\lambda_d^n, \qquad n = 1, 2, \ldots,$$

where the coefficients $a_1, \ldots, a_n \in \mathbb{Q}(\lambda_1, \ldots, \lambda_d)$ are determined by solving for the first d terms of $\boldsymbol{u}$.

This is a well-known result in number theory, and I have provided my version of the proof in the **Appendix**. From this it is implied that for a simple non-degenerate sequence $\mathbf{u}$, the growth of its terms is bounded by the power of the modulus of Λ, a dominant root of f. That is, $|u(n)| \ll |\Lambda|^n$. Van der Poorten and Schlickewei [8] further proved that for any $\varepsilon > 0$, the lower bound $|u(n)| \gg |\Lambda|^{n(1-\varepsilon)}$ is satisfied for $n \geqslant n_0(\mathbf{u}, \varepsilon)$, where $n_0(\mathbf{u}, \varepsilon)$ is a constant dependent on ε and $\mathbf{u}$.

A set of integers $\gamma_1, \ldots, \gamma_s$ is said to be *multiplicatively dependent (m.d.)* if there exists a solution to $\gamma_1^{k_1} \ldots \gamma_s^{k_s} = 1$ such that $k_1, \ldots, k_s$ are all integers and not all of them are 0. Additionally, $\gamma_1, \ldots, \gamma_s$ is *m.d. of maximal rank* if all solutions have $k_1 \ldots k_s \neq 0$. Finally, for any set of primes S, we define an integer *S-unit* as an integer whose prime factors are all in S. We define $A \ll B$ if there exists a positive constant c such that $|A| \leqslant cB$. Furthermore, we define $A \ll_{\mathbf{u}} B$ if the constant c depends on the roots of the characteristic polynomial of the sequence $\mathbf{u}$.

3.2 Theorems and Lemmas

Theorem 3.2. *Let $\mathcal{A}(S; N)$ be the number of S-units among the elements $u(1), u(4), \ldots, u(N^2)$, where $\boldsymbol{u}$ is a simple, non-degenerate sequence of order d and S is a set of r distinct primes. Then,*

$$\mathcal{A}(S; N) \ll_{\boldsymbol{u}} rN^{d/(d+1)}.$$

Lemma 3.3. *Let $\boldsymbol{u}$ be a simple recurrence sequence of order d and $\lambda_1, \ldots, \lambda_d$ be the roots of its characteristic polynomial in $\overline{\mathbb{F}}_q^*$ such that*

$$\prod_{i=1}^{d} \lambda_i^{k_i} \neq 1, \qquad 0 < \max_{i=1..d} |k_i| \leqslant \tau_q.$$

Then,

$$\sharp\{1 \leqslant x \leqslant N : u(x^2) = 0\} \ll_{\boldsymbol{u}} N(N^{-1/((d+1)!-d-1)} + \tau_q^{-1/((d+1)!-d)}).$$

This result comes from **Corollary 2.4** of [5].

Lemma 3.4. *Let $\mathbb{P}$ be the set of prime numbers. For $R \geqslant 2$, let us consider the set*

$$\mathcal{W}(R) = \{p \in \mathbb{P} : \tau_p \leqslant R\}.$$

Then, $\sharp\mathcal{W}(R) \ll_{\boldsymbol{u}} \frac{R^{d+1}}{\log R}$.

This result is an analogue of **Lemma 2.5** of [2].

Lemma 3.5. *Let G be a graph with vertex set $\mathcal{V}$ and with no isolated vertices. Then, there exists $\mathcal{V}_1 \subseteq \mathcal{V}$ with $\sharp\mathcal{V}_1 \leqslant \sharp\mathcal{V}/2$ such that every vertex in $\mathcal{V}_2 = \mathcal{V}\backslash\mathcal{V}_1$ has a neighbour in $\mathcal{V}_1$.*

This is a well known graph theoretical result, and a simple proof of it is given in **Lemma 2.7** of [2].

Theorem 3.6. *Let $M_s^*(N)$ be the number of multiplicatively-dependent s-tuples among $\{u(n^2) : 1 \leqslant n \leqslant N\}$. Then,*

$$M_s^*(N) \ll N^{s(1-1/(d+1)+2((d+1)!-d)/(1+2(d+1)(d+1)!-2d(d+1))}.$$

4 Multiplicative Dependence at Quadratic Terms

4.1 Proofs

In this section, we bound $M_s^*(N)$ the number of s-tuples among the terms $u(1), u(4), \ldots, u(N^2)$ that are m.d. of maximal rank, based on the proof techniques presented by Shparlinski [5] [6] [7] and Zannier [5].

4.1.1 Proof of Theorem 3.2

Let us define $U(S;N) = \{x \in [1,N] : u(x^2) \text{ is an } S\text{-unit}\}$. We consider the product

$$W(S;N) = \prod_{u(x^2)\in U(S;N)} \overline{u(x^2)}$$

where $\overline{a} = \max(1, |a|)$. This ensures that if any element in $U(S;N)$ equals 0, then the entire product would not become 0. Now, if we let $\alpha_p(W(S;N))$ be the p-adic order, which is the exponent of the greatest power of a prime p that divides an integer, of $W(S;N)$ for some $p \in S$. Hence, we may deduce that

$$\log W(S;N) = \sum_{\nu=1}^{r} \alpha_{p_\nu}(W(S;N)) \log p_\nu$$

where $p_1, \ldots, p_r$ are the primes of S. Now, we must bound $\alpha_p(W(S;N))$ for some prime p. Here, for each p, we can see that

$$\alpha_p(W(S;N)) = \sum_{\beta=1}^{\beta_p} \sharp\{x \in [1,N] : u(x^2) \equiv 0 \mod (p^\beta)\},$$

where β_p is the largest p-adic order among all the terms of $U(S;N)$. Next, by the simple and non-degenerate property of $\mathbf{u}$, we can see that $u(x^2) \ll (|\Lambda|+1)^{x^2}$ where Λ is a dominant root of the characteristic polynomial of $\mathbf{u}$. We must first have that

$$\begin{aligned} &p^{\beta_p} \ll u(N^2) \\ \Longrightarrow\ &p^{\beta_p} \ll (|\Lambda|+1)^{N^2} \\ \Longrightarrow\ &\beta_p \log p \ll N^2 \log(|\Lambda|+1), \end{aligned}$$

and therefore $\beta_p \ll \frac{N^2}{\log p}$. Furthermore, by Theorem 2.1 of Shparlinski and Zannier's paper [5], we have that

$$\sharp\{1 \leqslant x \leqslant N : u(Q(x)) \equiv 0 \mod \mathfrak{q}\} \ll_{\mathbf{u}} \frac{N}{(\log \mathrm{Nm}\mathfrak{q})^{1/(d+2)}} + N^{d/(d+1)}$$

where $Q(x)$ is a quadratic polynomial and $\mathfrak{q}$ is an integer ideal in an algebraic number field K. In $K = \mathbb{Q}$, the norm of an integer ideal $\mathfrak{a} = (a)$ is $|a|$. Hence, we can deduce that

$$\sharp\{1 \leqslant x \leqslant N : u(x^2) \equiv 0 \mod p^\beta\} \ll_{\mathbf{u}} \frac{N}{(\beta \log p)^{1/(d+2)}} + N^{d/(d+1)}.$$

As such, we obtain

$$\begin{aligned}
\alpha_p(W(S;N)) &\ll_{\mathbf{u}} \sum_{\beta=1}^{\beta_p} \left(\frac{N}{(\beta \log p)^{1/(d+2)}} + N^{d/(d+1)} \right) \\
&\ll_{\mathbf{u}} \beta_p N^{d/(d+1)} + \frac{N}{(\log p)^{1/(d+2)}} \sum_{\beta=1}^{\beta_p} \frac{1}{\beta^{1/(d+2)}} \\
&\ll \frac{N^2 \cdot N^{d/(d+1)}}{\log p} + \frac{N}{(\log p)^{1/(d+2)}} \cdot \beta^{(d+1)/(d+2)} \\
&\ll \frac{N^{(3d+2)/(d+1)}}{\log p} + \frac{N \cdot N^{2(d+1)/(d+2)}}{(\log p)^{1/(d+2)} \cdot (\log p)^{(d+1)/(d+2)}} \\
&\ll \frac{N^{(3d+2)/(d+1)} + N^{(3d+4)/(d+2)}}{\log p},
\end{aligned}$$

with the third line of reasoning following from $\beta_p \ll \frac{N^2}{\log p}$. Now, since $(3d+2)/(d+1) = 3 - 1/(d+1)$ and $(3d+4)/(d+2) = 3 - 2/(d+2)$, we can see that the former is larger for all integers $d > 0$. Hence, we get

$$\alpha_p(W(S;N)) \ll_{\mathbf{u}} \frac{N^{(3d+2)/(d+1)}}{\log p}.$$

Therefore, we arrive at

$$\begin{aligned}
\log W(S;N) &= \sum_{\nu=1}^{r} \alpha_{p_\nu}(W(S;N)) \log p_\nu \\
&\ll_{\mathbf{u}} \sum_{\nu=1}^{r} \frac{N^{(3d+2)/(d+1)}}{\log p_\nu} \log p_\nu \\
&= \sum_{\nu=1}^{r} N^{(3d+2)/(d+1)} \\
&= rN^{(3d+2)/(d+1)}.
\end{aligned}$$

Now, we want to find a lower bound for $\log W(S;N)$ involving $\mathcal{A}(S;N)$. First, let $U^* = \{x \in [N/2, N] : u(x^2) \text{ is an } S\text{-unit}\}$, and let $\mathcal{A}^*$ be the cardinality of U^*. For sufficiently large n, we have $\overline{u(n^2)} > |\Lambda|^{c(N/2)^2}$ for some constant $c > 0$ [8], and thus

$$W(S;N) \geqslant \prod_{u(x^2) \in U^*} \overline{u(x^2)} > |\Lambda|^{c\mathcal{A}^* N^2/4},$$

hence $\log W(S;N) > c\mathcal{A}^* N^2 \log|\Lambda|/4 \gg \mathcal{A}^* N^2$. Consequently, we have

$$\begin{aligned}
\mathcal{A}^* &\ll \log W(S;N)/N^2 \\
&\ll_{\mathbf{u}} rN^{(3d+2)/(d+1)-2} \\
&= rN^{d/(d+1)}.
\end{aligned}$$

This shows that the number of S-units within $[N/2, N]$ is $O(rN^{d/(d+1)})$. If we repeat this estimate for $[N/4, N/2]$ using $W(S; N/2)$, we obtain $r(N/2)^k$ S-units within that bound, where $k = d/(d+1)$. We then recursively obtain the estimate

$$\begin{aligned}\mathcal{A}(S;N) &\ll_{\mathbf{u}} \sum_{l=0}^{\infty} r\lfloor N/2^l \rfloor^k \\ &\leqslant rN^k \sum_{i=0}^{\infty} (1/2)^{kl} \\ &\ll rN^k \\ &= rN^{d/(d+1)},\end{aligned}$$

which concludes our proof.

□

4.1.2 Proof of Theorem 3.4

For $R \geqslant 2$, define

$$Q(R) = \prod_{0<\max_{i=1..d}|k_i| \leqslant R} \mathrm{Nm}_{K/\mathbb{Q}}\Big(\prod_{i=1}^{d} \lambda_i^{k_i} - 1\Big)$$

where $\mathrm{Nm}_{K/\mathbb{Q}}$ is the norm of the splitting field K of f to $\mathbb{Q}$. By our assumption, no value of $(\prod_{i=1}^{d} \lambda_i^{k_i} - 1)$ is equal to 0, and by the property of norms, $Q(R) \neq 0$. Furthermore, as all $\lambda_1, \ldots, \lambda_d$ are algebraic integers as they are roots of the monic polynomial f, we also have that $Q(R) \in \mathbb{Z}$. For any prime $p \nmid f(0)$ and $\tau_p \leqslant R$, there exists some $k_1, \ldots, k_d$ where $0 < \max_{i=1..d} |k_i| \leqslant R$ such that $\prod_{i=1}^{d} \lambda_i^{k_i} - 1 = 0$ in $\mathbb{F}_p$, and thus $p | Q(R)$. As such,

$$\sharp\mathcal{W}(R) \leqslant \omega(Q(R)) + O(1)$$

where $\omega(k)$ is the number of prime divisors of an integer $k \geqslant 1$. It is evident that $\omega(k)! \leqslant k$, and thus we can use Stirling's formula to obtain

$$\sharp\mathcal{W}(R) \ll \frac{\log Q(R)}{\log\log Q(R)}.$$

Since $Q(R)$ has $(2R+1)^d - 1 \ll R^d$ terms and the magnitude of the logarithm of each norm can be approximated by

$$\log \mathrm{Nm}_{K/\mathbb{Q}}\Big(\prod_{i=1}^{d} \lambda_i^{k_i} - 1\Big) \ll \sum_{i=1}^{d} k_i \leqslant dR.$$

Thus, $Q(R) \ll_{\mathbf{u}} R^{d+1}$ and

$$\sharp\mathcal{W}(R) \ll_{\mathbf{u}} \frac{R^{d+1}}{\log R}.$$

□

4.2 Obtaining the Bound for $M_s^*(N)$

With these lemmas and our bound for $\mathcal{A}(S; N)$, we may proceed in obtaining a bound for $M_s^*(N)$, the number of s-tuples among $u(1), u(4), \ldots, u(N^2)$ that are m.d. of maximal rank, using an analogue of the proof techniques presented in [2]. Suppose we have an s-tuple $u(n_1^2),\dots,u(n_s^2)$ that is m.d. of maximal rank. Choose $S = \mathcal{W}(R)$ for some $R \geqslant 2$ that will be specified later. Let r be the number of S-units among $u(n_1^2),\dots,u(n_s^2)$, and $t = s - r$ be the number of non S-units. Without loss of generality, we may order the terms $u(n_1^2),\dots,u(n_t^2),u(n_{t+1}^2),\dots,u(n_s^2)$ where the first t terms are the non S-units. By **Theorem 3.2** and **Lemma 3.4**, we find that the number K_1 of r-tuples $u(n_{t+1}^2),\dots,u(n_s^2)$ that are all S-units is bounded by

$$K_1 \ll_{\mathbf{u}} \left(\frac{R^{d+1}N^{d/(d+1)}}{\log R}\right)^r.$$

Let us fix the r-tuple $u(n_{t+1}^2),\dots,u(n_s^2)$. Next, we consider the t-tuple $u(n_1^2),\dots,u(n_t^2)$ of non S-units, that is, every element has a prime divisor $p \notin S$. Construct the graph $\mathcal{G}$ with vertices $u(n_1^2),\dots,u(n_t^2)$, and join vertices $u(n_i^2)$ and $u(n_j^2)$ if $\gcd(u(n_i^2),u(n_j^2))$ is not an S-unit. Because $u(n_1^2),\dots,u(n_s^2)$ is m.d. of maximal rank, there are no isolated vertices in $\mathcal{G}$. Hence, by **Lemma 3.5**, there exists a subset $\mathcal{J}$ of $u(n_1^2),\dots,u(n_t^2)$ with

$$m = \sharp\mathcal{J} \leqslant t/2$$

such that every element in $\{u(n_1^2),\dots,u(n_t^2)\}\backslash\mathcal{J}$ has a neighbour in $\mathcal{J}$. Without loss of generality, let $\mathcal{J} = \{u(n_1^2),\dots,u(n_m^2)\}$. The number K_2 of such m-tuples is

$$K_2 \ll N^m.$$

We now fix the m-tuple $u(n_1^2),\dots,u(n_m^2)$. For $\ell = t - m$, we seek to find the number K_3 of remaining ℓ-tuples $u(n_{m+1}^2),\dots,u(n_t^2)$. Since each remaining element shares an edge with an element in $\mathcal{J}$, therefore each $u(n_j^2)$ with $m+1 \leqslant j \leqslant t$ shares some prime factor $p \notin S$ with some $u(n_i^2)$ with $1 \leqslant i \leqslant m$. Observe that $\tau_p > R$. As such, we see that each element of $u(n_1^2),\dots,u(n_m^2)$ comes from a selection of a set $\mathcal{N}$ of non S-units of cardinality

$$\sharp\mathcal{N} \ll_{\mathbf{u}} N(N^{-1/((d+1)!-d-1)} + \tau_p^{-1/((d+1)!-d)}) \leqslant N(N^{-1/((d+1)!-d-1)} + R^{-1/((d+1)!-d)}),$$

a bound obtained from **Lemma 3.3**. Thus, we get

$$K_3 \leqslant \sharp(\mathcal{N})^\ell \ll (N(N^{-1/((d+1)!-d-1)} + R^{-1/((d+1)!-d)}))^{t-m}.$$

Finally, we multiply K_1, K_2, K_3 together to obtain the bound of $M_s^*(N)$. We get

$$\begin{aligned} M_s^*(N) &\leqslant K_1K_2K_3 \\ &\ll_{\mathbf{u}} \left(\frac{R^{d+1}N^{d/(d+1)}}{\log R}\right)^r N^m(N(N^{-1/((d+1)!-d-1)} + R^{-1/((d+1)!-d)}))^{t-m} \\ &\leqslant R^{r(d+1)}N^{t+rd/(d+1)}(N^{-1/((d+1)!-d-1)} + R^{-1/((d+1)!-d)})^{t/2}. \end{aligned}$$

Choosing $R = N^\eta$ for $0 < \eta < 1$, we obtain

$$\begin{aligned} M_s^*(N) &\ll_{\mathbf{u}} N^{\eta r(d+1)}N^{t+rd/(d+1)}(N^{-1/((d+1)!-d-1)} + N^{-\eta/((d+1)!-d)})^{t/2} \\ &\ll N^{\eta r(d+1)+t+rd/(d+1)}(N^{-\eta/((d+1)!-d)})^{t/2} \\ &\ll N^{\eta r(d+1)+t+rd/(d+1)-\eta t/2((d+1)!-d)}. \end{aligned}$$

Parameterising $t = sz$ and $r = s(1-z)$ where $0 \leqslant z \leqslant 1$, we get the exponent of the final expression to be

$$\begin{aligned}&\eta s(1-z)(d+1) + sz + s(1-z)d/(d+1) - \eta sz/2((d+1)! - d)\\ =&\frac{s}{2(d+1)}(2\eta(1-z)(d+1)^2 + 2z(d+1) + 2(1-z)d - \eta z(d+1)/((d+1)! - d))\\ =&\frac{s}{2(d+1)}(2\eta(d+1)^2 - 2\eta z(d+1)^2 + 2z + 2d - \eta z(d+1)/((d+1)! - d)).\end{aligned}$$

Setting

$$\eta = \frac{2((d+1)! - d)}{(d+1) + 2(d+1)^2(d+1)! - 2d(d+1)^2},$$

we eliminate all the z terms in the final exponent, and thus our exponent simplifies to

$$\begin{aligned}\frac{s}{2(d+1)}(2\eta(d+1)^2 + 2d) &= s\eta(d+1) + \frac{sd}{d+1}\\ &= \frac{2s((d+1)! - d)}{1 + 2(d+1)(d+1)! - 2d(d+1)} + s - \frac{s}{d+1}.\end{aligned}$$

Therefore,

$$M_s^*(N) \ll_{\mathbf{u}} N^{s(1 - \frac{1}{d+1} + \frac{2((d+1)! - d)}{1+2(d+1)(d+1)! - 2d(d+1)})}.$$

□

5 Discussion

In practice, the true number of multiplicatively dependent s-tuples from a chosen set of N terms of a linear recurrence of order $d \geqslant 2$ is much lower than the bounds obtained by previous papers published in the field. This is most evident in the Lucas numbers

$$L_{n+2} = L_{n+1} + L_n, \qquad n \geqslant 0,$$

where $L_0 = 2$ and $L_1 = 1$. It has been proven that for $n \geqslant 30$, every new Lucas number introduces a new prime factor that is not a divisor of any previous Lucas number [1], and consequently this restricts the number of multiplicatively dependent s-tuples. So naturally, one should consider the possibility of reducing current bounds with newer techniques.

Additionally, we may also explore multiplicative dependence of terms at higher ordered polynomials such as cubics or quartics, as that has not been proven yet. Furthermore, a natural extension of this problem would be the multiplicative dependence at exponential terms, such as $u(1), u(2), u(4), \ldots, u(2^N)$. Note that, by **Remark 3.1**, the closed-form expression of these terms are

$$\begin{aligned}u(1) &= a_1\lambda_1 + a_2\lambda_2 + \ldots + a_d\lambda_d\\ u(2) &= a_1\lambda_1^2 + a_2\lambda_2^2 + \ldots + a_d\lambda_d^2\\ &\vdots\\ u(2^N) &= a_1\lambda_1^{2^N} + a_2\lambda_2^{2^N} + \ldots + a_d\lambda_d^{2^N},\end{aligned}$$

with $a_1, \ldots, a_d \in \mathbb{C}$ being constants and $\lambda_1, \ldots, \lambda_d$ being the roots of the characteristic polynomial. It is evident that $|u(2^n)| \ll |u(2^{n-1})|^2$, and as such this sequence grows as a double exponential function, making the distribution of the terms more sparse and consequently reducing the number of multiplicative dependent s-tuples.

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6 Appendix

6.1 Proof of Remark 3.1

Proof. First, we note that for each root λ_i of f, the sequence $u_i(n) = \lambda_i^n$ solves the recurrence

$$u_i(n+d) = c_{d-1}u_i(n+d-1) + \ldots + c_0u_i(n).$$

Since $\mathbf{u}$ depends only on its first d terms, the solution space U_f of complex sequences whose characteristic polynomial is f has a dimension of d. It suffices to show that the d solutions $\lambda_1^n, \ldots, \lambda_d^n$ form the basis of this space.

Proposition 6.1. *Let U_f be the vector space of sequences satisfying the linear recurrence relation with the fixed characteristic polynomial f of degree d. The map $\varphi : U_f \to \mathbb{C}^d$ such that*

$$\varphi(\boldsymbol{u}) = \begin{pmatrix} u(1) \\ \vdots \\ u(d) \end{pmatrix}$$

is an isomorphism.

Proof. To show that φ is an isomorphism, we must show that φ is linear, injective and surjective.

1. **Linearity.** Consider two sequences $\mathbf{u}, \mathbf{v} \in U_f$ and some scalar $\alpha \in \mathbb{C}$. We can see that

$$\begin{aligned}\varphi(\alpha\mathbf{u} + \mathbf{v}) &= \begin{pmatrix} \alpha u(1) + v(1) \\ \vdots \\ \alpha u(d) + v(d) \end{pmatrix} \\ &= \alpha \begin{pmatrix} u(1) \\ \vdots \\ u(d) \end{pmatrix} + \begin{pmatrix} v(1) \\ \vdots \\ v(d) \end{pmatrix} \\ &= \alpha\varphi(\mathbf{u}) + \varphi(\mathbf{v}).\end{aligned}$$

   As such, the linearity condition of φ is fulfilled.

2. **Injectivity.** Suppose that there exist two sequences $\mathbf{u}, \mathbf{v} \in U_f$ such that $\varphi(\mathbf{u}) = \varphi(\mathbf{v})$. Consequently, both $\mathbf{u}$ and $\mathbf{v}$ have the same beginning d terms, and since their recurrence relations have the same characteristic polynomial, the rest of the terms of $\mathbf{u}$ will be identical to those of $\mathbf{v}$, and hence $\mathbf{u} = \mathbf{v}$, thus φ must be injective.

3. **Surjectivity.** Take any vector $\vec{v} \in \mathbb{C}^d$ such that

$$\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix}.$$

   There always exists a sequence $\mathbf{u} \in U_f$ such that the first d terms of $\mathbf{u}$ are equal to the components of $\vec{v}$ as U_f contains all sequences whose recurrence has characteristic polynomial f of degree d. As such, φ is surjective.

□

Consequently, the preimage of any set of basis of $\mathbb{C}^d$ is a basis of U_f.

Proposition 6.2. *The set of d vectors*

$$\left\{ \begin{pmatrix} \lambda_1 \\ \lambda_1^2 \\ \vdots \\ \lambda_1^d \end{pmatrix}, \begin{pmatrix} \lambda_2 \\ \lambda_2^2 \\ \vdots \\ \lambda_2^d \end{pmatrix}, \dots, \begin{pmatrix} \lambda_d \\ \lambda_d^2 \\ \vdots \\ \lambda_d^d \end{pmatrix} \right\}$$

form a basis of $\mathbb{C}^d$.

Proof. We observe that

$$\text{Det} \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_d \\ \lambda_1^2 & \lambda_2^2 & \dots & \lambda_d^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^d & \lambda_2^d & \dots & \lambda_d^d \end{pmatrix} = \lambda_1 \lambda_2 \dots \lambda_d \text{Det} \begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_d \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{d-1} & \lambda_2^{d-1} & \dots & \lambda_d^{d-1} \end{pmatrix}.$$

The latter matrix is the transpose of the Vandermonde matrix[2] $V(\lambda_1, \lambda_2, \dots, \lambda_d)$, whose determinant is

$$\prod_{1 \leqslant i < j \leqslant d} (\lambda_j - \lambda_i).$$

Because f does not have any repeating roots as per our assumption, the determinant of $V(\lambda_1, \lambda_2, \dots, \lambda_d)$ is nonzero. Consequently, the aforementioned vectors must be linearly independent, and thus they form a basis for $\mathbb{C}^d$.

□

Consequently, the set of sequences $u_i(n) = \lambda_i^n$ for $i = 1 \dots d$ forms the basis of U_f, and thus the closed form solution to $u(n)$ is a linear combination of $\lambda_1^n, \dots, \lambda_d^n$.

□

[2]The Vandermonde matrix is a matrix where the elements in each row form a geometric progression.