The Hopf Fibration

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Abstract

The Hopf fibration is a particular map which shows up in many areas of mathematics including differential geometry and homotopy theory. In this report, we will make sense of the nontriviality of the Hopf map as a fibre bundle, and then investigate its higher dimensional analogues. We will also be discussing the Hopf map from an algebraic perspective, particularly how it relates to spin group actions on spheres.

Contents

1 Introduction

The Hopf fibration is a particular map $S^3 \to S^2$ which has many interesting properties which do not have clear lower dimensional analogues, and is relevant in many areas of mathematics, such as differential geometry or homotopy theory. In this report, we first discuss what it means for the Hopf fibration to be a nontrivial fibre bundle, then we will investigate how we can construct higher dimensional analogues of the Hopf map. Finally, we will shift our attention to spin groups, and we will discuss how certain spin actions on n-spheres give rise to the various Hopf maps.

1.1 Statement of authorship

All results presented in this report are already attested in the literature, although the author has proven some known results independently. Sections 2 and 3 mainly follow the work of Lyons (2003), Section 4 mainly follows Gluck, Warner, and Ziller (1985) and Section 5 mainly follows Porteous (2009), with many minor results and definitions taken from various sources, each cited appropriately.

2 Background

We begin this report by introducing some ideas and definitions which will aid us in constructing the Hopf fibration.

2.1 Fibre bundles

Definition 1 (Weisstein n.d.)**.** A function between two topological spaces is a *homeomorphism* if it is a continuous bijection and its inverse is also continuous. We say that two topological spaces X and Y are *homeomorphic* if there exists a homeomorphism from X to Y, and we denote this by writing $X \cong Y$.

Definition 2 (Totaro 2004)**.** A map $\pi : E \to B$ between topological spaces is a *fibre bundle* with *fibre* F if for every point $b \in B$, there is an open set U containing b such that $\pi^{-1}(b)$ is homeomorphic to $U \times F$. We call E the *total space* and B the *base space*. A fibre bundle can also be denoted as $F \hookrightarrow E \overset{\pi}{\twoheadrightarrow} B$ if it is necessary to specify the fibre.

This is a relatively technical definition of a fibre bundle coming from the field of topology. In this report, we will rarely be using the full definition provided above, and we will instead think of fibre bundles in a more intuitive way.

One way to think about fibre bundles intuitively is to imagine taking the base space, and replacing each point on it with a homeomorphic copy of the fibre, while 'gluing' all the fibres together continuously to form a larger space, which would be the aforementioned total space. We provide some low-dimensional examples below to illustrate this picture. Note also that with a given base space and fibre, we may be able to form more than one possible total space.

Definition 3. The n-sphere, denoted S^n, is the subset of points in $\mathbb{R}^{n+1}$ given by $\{\vec{x} \mid \|\vec{x}\| = 1, \vec{x} \in \mathbb{R}^{n+1}\}$.

Remark. Since we will often be thinking of n-spheres as topological spaces, we may also use the notation S^n to denote a space homeomorphic to the n-sphere.

Example 1. Suppose we let the base space by S^1 and the fibre be S^0. A possible total space that we can construct is $S^0 \times S^1$, with the corresponding fibre bundle being the projection map $\text{proj}_2 : S^0 \times S^1 \to S^1$ where $\text{proj}_2(x, y) = y$. We illustrate this in Figure 1 below.

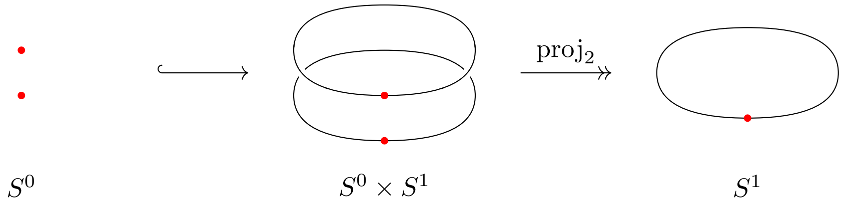


Figure 1: A trivial fibre bundle $S^0 \hookrightarrow S^0 \times S^1 \twoheadrightarrow S^1$

Definition 4 (Totaro 2004)**.** A fibre bundle $F \hookrightarrow E \overset{\pi}{\twoheadrightarrow} B$ is a *trivial* fibre bundle if $E \cong F \times B$ and $\pi : F \times B \to B$ is the projection map. A *nontrivial* fibre bundle is a fibre bundle which is not isomorphic to a trivial bundle.

The fibre bundle in Example 1 is an example of a trivial fibre bundle. This is a relatively uninteresting example of a fibre bundle as we could theoretically construct such a bundle for any fibre and base space. Instead, what we are more interested in are nontrivial fibre bundles, such as the one in Example 2 below.

Example 2 (real Hopf fibration)**.** Another total space we can form using S^0 as a fibre and S^1 as a base space is S^1, with the fibre bundle being the map $\pi : S^1 \to S^1$ defined by $\pi(z) = z^2$, where we think of the points on S^1 as unit complex numbers. We illustrate this in Figure 2 below.

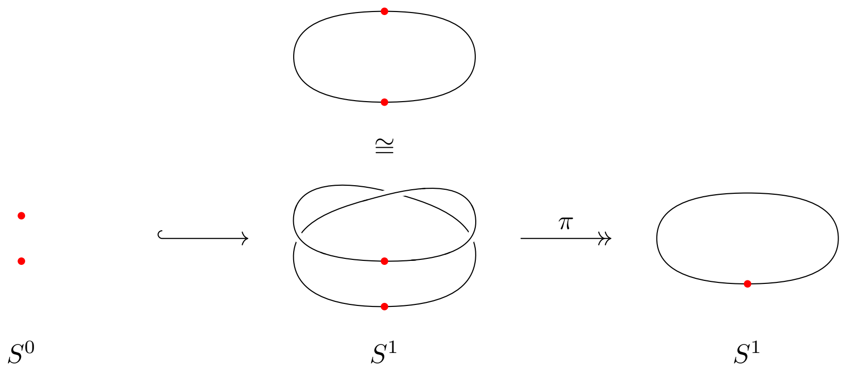


Figure 2: A nontrivial fibre bundle $S^0 \hookrightarrow S^1 \twoheadrightarrow S^1$

In this example, we see that there is a 'twist' in the total space which did not occur in Example 1. It is for this reason that it is sometimes useful to think of nontrivial fibre bundles as a kind of 'twisted' product, and the specific way in which the product is twisted determines which total space we get.

2.2 Projective spaces over fields

Definition 5 (Glasser 2005)**.** The n-dimensional *projective space* over a field K, denoted $K\mathbb{P}^n$, is the set of 1-dimensional subspaces in the vector space K^{n+1}.

When discussing projective spaces, we may want to refer to its elements. First notice that each nonzero element $v \in K^{n+1}$ is a member of a unique 1-dimensional subspace, given by the set of all vectors of the form kv where $k \in K$. As such, we can represent the elements of $K\mathbb{P}^n$ using points in K^{n+1}. However, this creates redundancy as elements in K^{n+1} which are scalar multiples of each other represent the same element in $K\mathbb{P}^n$. As such, we notate the elements of $K\mathbb{P}^n$ as a ratio of scalars $[x_1 : x_2 : \cdots : x_{n+1}]$ with at least one of the x_i's being nonzero, where $[x_1 : x_2 : \cdots : x_{n+1}] = [kx_1 : kx_2 : \cdots : kx_{n+1}]$ for any nonzero $k \in K$. This notation is known as *homogeneous coordinates*.

In the specific case where $n = 1$, we call the space $K\mathbb{P}^1$ the *projective line* over K. In this case, we have an alternative way of thinking about its elements.

Since the elements of $K\mathbb{P}^1$ are simply a ratio of two scalars $[x_1 : x_2]$, we can simply identify it with elements of K plus an additional *point at infinity* via the map $\iota : K\mathbb{P}^1 \to K \cup \{\infty\}$ defined by

$$\iota([x_1 : x_2]) = \begin{cases} x_2 x_1^{-1} & \text{if } x_1 \neq 0 \\ \infty & \text{if } x_1 = 0 \end{cases}.$$

Theorem 6. The map ι defined above is well-defined and a bijection.

Proof. To show that ι is well-defined, suppose we have two representations of the same point in $K\mathbb{P}^1$, given by $[x_1 : x_2]$ and $[kx_1 : kx_2]$, where $k \neq 0$. If $x_1 \neq 0$, then $\iota([x_1 : x_2]) = x_2 x_1^{-1} = kk^{-1} x_2 x_1^{-1} = (kx_2)(kx_1)^{-1} = \iota([kx_1 : kx_2])$. If $x_1 = 0$, then $kx_1 = 0$ and so $\iota([x_1 : x_2]) = \iota([kx_1 : kx_2]) = \infty$. To show bijectivity, define $\kappa : K \cup \{\infty\} \to K\mathbb{P}^1$ such that $\kappa(x) = \begin{cases} [1 : x] & \text{if } x \neq \infty \\ [0 : 1] & \text{if } x = \infty \end{cases}$. Then,

$$\kappa(\iota([x_1 : x_2])) = \begin{cases} \kappa(x_2 x_1^{-1}) & \text{if } x_1 \neq 0 \\ \kappa(\infty) & \text{if } x_1 = 0 \end{cases} = \begin{cases} [1 : x_2 x_1^{-1}] & \text{if } x_1 \neq 0 \\ [0 : 1] & \text{if } x_1 = 0 \end{cases} = \begin{cases} [x_1 : x_2] & \text{if } x_1 \neq 0 \\ [0 : x_2] & \text{if } x_1 = 0 \end{cases} = [x_1 : x_2]$$

and

$$\iota(\kappa(x)) = \begin{cases} \iota([1 : x]) & \text{if } x \neq \infty \\ \iota([0 : 1]) & \text{if } x = \infty \end{cases} = \begin{cases} x & \text{if } x \neq \infty \\ \infty & \text{if } x = \infty \end{cases} = x,$$

thus showing κ is a two-sided inverse of ι, meaning that ι is a bijection. □

Due to the correspondence given by ι, for the sake of convenience, we may notate elements of $K\mathbb{P}^1$ directly as if they were elements of $K \cup \{\infty\}$.

2.3 Stereographic projection

Stereographic projection is a way of identifying elements of S^n (with the exception of a single point) with elements of $\mathbb{R}^n$ (Lyons 2003). Geometrically, we can imagine selecting a fixed point N in S^n, usually taken to be the north pole, then for any other point A on S^n, we draw a line connecting N and that point. Then, where the line intersects the n-dimensional subspace passing through the equator of the sphere is where A is mapped to under stereographic projection. We illustrate this for the case with S^1 and $\mathbb{R}^1$ in Figure 3 below. Note that the stereographic projection of the point N itself is undefined.

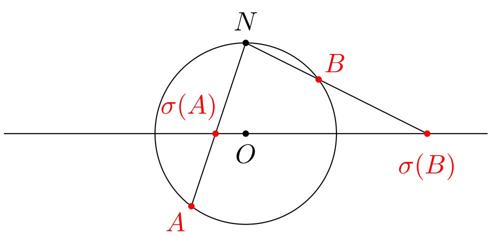


Figure 3: Stereographic projection from $S^1 \setminus \{N\}$ to $\mathbb{R}^1$

We also provide an explicit formula for stereographic projection below.

Definition 7 (Lyons 2003)**.** Stereographic projection is a map $\sigma : S^n \setminus \{N\} \to \mathbb{R}^n$, where $N = (1, 0, \ldots, 0)$, given by

$$\sigma(x_0, x_1, \ldots, x_n) = \frac{1}{1 - x_0}(x_1, \ldots, x_n)$$

Note that we have used a different coordinate convention compared to Lyons (ibid.) for convenience.

Theorem 8. The stereographic projection map is a bijection, with the inverse map being given by

$$\sigma^{-1}(X_1, \ldots, X_n) = \frac{1}{1 + R^2}(R^2 - 1, 2X_1, \ldots, 2X_n)$$

where $R^2 = \sum_{k=1}^{n} X_k^2$.

Proof. First, we show that $\sigma^{-1} \circ \sigma = \mathrm{id}_{S^n \setminus \{N\}}$.

$$\sigma^{-1}(\sigma(x_0, x_1, \ldots, x_n)) = \sigma^{-1}\left(\frac{x_1}{1 - x_0}, \ldots, \frac{x_n}{1 - x_0}\right)$$

In this case, since $x_0 \neq 1$ for any point in $S^n \setminus \{N\}$, we have the following. Here, we use the fact that $\sum_{k=0}^{n} x_k^2 = 1$ due to $(x_0, x_1, \ldots, x_n)$ being a point on S^n.

$$\begin{aligned} 1 + R^2 &= 1 + \frac{1}{(1 - x_0)^2}\sum_{k=1}^{n} x_k^2 = 1 + \frac{1 - x_0^2}{(1 - x_0)^2} = \frac{2}{1 - x_0} \\ R^2 - 1 &= (-1) + \frac{1}{(1 - x_0)^2}\sum_{k=1}^{n} x_k^2 = \frac{1 - x_0^2}{(1 - x_0)^2} - 1 = \frac{2x_0}{1 - x_0} \end{aligned}$$

As such,

$$\sigma^{-1}\left(\frac{x_1}{1 - x_0}, \ldots, \frac{x_n}{1 - x_0}\right) = \frac{1 - x_0}{2}\left(\frac{2x_0}{1 - x_0}, \frac{2x_1}{1 - x_0}, \ldots, \frac{2x_n}{1 - x_0}\right) = (x_0, x_1, \ldots, x_n).$$

Next, we show that $\sigma \circ \sigma^{-1} = \mathrm{id}_{\mathbb{R}^n}$.

$$\begin{aligned} \sigma(\sigma^{-1}(X_1, \ldots, X_n)) &= \sigma\left(\frac{R^2 - 1}{1 + R^2}, \frac{2X_1}{1 + R^2}, \ldots, \frac{2X_n}{1 + R^2}\right) = \left(1 - \frac{R^2 - 1}{1 + R^2}\right)^{-1}\left(\frac{2X_1}{1 + R^2}, \ldots, \frac{2X_n}{1 + R^2}\right) \\ &= \frac{1 + R^2}{2}\left(\frac{2X_1}{1 + R^2}, \ldots, \frac{2X_n}{1 + R^2}\right) = (X_1, \ldots, X_n) \end{aligned}$$

□

Theorem 9. Stereographic projection maps circles in S^n to either circles or lines in $\mathbb{R}^n$.

Proof. See Theorem 24.1 of Trettel (n.d.). □

Theorem 10. $S^1 \cong \mathbb{RP}^1$ and $S^2 \cong \mathbb{CP}^1$.

Proof. Since $\mathbb{RP}^1 \cong \mathbb{R} \cup \{\infty\}$ and $\mathbb{CP}^1 \cong \mathbb{C} \cup \{\infty\} \cong \mathbb{R}^2 \cup \{\infty\}$, by using stereographic projection on $S^1 \setminus \{N\}$ and $S^2 \setminus \{N\}$ while additionally mapping N to the point at infinity in each case, we obtain a homeomorphism between the aforementioned spaces. □

Remark. The astute reader may have noticed that we have not properly defined a topology on $\mathbb{RP}^1$ or $\mathbb{CP}^1$. For our purposes, we are considering the topology induced by the one-point compactification on $\mathbb{R}$ and $\mathbb{R}^2$ respectively. See Khatchatourian (2018) for more information; however, this is not a focus of this report.

3 The Hopf fibration

3.1 Defining the Hopf fibration

Definition 11 (Santos 2024)**.** The *complex Hopf map*, or just the *Hopf map*, is a specific map $h_{\mathbb{C}} : S^3 \to \mathbb{CP}^1$. Since $S^3 \subseteq \mathbb{R}^4 \cong \mathbb{C}^2$, we can write the elements of S^3 as pairs of complex numbers (z_1, z_2) which satisfy the condition $|z_1|^2 + |z_2|^2 = 1$. Then, for each $(z_1, z_2) \in S^3$, we define $h_{\mathbb{C}}(z_1, z_2) = [z_1 : z_2]$.

Theorem 12 (Hopf fibration)**.** There exists a fibre bundle of the form $S^1 \hookrightarrow S^3 \twoheadrightarrow S^2$.

Proof. We first show that the complex Hopf map induces a fibre bundle of the form $S^1 \hookrightarrow S^3 \overset{h_{\mathbb{C}}}{\twoheadrightarrow} \mathbb{CP}^1$. We do this by showing that the preimage of every point under $h_{\mathbb{C}}$ is homeomorphic to S^1. For any given point in $\mathbb{CP}^1$, we can represent it as $[z_1 : z_2]$ where we assume that the homogeneous coordinates are normalised, that is, we suppose that $|z_1|^2 + |z_2|^2 = 1$ without loss of generality. Now, we suppose that $h(a, b) = [z_1 : z_2]$. This would mean $[a : b] = [z_1 : z_2]$, and so there exists a nonzero $z \in \mathbb{C}$ such that $a = zz_1$ and $b = zz_2$. Moreover, since $(a, b) \in S^3$, we must have $|zz_1|^2 + |zz_2|^2 = 1$, which implies that $|z|^2(|z_1|^2 + |z_2|^2) = 1$ and so $|z| = 1$. Thus, the preimage of a given point in $\mathbb{CP}^1$ under $h_{\mathbb{C}}$ is always of the form $\{(zz_1, zz_2) \mid z \in \mathbb{C}, |z| = 1\}$ for a fixed $(z_1, z_2) \in S^3$. Then, by mapping each point $(zz_1, zz_2) \in h^{-1}([z_1, z_2])$ to $z \in S^1$, we show that each fibre is homeomorphic to S^1.

Finally, since stereographic projection (with N mapping to ∞) gives a homeomorphism $\sigma_* : S^2 \to \mathbb{CP}^1$ by Theorem 10, we see that the map $\sigma_*^{-1} \circ h_{\mathbb{C}} : S^3 \to S^2$ is also a fibre bundle with fibres homeomorphic to S^1. □

Remark. We can also define a *real Hopf map* $h_{\mathbb{R}} : S^1 \to \mathbb{RP}^1$ analogously to Definition 11. When composed with an inverse stereographic projection, this induces a fibre bundle $S^0 \hookrightarrow S^1 \twoheadrightarrow S^1$, which is the real Hopf fibration in Example 2.

Theorem 13. The fibres of the complex Hopf map are *great circles* of S^3. That is, each of them is the intersection of S^3 and a 2-dimensional subspace of $\mathbb{R}^4$.

Proof. Each point in $\mathbb{CP}^1$ can be written in the form $[1 : z]$ for some nonzero $z \in \mathbb{C}$ or as $[0 : 1]$. We consider the two cases separately. Note that we will be representing points as if they were in $\mathbb{C}^2$ or $\mathbb{R}^4$ interchangeably.

For points of the form $[1 : z]$, where $z = a + bi$ and $a, b \in \mathbb{R}$,

$$\begin{aligned} h_{\mathbb{C}}^{-1}([1 : z]) &= \left\{ \frac{1}{\sqrt{1 + |z|^2}}(1, a + bi)e^{i\theta} \,\middle|\, \theta \in [0, 2\pi) \right\} \\ &= \left\{ \frac{1}{\sqrt{1 + |z|^2}}(\cos\theta, \sin\theta, a\cos\theta - b\sin\theta, b\cos\theta + a\sin\theta) \,\middle|\, \theta \in [0, 2\pi) \right\} \\ &\subseteq \operatorname{span}\left\{ \frac{1}{\sqrt{1 + |z|^2}}(1, 0, a, b), \frac{1}{\sqrt{1 + |z|^2}}(0, 1, -b, a) \right\}. \end{aligned}$$

For $[0 : 1]$,

$$h_{\mathbb{C}}^{-1}([0 : 1]) = \left\{(0, 1)e^{i\theta} \,\middle|\, \theta \in [0, 2\pi)\right\} = \{(0, 0, \cos\theta, \sin\theta) \mid \theta \in [0, 2\pi)\} \subseteq \operatorname{span}\{(0, 0, 1, 0), (0, 0, 0, 1)\}.$$

In either case, each fibre is contained within a 2-dimensional subspace of $\mathbb{R}^4$. Moreover, it can be easily verified that the unit vectors in each of those subspaces are exactly the fibres, by observing that all the unit vectors in a 2-dimensional subspace are given by $(\cos\theta)\vec{e}_1 + (\sin\theta)\vec{e}_2$, where $\{\vec{e}_1, \vec{e}_2\}$ is an orthonormal basis (which each of the spanning sets given above is). Thus, each fibre is the intersection of S^3 and its corresponding 2-dimensional subspace given above. □

3.2 Nontriviality of the Hopf fibration

One of the things that make the Hopf fibration notable is that it is a nontrival fibre bundle, however, the fact that the 'twisting' occurs in higher than 3-dimensions makes it difficult to visualise. In this section, we will discuss some special properties of the Hopf fibration which arise from its nontriviality.

It turns out that any pair of fibres of the Hopf fibration are 'linked' in S^3 (Lyons 2003). We will not be providing a rigorous definition of linking here, and will instead be providing a geometric argument for why this is true. However, the curious reader may be interested in looking into the idea of 'ambient isotopy' for a more rigorous notion of linking. Note that when discussing linking, it is important to keep in mind the ambient space in which the linked objects are embedded in. For example, a copy of S^0 and S^1 can be linked in $\mathbb{R}^2$ as shown in Figure 4, but not in $\mathbb{R}^3$ as unlinking the objects necessarily involves moving points out of the plane.

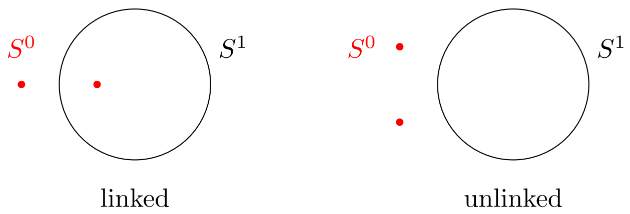


Figure 4: Linked and unlinked configurations of S^0 and S^1

Lemma 14. Any fibre of the Hopf fibration is linked with the fibre $h_{\mathbb{C}}^{-1}([0:1])$.

Proof. Here is the outline of the proof. First, we use stereographic projection σ to project all of S^3 except $N = (1,0,0,0)$ onto $\mathbb{R}^3$. By Theorem 2.3, all the fibres stay circles after stereographic projection, with the exception of the fibre $h_{\mathbb{C}}^{-1}([1:0])$ containing N, which is mapped to a line. Since this is a homeomorphism of the ambient spaces, this will not change the linkedness of any fibres (ibid.). We then show that each fibre (except $h_{\mathbb{C}}^{-1}([1:0])$) intersects the plane spanned by $(0,1,0)$ and $(0,0,1)$ in which $h_{\mathbb{C}}^{-1}([0:1])$ is contained (we call this plane E) at exactly two points, one inside $h_{\mathbb{C}}^{-1}([0:1])$ and one outside $h_{\mathbb{C}}^{-1}([0:1])$, where the dark blue points are located in Figure 5. Since the fibres are circles, this is sufficient to show that the fibre is linked with $h_{\mathbb{C}}^{-1}([0:1])$. For $h_{\mathbb{C}}^{-1}([1:0])$, we simply need to show that it intersects the same plane at one point inside $h_{\mathbb{C}}^{-1}([0:1])$, where the cyan point is in Figure 5.

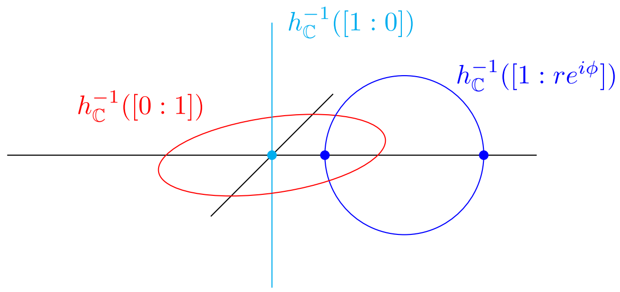


Figure 5: Linking of stereographically projected Hopf fibres in $\mathbb{R}^3$

First, observe that every point on $\mathbb{CP}^1$ can be written as $[0:1]$ or in the form $[1:re^{i\phi}]$. The stereographic projection of $h_{\mathbb{C}}^{-1}([0:1])$ is as follows. For convenience, in this proof and in Theorem 3.2, whenever we write down a coordinate containing the variable θ, we really mean the set of all points of that form where $\theta \in [0, 2\pi)$.

$$\sigma(h_{\mathbb{C}}^{-1}([0:1])) = \sigma(0, e^{i\theta}) = \sigma(0,0,\cos\theta,\sin\theta) = (0,\cos\theta,\sin\theta)$$

We first show that the fibre $h_{\mathbb{C}}^{-1}([1 : re^{i\phi}])$ is linked with $h_{\mathbb{C}}^{-1}([0 : 1])$ whenever $r \neq 0$. First, stereographically projecting the fibre gives

$$\begin{aligned}\sigma(h_{\mathbb{C}}^{-1}([1 : re^{i\phi}])) &= \sigma\left(e^{i\theta}\left(\frac{1}{\sqrt{1+r^2}}, \frac{re^{i\phi}}{\sqrt{1+r^2}}\right)\right) = \sigma\left(\frac{\cos\theta}{\sqrt{1+r^2}}, \frac{\sin\theta}{\sqrt{1+r^2}}, \frac{r\cos(\theta+\phi)}{\sqrt{1+r^2}}, \frac{r\sin(\theta+\phi)}{\sqrt{1+r^2}}\right)\\ &= \left(\frac{\frac{\sin\theta}{\sqrt{1+r^2}}}{1-\frac{\cos\theta}{\sqrt{1+r^2}}}, \frac{\frac{r\cos(\theta+\phi)}{\sqrt{1+r^2}}}{1-\frac{\cos\theta}{\sqrt{1+r^2}}}, \frac{\frac{r\sin(\theta+\phi)}{\sqrt{1+r^2}}}{1-\frac{\cos\theta}{\sqrt{1+r^2}}}\right)\\ &= \left(\frac{\sin\theta}{\sqrt{1+r^2}-\cos\theta}, \frac{r\cos(\theta+\phi)}{\sqrt{1+r^2}-\cos\theta}, \frac{r\sin(\theta+\phi)}{\sqrt{1+r^2}-\cos\theta}\right).\end{aligned}$$

Then, to find where the fibre intersects E, we set the first coordinate to 0. Note that since $r \neq 0$, this means $\sqrt{1+r^2} > 1$ and so $\sqrt{1+r^2} - \cos\theta > 1 - 1 = 0$. Thus,

$$\frac{\sin\theta}{\sqrt{1+r^2}-\cos\theta} = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0 \text{ or } \pi$$

and so the points of intersection occur at

$$\vec{u}_1 = \left(0, \frac{r\cos\phi}{\sqrt{1+r^2}-1}, \frac{r\sin\phi}{\sqrt{1+r^2}-1}\right) \text{ and } \vec{u}_2 = \left(0, \frac{-r\cos\phi}{\sqrt{1+r^2}+1}, \frac{-r\sin\phi}{\sqrt{1+r^2}+1}\right).$$

Since $h_{\mathbb{C}}^{-1}([0 : 1])$ is just a unit circle centred at the origin, we simply have to verify that one of the intersection points is more than 1 unit from the origin and the other is less than 1 unit from the origin, as shown below.

$$\begin{aligned}\|\vec{u}_2\| &= \sqrt{\frac{r^2\cos^2\phi + r^2\sin^2\phi}{(\sqrt{1+r^2}+1)^2}} = \frac{r}{\sqrt{1+r^2}+1} < \frac{r}{\sqrt{r^2}} = 1\\ \|\vec{u}_1\| &= \sqrt{\frac{r^2\cos^2\phi + r^2\sin^2\phi}{(\sqrt{1+r^2}-1)^2}} = \frac{r}{\sqrt{1+r^2}-1} = \frac{r(\sqrt{1+r^2}+1)}{(\sqrt{1+r^2}-1)(\sqrt{1+r^2}+1)} = \frac{r(\sqrt{1+r^2}+1)}{r^2}\\ &= \frac{\sqrt{1+r^2}+1}{r} = \frac{1}{\|\vec{u}_2\|} > 1\end{aligned}$$

Finally, for the case when $r = 0$,

$$\sigma(h_{\mathbb{C}}^{-1}([1 : 0])) = \sigma(\cos\theta, \sin\theta, 0, 0) = \left(\frac{\sin\theta}{1-\cos\theta}, 0, 0\right)$$

which only intersects E at the origin when $\theta = \frac{\pi}{2}$, as required. □

Theorem 15. Any two fibres of the Hopf fibration are linked.

Proof. For any $\vec{v} = (z_1, z_2) \in S^3$, we can map it to $(1, 0) \in S^3$ via the unitary matrix $A = \begin{bmatrix} z_1^* & z_2^* \\ -z_2 & z_1 \end{bmatrix}$ as

$$\begin{bmatrix} z_1^* & z_2^* \\ -z_2 & z_1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} z_1^* z_1 + z_2^* z_2 \\ -z_2 z_1 + z_1 z_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Notice that A always maps Hopf fibres to Hopf fibres. Since Hopf fibres in S^3 are of the form $e^{i\theta}\vec{v}$ where $\vec{v}^{\mathsf{H}}\vec{v} = 1$, where $\vec{v}^{\mathsf{H}}$ denotes the conjugate transpose of $\vec{v}$, this means by linearity, $A(e^{i\theta}\vec{v}) = e^{i\theta}(A\vec{v})$. This is once again a Hopf fibre as $(A\vec{v})^{\mathsf{H}}(A\vec{v}) = \vec{v}^{\mathsf{H}}A^{\mathsf{H}}A\vec{v} = \vec{v}^{\mathsf{H}}\vec{v} = 1$, using the fact that A is unitary.

Finally, whenever we have two Hopf fibres, we can apply a unitary matrix to S^3 so that one of the fibres is mapped to $h_{\mathbb{C}}^{-1}([1 : 0])$. Then by invoking Lemma 14, along with the fact A is always a homeomorphism (unitary matrices are invertible) and thus preserves linking, we can show that any two Hopf fibres are linked. □

Theorem 16. The 3-sphere with a single Hopf fibre removed $S^3 \setminus h_{\mathbb{C}}^{-1}(N)$ is homeomorphic to $\mathbb{C} \times S^1$.

Proof. Since $S^2 \setminus \{N\} \cong \mathbb{C}$ by stereographic projection, if the theorem statement is true, there should exists a homeomorphism φ such that the following diagram commutes.

$$\begin{array}{ccc} S^3 \setminus h^{-1}(N) & \xrightarrow{\ \varphi\ } & \mathbb{C} \times S^1 \\ {\scriptstyle \sigma^{-1}\circ h_{\mathbb{C}}}\downarrow & & \downarrow{\scriptstyle \mathrm{proj}_1} \\ S^2 \setminus \{N\} & \xrightarrow[\ \sigma\ ]{} & \mathbb{C} \end{array}$$

We now explicitly construct φ. Notice that the first component of $\varphi(z_1, z_2)$ is simply given by $h_{\mathbb{C}}$, and we know that $h_{\mathbb{C}}(z_1, z_2) = [z_1 : z_2] \in \mathbb{CP}^1 \setminus \{[1 : 0]\} \cong \mathbb{R}^2$. Since the preimage of each point under the Hopf map is a circle, to make sure φ is a homeomorphism, we need to include the lost information about the location of the point on the fibre from which we have mapped, as the second component of the φ map. We can encode this information by using the phase of z_2. (The choice of including the phase of z_2 instead of z_1 is arbitrary. This is due to the fact that the relative phase of z_1 and z_2 is already included in the first component, so including the phase of either z_1 or z_2 is sufficient.) Thus, we define $\varphi(z_1, z_2) = \left(\frac{z_1}{z_2}, \frac{z_2}{|z_2|}\right)$.

We show that φ is a homeomorphism by showing that is has a continuous inverse given by $\varphi^{-1}(z, \lambda) = \left(\frac{\lambda z}{\sqrt{1+|z|^2}}, \frac{\lambda}{\sqrt{1+|z|^2}}\right)$ where $|\lambda| = 1$.

$$\varphi(\varphi^{-1}(z,\lambda)) = \varphi\left(\frac{\lambda z}{\sqrt{1+|z|^2}}, \frac{\lambda}{\sqrt{1+|z|^2}}\right) = \left(\frac{\lambda z}{\sqrt{1+|z|^2}}\frac{\sqrt{1+|z|^2}}{\lambda}, \frac{\lambda}{\sqrt{1+|z|^2}}\sqrt{1+|z|^2}\right) = (z, \lambda)$$

$$\begin{aligned}\varphi^{-1}(\varphi(z_1, z_2)) &= \varphi^{-1}\left(\frac{z_1}{z_2}, \frac{z_2}{|z_2|}\right) = \left(\frac{z_2}{|z_2|}\frac{z_1}{z_2}\sqrt{\frac{|z_2|^2}{|z_2|^2+|z_1|^2}}, \frac{z_1}{z_2}\sqrt{\frac{|z_2|^2}{|z_2|^2+|z_1|^2}}\right) = \left(\frac{z_2}{|z_2|}\frac{z_1}{z_2}|z_2|, \frac{z_2}{|z_2|}|z_2|\right) \\ &= (z_1, z_2)\end{aligned}$$

Note that since N is not included in the domain, $z_2 \neq 0$ and so both φ and φ^{-1} are continuous. □

The reason why we care about the result in Theorem 16 is that it implies that the Hopf fibration is 'locally trivial', in the sense that the only thing making it nontrivial is its global structure. Specifically, if we consider the Hopf map but with one single fibre removed, then the whole fibration becomes equivalent to the trivial fibration $S^1 \hookrightarrow \mathbb{C} \times S^1 \twoheadrightarrow \mathbb{C}$, as shown in the commutative diagram below.

$$\begin{array}{ccccc} S^1 & \hookrightarrow & S^3 \setminus h^{-1}(N) & \xrightarrow{\ \sigma^{-1}\circ h_{\mathbb{C}}\ } & S^2 \setminus \{N\} \\ & \searrow & {\scriptstyle \varphi^{-1}}\uparrow\downarrow{\scriptstyle \varphi} & & {\scriptstyle \sigma^{-1}}\uparrow\downarrow{\scriptstyle \sigma} \\ & & \mathbb{C} \times S^1 & \xrightarrow{\ \mathrm{proj}_1\ } & \mathbb{C} \end{array}$$

4 Generalisations of the Hopf fibration

In this section, we will be investigating certain higher dimensional analogues of the Hopf fibration.

4.1 The Cayley-Dickson algebras

The Cayley-Dickson algebras are a family of algebras, the first four of which have strong connections to the higher Hopf maps.

Definition 17 (nLab authors 2026b)**.** An *algebra* over a field K, or a K-algebra, is a K-vector space A equipped with a bilinear binary operation $\cdot : A \times A \to A$, which can be thought of as vector multiplication. An $\mathbb{R}$-*-algebra is an $\mathbb{R}$-algebra A equipped with an $\mathbb{R}$-linear map $^* : A \to A$, called the *anti-involution*, where for any $x, y \in A$, the following properties are satisfied.

$$(x^*)^* = x \qquad (xy)^* = y^*x^* \qquad 1^* = 1$$

Remark. *-algebras can be defined over arbitrary fields, but this is not necessary for our purposes.

Definition 18 (Cayley-Dickson construction (nLab authors 2026a))**.** The *Cayley-Dickson double* of an $\mathbb{R}$-*-algebra A is an $\mathbb{R}$-*-algebra $\mathrm{CD}(A)$ where the underlying vector space is $A \oplus A$ where vector multiplication is given by $(a, b) \cdot (c, d) = (ac - db^*, a^*d + cb)$ and the anti-involution is given by $(a, b)^* = (a^*, -b)$.

Example 3. First, by considering $\mathbb{R}$ as an $\mathbb{R}$-*-algebra with anti-involution given by the identity map, we can construct the Cayley-Dickson double, which gives $\mathrm{CD}(\mathbb{R}) \cong \mathbb{C}$, where the anti-involution corresponds to taking the complex conjugate.

By continuing this process, we define the *quaternions* to be $\mathbb{H} = \mathrm{CD}(\mathbb{C})$ and the *octonions* to be $\mathbb{O} = \mathrm{CD}(\mathbb{H})$. In each step, the dimension doubles when viewed as an algebra over $\mathbb{R}$. We will omit the proof here, but it turns out that unlike $\mathbb{R}$ and $\mathbb{C}$, $\mathbb{H}$ is not commutative, and $\mathbb{O}$ is neither commutative nor associative. $\mathbb{O}$ is, however, alternative, which means that $(xx)y = x(xy)$ and $(xy)y = x(yy)$ for any $x, y \in \mathbb{O}$.

A norm can also be defined in the algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, where $|x| = \sqrt{x^*x}$, which is why these are known as the *real normed division algebras*. Note that this norm agrees with the Euclidean norm in $\mathbb{R}^n$ where $n = 1, 2, 4, 8$ respectively. Inverses can also be taken in these algebras, with $x^{-1} = \frac{x^*}{|x|^2}$. Taking further Cayley-Dickson doubles yields algebras which contain zero divisors, causing them to no longer be division algebras.

4.2 Higher Hopf maps between spheres

It turns out that there two additional fibrations between n-spheres in higher dimensions, which are $S^3 \hookrightarrow S^7 \twoheadrightarrow S^4$ and $S^7 \hookrightarrow S^{15} \twoheadrightarrow S^8$, called the quaternionic and octonionic Hopf fibrations respectively. We can define them similarly to the real and complex Hopf fibrations. But first, we have to properly define the quaternionic and octonionic projective spaces. Since $\mathbb{H}$ and $\mathbb{O}$ are not fields, Definition 5 does not apply here.

The quaternions are a *skew-field*, which means it is almost a field except multiplication is not necessarily commutative. Thus, the definition of quaternionic projective spaces closely matches Definition 5, except we have to pay attention to the direction of multiplication.

Definition 19 (Gluck, Warner, and Ziller 1985)**.** The n-dimensional *quaternionic projective space* $\mathbb{HP}^n$ is the set of all quaternionic lines in $\mathbb{H}^{n+1} \setminus \{0\}$. We define a *quaternionic line* $[q_1 : q_2 : \cdots : q_{n+1}]$, where at least one $q_i \neq 0$, to be the subset $\{(qq_1, qq_2, \ldots, qq_{n+1}) \mid q \in \mathbb{H}\}$ in $\mathbb{H}^{n+1}$.

Remark. Note that $[q_1 : q_2 : \cdots : q_{n+1}]$ is equal to $[qq_1 : qq_2 : \cdots : qq_{n+1}]$ for any nonzero $q \in \mathbb{H}$, but is not necessarily equal to $[q_1q : q_2q : \cdots : q_{n+1}q]$ due to the noncommutativity of quaternionic multiplication.

Definition 20 (ibid.)**.** The *octonionic projective line* $\mathbb{OP}^1$ is the set of all octonionic lines in $\mathbb{O}^2$. We define an *octonionic line* to be one of the following subsets of $\mathbb{O}^2$.

- For each $m \in \mathbb{O}$, define $L_m = \{(u, mu) \mid u \in \mathbb{O}\}$.
- $L_\infty = \{(0, u) \mid u \in \mathbb{O}\}$.

Remark. Note that octonionic projective spaces cannot be defined in general for any dimension. In particular, $\mathbb{OP}^2$ exists but is defined much differently than $\mathbb{OP}^1$, and $\mathbb{OP}^n$ does not exist for $n \geq 3$. See Lackmann (2019) for more information. It is also not possible to define projective spaces for further Cayley-Dickson doubles beyond the octonions due to the presence of zero divisors.

It turns out that the octonionic lines in Definition 20 are not closed under octonionic multiplication due to the nonassociativity of the octonions. Thus, we have to make sure we are able to properly define the octonionic Hopf map later on, and so we require the result in Theorem 21.

Theorem 21. Every nonzero point in $\mathbb{O}^2$ is on a unique octonionic line.

Proof. This proof has been adapted from Gluck, Warner, and Ziller (1985).

We first show that every nonzero point in $\mathbb{O}^2$ is on at least one octonionic line. According to Réka (2022), it can be shown that $(vu)u^* = |u|^2 v$ is satisfied for any $u, v \in \mathbb{O}$. Thus, for each point $(u, v) \in \mathbb{O}^2$, if $u \neq 0$, we see that

$$(u, v) = \left(u, \frac{|u^*|^2}{|u^*|^2} v\right) = \left(u, \frac{1}{|u^*|^2}(vu^*)u\right) = (u, (vu^{-1})u) \in L_{vu^{-1}}.$$

Otherwise, if $u = 0$, then $(u, v) \in L_\infty$.

Next, we show that any two distinct octonionic lines are disjoint except at $(0, 0)$. We can see that L_∞ intersects any other octonionic line $L_m = \{(u, mu) \mid u \in \mathbb{O}\}$ only when $u = 0$, which corresponds to the origin. For any two distinct octonionic lines L_m and L_n where neither are L_∞, suppose that they intersect at a nonzero point (u, v), then $v = mu = nu$ implies that $m = n$ which contradicts our premise. □

Now, we can define the higher Hopf maps analogously to the real and complex Hopf maps.

Definition 22 (Gluck, Warner, and Ziller 1985)**.** The *quaternionic Hopf map* is a map $h_{\mathbb{H}} : S^7 \to \mathbb{HP}^1$. Since $S^7 \subseteq \mathbb{R}^8 \cong \mathbb{H}^2$, we can write the elements of S^7 as (q_1, q_2) where $q_1, q_2 \in \mathbb{H}$ and $|q_1|^2 + |q_2|^2 = 1$. Then we define $h_{\mathbb{H}}(q_1, q_2) = [q_1 : q_2]$.

Theorem 23. The fibres of $h_{\mathbb{H}}$ are homeomorphic to S^3.

Proof. We can write any point on $\mathbb{HP}^1$ in the form $[q_1 : q_2]$ where $|q_1|^2 + |q_2|^2 = 1$ without loss of generality. Then, it is precisely the points of the form $(\lambda q_1, \lambda q_2) \in \mathbb{H}^2$ which are mapped to $[q_1 : q_2]$. Since the domain of $h_{\mathbb{H}}$ is S^7, we have $|\lambda q_1|^2 + |\lambda q_2|^2 = 1$ which implies $|\lambda| = 1$. Since the quaternionic norm agrees with the Euclidean norm in $\mathbb{R}^4$, λ can be thought of as a point on S^3. Thus, the map $(\lambda q_1, \lambda q_2) \mapsto \lambda$ is a homeomorphism $h_{\mathbb{H}}^{-1}([q_1 : q_2]) \to S^3$. □

Definition 24 (ibid.)**.** The *octonionic Hopf map* is a map $h_{\mathbb{H}} : S^{15} \to \mathbb{OP}^1$. Since $S^{15} \subseteq \mathbb{R}^{16} \cong \mathbb{O}^2$, we can write the elements of S^{15} as (o_1, o_2) where $o_1, o_2 \in \mathbb{O}$ and $|o_1|^2 + |o_2|^2 = 1$. Then we define $h_{\mathbb{O}}(o_1, o_2) = \begin{cases} L_{o_2 o_1^{-1}} & \text{if } o_1 \neq 0 \\ L_\infty & \text{if } o_1 = 0 \end{cases}$. This is a well-defined map as each pair of octonions on S^{15} (which are thus nonzero) lies on a unique octonionic line, as per Theorem 21.

Theorem 25. The fibres of $h_{\mathbb{O}}$ are homeomorphic to S^7.

Proof. The preimage of any $L_m \in \mathbb{OP}^1$ is precisely the set of points on L_m in $\mathbb{O}^2 \cong \mathbb{R}^{16}$ which are also on S^{15} (i.e. they are a unit distance from the origin). Since each L_m is an 8-dimensional subspace of $\mathbb{R}^{16}$, the preimage of L_m is homeomorphic to a copy of S^7. □

Similarly to the argument given in Theorem 10, we can say that $S^4 \cong \mathbb{HP}^1$ and $S^8 \cong \mathbb{OP}^1$ via stereographic projection. Thus, given this fact along with Theorem 23 and 25, we have shown the existence of fibre bundles $S^3 \hookrightarrow S^7 \twoheadrightarrow S^4$ and $S^7 \hookrightarrow S^{15} \twoheadrightarrow S^8$. Since further Cayley-Dickson algebras do not have corresponding projective spaces, we cannot construct further Hopf fibrations using this process. In fact, these, in addition to the aforementioned real and complex Hopf fibrations, are the only possible fibrations between spheres, and this fact can be deduced from Adams's theorem (Price 2010).

4.3 Hopf maps onto projective spaces

So far, we have looked at Hopf fibrations which have an n-sphere as the base space. However, in all the previous examples, what actually happened is that we have constructed Hopf maps which map to projective lines, and used a stereographic projection to identify it with a sphere. Here, the stereographic projection is not an essential part of the fibration, and only serves to turn the base space into an n-sphere. If we ignore the stereographic projection, we are now able to construct related fibrations out of higher dimensional projective spaces, which are not homeomorphic to n-spheres.

In particular, we can construct the following family of fibrations for any $n \geq 0$.

$$S^0 \hookrightarrow S^k \twoheadrightarrow \mathbb{RP}^k$$
$$S^1 \hookrightarrow S^{2k+1} \twoheadrightarrow \mathbb{CP}^k$$
$$S^3 \hookrightarrow S^{4k+3} \twoheadrightarrow \mathbb{HP}^k$$

In each case, the fibre bundle is given by writing the points in the respective n-spheres in the total space as a $(k+1)$-tuple $(x_1, x_2, \ldots, x_{k+1})$ where each $x_i \in \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, then mapping each point to the point $[x_1 : x_2 : \cdots : x_{k+1}]$ in $\mathbb{RP}^k$, $\mathbb{CP}^k$ or $\mathbb{HP}^k$ respectively. It is then easy to see that the fibres are S^0, S^1 and S^3 as they are each homeomorphic to the set of unit real numbers, unit complex numbers and unit quaternions respectively. Note that due to the unusual structure of $\mathbb{OP}^2$ and the nonexistence of $\mathbb{OP}^n$ for $n \geq 3$, the only Hopf fibration with fibre S^7 is the aforementioned one given by $S^7 \hookrightarrow S^{15} \twoheadrightarrow \mathbb{OP}^1 \cong S^8$.

5 Spin actions on spheres

In this section, we will investigate how Hopf maps arise from the actions of spin groups on n-spheres.

5.1 Clifford algebras and spin groups

We begin with defining certain relevant groups.

Definition 26 (Baez 2014)**.** The *orthogonal group* $\mathrm{O}(n)$ is the group of $n \times n$ real orthogonal matrices with matrix multiplication as the group operation. That means for all $A \in \mathrm{O}(n)$, $AA^\mathsf{T} = A^\mathsf{T}A = I_n$, where A^T denotes the transpose of A. The *special orthogonal group* $\mathrm{SO}(n)$ is the subgroup of $\mathrm{O}(n)$ containing only the elements with determinant 1. $\mathrm{SO}(n)$ can be thought of as the group of rotations in $\mathbb{R}^n$.

Definition 27 (ibid.)**.** The *unitary group* $\mathrm{U}(n)$ is the group of $n \times n$ complex unitary matrices with matrix multiplication as the group operation. That means for all $A \in \mathrm{U}(n)$, $AA^\mathsf{H} = A^\mathsf{H}A = I_n$, where A^H denotes the conjugate transpose of A. The *special unitary group* $\mathrm{SU}(n)$ is the subgroup of $\mathrm{U}(n)$ containing only the elements with determinant 1.

Definition 28 (ibid.)**.** The *symplectic group* $\mathrm{Sp}(n)$ is the group of $n \times n$ quaternionic symplectic matrices with matrix multiplication as the group operation. That means for all $A \in \mathrm{Sp}(n)$, $AA^\dagger = A^\dagger A = I_n$, where $A^\dagger$ denotes the quaternionic conjugate transpose of A.

To be able to define spin groups, we first have to define Clifford algebras.

Definition 29 (Renaud 2020)**.** A real Clifford algebra with signature (p, q), denoted $\mathrm{Cl}(p, q)$, is an algebra over $\mathbb{R}$ generated by $(p + q)$ symbols, denoted $e_1, e_2, \ldots, e_{p+q}$, where the vector multiplication (sometimes called the *geometric product*) satisfies the following properties.

$$e_i e_j = -e_j e_i \text{ where } i \neq j$$
$$e_i^2 = \begin{cases} +1 & \text{if } 1 \leq i \leq p \\ -1 & \text{if } p+1 \leq i \leq p+q \end{cases}$$

In general, the elements of $\mathrm{Cl}(p, q)$ are called *multivectors*. The subset of $\mathrm{Cl}(p, q)$ consisting only of real linear combinations of $\{e_1, e_2, \ldots, e_{p+q}\}$ are vectors and can be identified directly with the vector space $\mathbb{R}^{p+q}$.

Clifford algebras allow us to rotate and reflect vectors with relative ease. For instance, the reflection of a vector v across a vector b is given simply by bvb^{-1} (Dechant 2012). To rotate a vector within a plane, we can simply compose two reflections together. In higher dimensions, rotations may occur in multiple planes at once, but we can do this by simply composing multiple planar rotations together. This motivates our definition of a versor below.

Definition 30 (ibid.)**.** An element $R \in \mathrm{Cl}(p, q)$ is a *versor* if R can be written as a geometric product of any number of vectors. A *unit versor* is a geometric product of unit vectors, and an *even versor* is a geometric product of an even number of vectors.

Remark. Notice that we can always normalise any versor to get a unit versor, and it would represent the same transformation when a vector is conjugated by it. Also, general versors represent any orthogonal transformation, while even versors only represent orientation preserving ones, i.e. rotations.

Definition 31 (ibid.)**.** The *spin group* $\mathrm{Spin}(n)$ is the group of all even unit versors in the Clifford algebra $\mathrm{Cl}(n, 0)$, with the geometric product as the group operation.

Remark. The elements of $\mathrm{Spin}(n)$ are even unit versors, which correspond to rotations in $\mathbb{R}^n$. However, unit versors with opposite signs R and $-R$ represent the same transformation, as given any vector $v \in \mathbb{R}^n$, $RvR^{-1} = (-R)v(-R)^{-1}$. As such, each element of $\mathrm{SO}(n)$ corresponds to two elements with opposite signs in $\mathrm{Spin}(n)$. It is for this reason that $\mathrm{Spin}(n)$ is sometimes called a *double cover* of $\mathrm{SO}(n)$, as there exists a short exact sequence of groups $\mathbb{Z}_2 \hookrightarrow \mathrm{Spin}(n) \twoheadrightarrow \mathrm{SO}(n)$.

5.2 Homogeneous spaces

Definition 32. An *action* of a group G on a set S is a map $\cdot : G \times S \to S$ satisfying $\mathrm{id}_G \cdot x = x$ and $(gh) \cdot s = g \cdot (h \cdot x)$ for any $g, h \in G$ and $x \in S$. In this case, we call S a G-set.

Definition 33. The *orbit* of an element x in a G-set S is the set $\{g \cdot x \mid g \in G\}$. The *isotropy subgroup* of an element $x \in S$ is the subgroup $\{g \in G \mid g \cdot x = x\}$. If the orbit of any given $x \in G$ is the entire set S, then G is said to act *transitively* on S.

Definition 34 (Rowland n.d.)**.** A *homogeneous space* M is a space with a transitive group action by a Lie group. (We will not define what a Lie group is rigorously, but it can be thought of as a 'continuous' group.) We can describe M as the space of cosets of an isotropy subgroup H in G which fixes a given element x, and we write this as a left-coset exact sequence $H \hookrightarrow G \twoheadrightarrow M$. It turns out that since G is a transitive group action, the choice of x does not change the resulting homogeneous space up to isomorphism.

Lemma 35. S^n is a homogeneous space given by the left-coset exact sequence $\mathrm{SO}(n-1) \hookrightarrow \mathrm{SO}(n) \twoheadrightarrow S^{n-1}$ when $n > 1$.

Proof. We first show that $\mathrm{SO}(n)$ acts transitively on S^{n-1}. That is, we need to show that if $\|u\| = 1$ and $\|v\| = 1$ for any given $u, v \in \mathbb{R}^n$, then there exists an element $A \in \mathrm{SO}(n)$ such that $Au = v$. Clearly, there exists an isometry $T : \mathrm{span}\{u\} \to \mathrm{span}\{v\}$ by letting $Tu = v$ and extending linearly. By Witt's theorem (Gross 1979), we can extend T to be an isometry $\mathbb{R}^n \to \mathbb{R}^n$. This gives is an element $A \in \mathrm{O}(n)$ satisfying the given property. If $\det A = 1$, then $A \in \mathrm{SO}(n)$ and so we are done. If $\det A = -1$, then we can define B to be a reflection across an $(n-1)$ dimensional subspace which contains v (this is always possible as $n > 1$). We can see that $Bv = v$ and $\det B = -1$, and so the matrix BA is in $\mathrm{SO}(n)$ and also satisfies the required property.

Next, we show that the isotropy subgroup of any given $u \in S^{n-1}$ is isomorphic to $\mathrm{SO}(n-1)$. We first show that this is true for a particular point $u = e_1 = (1, 0, \ldots, 0)$. Then, we suppose that $A \in SO(n)$ satisfies $Ae_1 = e_1$. If we let $A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix}$, then we see that $\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ implies that $a_{1,1} = 0$ and $a_{k,1} = 0$ for $2 \leq k \leq n$ by considering the dot product of each row of A with e_1. Since A is an orthogonal matrix, $A^\intercal A = I_n$ and so $\begin{bmatrix} 1 & 0 & \cdots & 0 \\ a_{1,2} & a_{2,2} & \cdots & a_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,n} & a_{2,n} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} 1 & a_{1,2} & \cdots & a_{1,n} \\ 0 & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$, which implies that $a_{1,k} = 0$ for $2 \leq k \leq n$ by considering the dot product of each row of $A^\intercal$ with the first column of A. Thus, we can write A as a block diagonal matrix $A = I_1 \oplus B$ where B is an $(n-1) \times (n-1)$ matrix. Now, we can notice that

$$
\begin{aligned}
&A^\intercal A = I_n \\
&\Rightarrow (I_1^\intercal \oplus B^\intercal)(I_1 \oplus B) = I_n \\
&\Rightarrow I_1 \oplus B^\intercal B = I_n \\
&B^\intercal B = I_{n-1}.
\end{aligned}
$$

which means $B \in \mathrm{O}(n-1)$.

Moreover, $\det A = \det(I_1 \oplus B) = \det I_1 \det B = \det B = 1$, which means that $B \in \mathrm{SO}(n-1)$. As such, every element in the isotropy subgroup of e_1 corresponds to an element of $\mathrm{SO}(n-1)$. The reverse is also easy to show by simply showing that $I_1 \oplus B$ is an orthogonal matrix with determinant 1 which also fixes e_1, given any orthogonal matrix B. Thus, the isotropy subgroup of e_1 is isomorphic to $\mathrm{SO}(n-1)$.

Finally, since $\mathrm{SO}(n)$ acts transitively on S^{n-1}, for any $u \in S^{n-1}$, there exists some $T \in \mathrm{SO}(n)$ such that $Tu = e_1$. Then, the elements of $\mathrm{SO}(n)$ which fix u are the elements $T^{-1}AT$ where A fixes e_1. Thus, the isotropy subgroup of any point u is isomorphic to $\mathrm{SO}(n-1)$. □

Theorem 36. S^n is a homogeneous space given by the left-coset exact sequence $\mathrm{Spin}(n-1) \hookrightarrow \mathrm{Spin}(n) \twoheadrightarrow S^{n-1}$ when $n > 1$.

Proof. See Proposition 24.1 of Porteous (2009). □

Remark. Lemma 35 provides us with an intuitive way to understand the result in Theorem 36. Since $\mathrm{Spin}(n)$ is a double cover of $\mathrm{SO}(n)$, we can deduce that for each element of $\mathrm{SO}(n)$ which fixes a particular point on S^{n-1}, there are two in $\mathrm{Spin}(n)$ which fix the same point. As such, the isotropy subgroup of a point on S^{n-1} as a $\mathrm{Spin}(n)$-set is a double cover of the isotropy subgroup of the same point where S^{n-1} is thought of as an $\mathrm{SO}(n)$-set. As such, one may guess that the isotropy subgroup of a point on S^{n-1} as a $\mathrm{Spin}(n)$-set is isomorphic to $\mathrm{Spin}(n-1)$, and this turns out to be precisely the case.

Theorem 37. S^7 is a homogeneous space given by the left-coset exact sequence $\mathrm{Sp}(1) \hookrightarrow \mathrm{Sp}(2) \twoheadrightarrow S^7$.

Proof. We know from Section 3 of Bauer and Laaroussi (2022) that Sp(2) acts transitively on S^7. Then, the rest of the proof follows a similar structure to the proof of Theorem 35.

Recall that we can write points on S^7 as pairs of quaternions (q_1, q_2) satisfying $|q_1|^2 + |q_2|^2 = 1$. We first consider the elements $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{Sp}(2)$ which fix the point $e_1 = (1, 0) \in S^7$. Since $Ae_1 = e_1$, this means $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ which implies $a = 1$ and $c = 0$. Next, since A satisfies $AA^\dagger = I_2$, this means $\begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ b^* & d^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ which means $b = 0$ and $dd^* = |d|^2 = 1$. On the other hand, given a unit quaternion d, it is easy to see that $\begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix}$ is an element of Sp(2) which fixes e_1. As such, the map $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to d$ is an isomorphism from the isotropy subgroup of e_1 to Sp(1).

Finally, for any other point $u \in S^7$, since Sp(2) acts transitively on S^7, we can suppose $T \in \mathrm{Sp}(2)$ satisfies $T(p) = e_1$. Then, $T^{-1}AT$ fixes u for any A which fixes e_1. Thus, the isotropy subgroup of any point in S^7 is isomorphic to Sp(1). □

5.3 How Hopf maps arise from spin actions

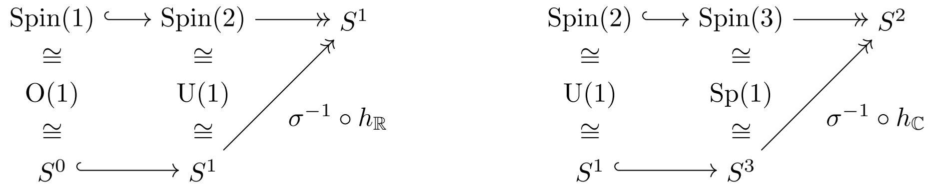
In this section, we will be summarising some results from Porteous (2009), which show how the Hopf maps arise from spin actions on S^1, S^2 and S^4, while giving informal justifications using the results we have discussed so far.

Theorem 38 (Exceptional isomorphisms of spin groups)**.** According to Porteous (ibid.), there exist the following exceptional isomorphisms for Spin(n) for $n \leq 6$ (i.e. Spin(n) is isomorphic to certain other known groups when $n \leq 6$ with no particular pattern in the correspondences).

$$\mathrm{Spin}(1) \cong \mathrm{O}(1) \qquad \mathrm{Spin}(4) \cong \mathrm{Sp}(1) \times \mathrm{Sp}(1)$$
$$\mathrm{Spin}(2) \cong \mathrm{U}(1) \qquad \mathrm{Spin}(5) \cong \mathrm{Sp}(2)$$
$$\mathrm{Spin}(3) \cong \mathrm{Sp}(1) \qquad \mathrm{Spin}(6) \cong \mathrm{SU}(4)$$

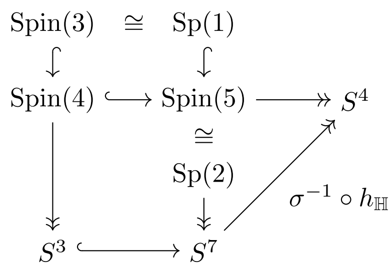
We first discuss the spin actions on S^1 and S^2, which gives us the following left-coset exact sequences $\mathrm{Spin}(1) \hookrightarrow \mathrm{Spin}(2) \twoheadrightarrow S^1$ and $\mathrm{Spin}(2) \hookrightarrow \mathrm{Spin}(3) \twoheadrightarrow S^2$ according to Theorem 36.

By the exceptional isomorphisms laid out in Theorem 38, and the fact that O(1), U(1) and Sp(1) are isomorphic to S^0, S^1 and S^3 respectively, we can show that there exist exact sequences of the form $S^0 \hookrightarrow S^1 \twoheadrightarrow S^1$ and $S^1 \hookrightarrow S^3 \twoheadrightarrow S^2$ via the following diagrams. This necessarily give us the Hopf maps, because according to Price (2010), the only fibre bundles between spheres are the Hopf maps.



We finish off with a discussion on the spin actions on S^4, which gives the left-coset exact sequence $\mathrm{Spin}(4) \hookrightarrow \mathrm{Spin}(5) \twoheadrightarrow S^4$.

Unlike for the spin actions on S^1 and S^2, the groups Spin(4) and Spin(5) are not themselves isomorphic to n-spheres. However, from Theorems 36 and 37, we know that there exist exact sequences $\mathrm{Spin}(3) \hookrightarrow \mathrm{Spin}(4) \twoheadrightarrow S^3$ and $\mathrm{Sp}(1) \hookrightarrow \mathrm{Sp}(2) \twoheadrightarrow S^7$. Since Theorem 38 give us the exceptional isomorphisms $\mathrm{Spin}(3) \cong \mathrm{Sp}(1)$ and $\mathrm{Spin}(4) \cong \mathrm{Sp}(2)$, we can factor out a subgroup isomorphic to Spin(3) from both Spin(4) and Spin(5), which we can represent in the diagram below. Then, it turns out that this projects to the exact sequence $S^3 \hookrightarrow S^7 \twoheadrightarrow S^4$ (Porteous 2009). And once again by Price (2010), this must necessarily be the quaternionic Hopf map.



As for the octonionic Hopf map, it does in fact arise from the left-coset exact sequence $\mathrm{Spin}(8) \hookrightarrow \mathrm{Spin}(9) \twoheadrightarrow S^8$. However, since these higher spin groups no longer exhibit exceptional isomorphisms, it is much more difficult to see its relation to the Hopf map, which involves a deeper analysis into the structure of $\mathbb{O}$ and Lie theory (Porteous 2009). As such, we will leave this discussion out of the scope of this report.

6 Conclusion

We have first investigated some properties of the complex Hopf fibration, and what it means for it to be a nontrivial fibre bundle. After this, we have looked at how the complex Hopf fibration generalises to higher dimensions, and the relationship between these higher Hopf maps and the real normed division algebras. We have also seen how spin groups can act on n-spheres, which gives us an alternative way of understanding the real, complex and quaternionic Hopf maps. As for future research directions, it may be interesting to investigate how defining a spin action on S^8 relates to the octonionic Hopf map, and why attempting to do the same for higher n-spheres fails to produce a higher Hopf map.

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