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Diagram Bethe Ansatz

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1 Abstract

In this project we diagonlise an operator of the Temperley-Lieb loop model with open boundaries, using a Bethe ansatz method. For the $L = 2$ case we find that solutions similar to that already know exist for other specalisations of the b parameter and that these solutions can be generalised to generic b.

2 Introduction

Many integrable systems such as the Simple Exclusion Process and the XXZ-spinchain are representations of the Temperley-Lieb algebra. By considering the loop representation of the Temperley-Lieb algebra we find the spectrum of the generator/Hamiltonian of these systems in a more general algebraic setting. Previously the entirety of the spectrum had been found using a Bethe ansatz under a certain constraint on a parameter in the algebra [2]. In our research we explore how this constraint can be removed for the $L = 2$ case in the hopes of finding a general solution in a simple factorised form. First we will introduce the Temperley-Lieb algebra, the loop representation and the solutions found in [2]. Then we outline solutions for $L = 2$ under a set of alternative constraints found in [1]. Finally we generalise these results to solutions with no constraints.

3 Statement of Authorship

The discussion and results presented in section 4, subsection 5.1, subsection 5.2 and subsection 5.3 are taken from [2]. The results in subsection 5.4 and subsection 5.5 are my own work excluding where otherwise stated. The diagrams used in this report are modifications of diagrams found in [2].

4 The Setting

4.1 Temperley-Lieb algebra

The Temperley-Lieb algebra with boundaries, T_L, can be defined by the generators $\{f_-, f_+, e_1, \ldots, e_{L-1}\}$ with relations

$$\begin{aligned} &e_j^2 = te_j && f_- = s_- f_- \\ &e_j e_{j\pm1} e_j = e_j && f_+^2 = s_+ f_+ \\ &e_j e_i = e_i e_j \quad \text{for } |i-j| \leq 2 && f_- f_+ = f_+ f_- \\ &e_1 f_- e_1 = e_1 && e_j f_- = f_- e_j \quad \text{for } j > 1 \\ &e_{L-1} f_+ e_{L-1} = e_{L-1} && f_+ e_j = e_j f_+ \quad \text{for } j < L-1, \end{aligned} \tag{4.1}$$

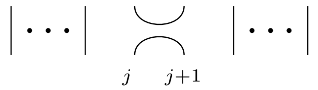
and the additional relations

$$I_L J_L I_L = bI_L, \quad J_L I_L J_L = bJ_L \tag{4.2}$$

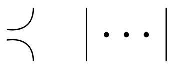
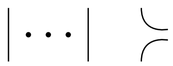
where

$$\begin{aligned} I_{2n} &= \prod_{j=0}^{n-1} e_{2j+1} \qquad J_{2n} = f_- \prod_{j=1}^{n-1} e_{2j} f_+ \\ I_{2n+1} &= f_- \prod_{j=1}^{n} e_{2j} \quad J_{2n+1} = \prod_{j=0}^{n-1} e_{2j+1} f_+ . \end{aligned} \tag{4.3}$$

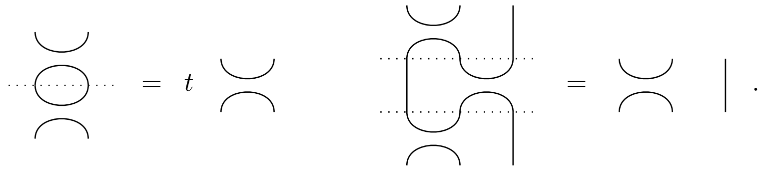
Where t, b, s_-, s_+ are constants. The Temperley-Lieb algebra is a *diagram algebra*, meaning we can represent its elements using diagrams. We depict the generators of the Temperley-Lieb algebra as

$e_j \quad =$  , (4.4)

and

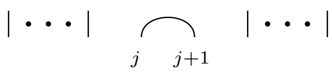
$f_- =$  $\qquad f_+ =$  . (4.5)

We can compose two elements by putting one diagram on top of the other and then connecting lines. As an example we can rewrite the relations $e_j^2 = te_j$ and $e_j e_{j+1} e_j = e_j$ as

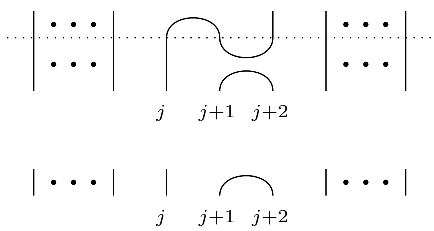
 . (4.6)

4.2 Loop Representation

We will not detail the precise way in which the loop representation is defined, that has been done here [3], instead we will give you an intuition as to how this representation works. We depict the elements of the loop representation as the bottom halves of the diagrams we have in the Temperley-Lieb algebra. A useful property of the loop representation is that it contains a highest weight vector, which we denote as $|\rangle$. As a diagram this vector looks like L straight lines. We can then produce any other vector by applying some element of T_L to $|\rangle$. Using diagrams we do this by taking an element of T_L, placing this element underneath the highest weight state and removing disconnected regions from the top. For example

$e_j|\rangle \quad =$  . (4.7)

Another useful demonstration is the action of e_{j+1} on $e_j|\rangle$,

$e_{j+1}|j\rangle \quad =$  . (4.8)

When in this representation each position where a line can be is called a *site*, so there are L sites in every vector. We will define the following notation $|l_1, \ldots, l_n; x_1, \ldots, x_m; r_1, \ldots, r_k\rangle$ for the vector that has sites $l_1, \ldots, l_n$ connected to the left boundary, sites $x_1, \ldots, x_m$ as the left site of a loop and, sites $r_1, \ldots, r_k$ connected to the right boundary. For example when $L = 4$ we have

$$| 1; 2; 4 \rangle \quad = \tag{4.9}$$

Define the following operator acting on the loop representation,

$$H_L = a_+ f_+ + a_- f_- + \sum_{j=1}^{L-1} e_j, \tag{4.10}$$

with constants a_- and a_+. For the remainder of this report we will be working to find the spectrum of this operator. We wish to do this as in the representations that are associated with physical systems this operator becomes the generator/Hamiltonian of that system. So finding its eigenvalues tells a physcist important information about that system.

5 Eigenvalues

For later convenience we define the functions

$$\begin{aligned} \lambda(z) &= t + z + z^{-1}, \\ K_\pm(z) &= \lambda(z) - a_\pm(s_\pm + z), \\ \tilde{K}_\pm(z) &= \lambda(z) + a_\pm(s_\pm + z(s_\pm t - 1)), \\ S(z, w) &= 1 + tw + zw. \end{aligned} \tag{5.1}$$

Observe that if you break the loop representation up into subspaces based on the number of *excited sites*, that is the number of sites that are not straight lines, then H_L takes a block triangular structure. This observation leads us to examine the left eigenvector equations, as we would expect them to be less complex.

5.1 2 Excited Sites

To simplify the problem we will suppose we have a trial eigenvector that consists of vectors with at most 2 excited sites, we will call this vector $\langle\psi|$ and write

$$\begin{aligned} \langle\psi| = \sum_{j=1}^{L-1} \psi(;j;)\langle\, ;j;\, | + \psi(1;;L)\langle\, 1;;L\, | + \psi(1,2;;)\langle\, 1,2;;\, | \\ + \psi(;;L-1,L)\langle\, ;;L-1,L\, | + \psi(1;;)\langle\, 1;;\, | + \psi(;;L)\langle\, ;;L\, | + \psi()\langle|. \end{aligned} \tag{5.2}$$

We wish to solve

$$\Lambda\langle\psi| = \langle\psi|H_L. \tag{5.3}$$

Assuming $L \geq 4$, this trial eigenvector gives eigenvector equations

$$\Lambda\psi() = \sum_{j=1}^{L-1} \psi(;j;) + a_-\psi(1;;) + a_+\psi(;;L) \tag{5.4}$$

$$\Lambda\psi(1;;) = a_-s_-\psi(1;;) + a_+\psi(1;;L) \tag{5.5}$$

$$\Lambda\psi(;;L) = a_+s_+\psi(;;L) + a_-\psi(1;;L) \tag{5.6}$$

and

$$\Lambda\psi(;x;) = t\psi(;x;) + \psi(;x-1;) + \psi(;x+1;) \quad \text{for } 1 < x < L-1 \tag{5.7}$$

$$\Lambda\psi(;1;) = t\psi(;1;) + \psi(;2;) + a_-\psi(1,2;;) \tag{5.8}$$

$$\Lambda\psi(;L-1;) = t\psi(;L-1;) + \psi(;L-2;) + a_+\psi(;;L-1,L) \tag{5.9}$$

$$\Lambda\psi(1,2;;) = a_-s_-\psi(1,2;;) + \psi(;1;) \tag{5.10}$$

$$\Lambda\psi(;;L-1,L) = a_+s_+\psi(;;L-1,L) + \psi(;L-1;) \tag{5.11}$$

$$\Lambda\psi(1;;L) = (a_-s_- + a_+s_+)\psi(1;;L). \tag{5.12}$$

A consequence of the block triangular structure of H_L is that Equation 5.7 - 5.12 depend only on one another, we can thus solve them independently of the remaining equations. Looking at these equations we notice that $\psi(1;;L)$ appears only in Equation 5.12, so we can split our solutions into cases.

- Case 1: if $\psi(1;;L) \neq 0$ then $\psi(1,2;;) = \psi(;;L-1,L) = \psi(;x;) = 0$ for $x = 1,\ldots,L-1$, is a solution with eigenvalue

$$\Lambda = a_-s_- + a_+s_+. \tag{5.13}$$

- Case 2: We set $\psi(1;;L) = 0$. We will now use what is called a *Bethe ansatz*, setting

$$\psi(;x;) = f(x) = A^+z^x + A^-z^{-x} \tag{5.14}$$

  with constants A^+, A^-, z. Plugging this ansatz into Equation 5.7 gives us the result

$$\Lambda = \lambda(z), \tag{5.15}$$

  so we have found the form of our eigenvalue but we still must find z. Using this value of Λ, Equation 5.8 and Equation 5.9 give

$$\begin{aligned} \psi(1,2;;) &= a_-^{-1}f(0) \\ \psi(;;L-1,L) &= a_+^{-1}f(L). \end{aligned} \tag{5.16}$$

  We can then apply all of these results to Equation 5.10 and Equation 5.11, which become the conditions on our constants

$$\frac{A^-}{A^+} = -\frac{K_-(z)}{K_-(z^{-1})} \quad \text{and} \quad z^{2L} = -\frac{K_+(z)}{K_+(z^{-1})}\frac{A^-}{A^+}. \tag{5.17}$$

  These conditions can then be put together to get the equation

$$z^{2L} = \frac{K_+(z)K_-(z)}{K_+(z^{-1})K_-(z^{-1})}, \tag{5.18}$$

which we call the *Bethe equation*. This equation is only deponent on z and the constants of our algebra and can thus be solved. So if z is a solution to the Bethe equation then $\lambda(z)$ is an eigenvalue.

It can be verified that the eigenvectors calculated above are eigenvectors for $L > 2$. When $L = 2$ we get the eigenvector equations

$$\Lambda\psi() = \psi(;1;) + a_-\psi(1;;) + a_+\psi(;;2) \tag{5.19}$$

$$\Lambda\psi(1;;) = a_-s_-\psi(1;;) + a_+\psi(1;;2) \tag{5.20}$$

$$\Lambda\psi(;;2) = a_+s_+\psi(;;2) + a_-\psi(1;;2) \tag{5.21}$$

and

$$\Lambda\psi(;1;) = t\psi(;1;) + a_-\psi(1,2;;) + a_+\psi(;;1,2) \tag{5.22}$$

$$\Lambda\psi(1,2;;) = a_-s_-\psi(1,2;;) + \psi(;1;) \tag{5.23}$$

$$\Lambda\psi(;;1,2) = a_+s_+\psi(;;1,2) + \psi(;1;) \tag{5.24}$$

$$\Lambda\psi(1;;2) = (a_-s_- + a_+s_+)\psi(1;;2) + b\psi(;1;). \tag{5.25}$$

Unlike previously, in Equation 5.25 we see the appearance of the constant b, which is a result of the boundary interaction relation 4.2. This relation will only ever come into play when we consider vectors that have every site excited. Apart from this b term, Equation 5.25 is the same as Equation 5.12 so one my wonder what happens when we set $b = 0$. One can verify that for this constraint on b our eigenvectors still work, however they do not for general b.

5.2 4 Excited Sites

We will not go over the full calculation for 4 excited sites as it is a bit laborious, however part of it will be shown to add some further context to the results in subsection 5.3. As before we will have a trial eigenvector $\langle\psi|$, this time containing vectors with at most 4 excited sites. We will also introduce the notation $\varphi(x)$, being the weight of the co-vector of $e_{x+1}e_{x+2}e_x|\rangle$, or pictorially the contribution of the 'nested loop' diagram beginning at the site x. As before thanks to the block triangular structure, we only need to solve the system of equations that refer exclusively to the vectors with exactly 4 excited sites. We consider of these equations

$$\Lambda\psi(;x_1,x_2;) = 2t\psi(;x_1,x_2;) + \psi(;x_1-1,x_2;) + \psi(;x_1,x_2-1;) + \psi(;x_1+1,x_2;) + \psi(;x_1,x_2+1;) \tag{5.26}$$

for $1 < x_1 < x_2 - 2 < L - 3$ and,

$$\Lambda\psi(;x,x+2;) = 2t\psi(;x,x+2;) + \psi(;x-1,x+2;) + \psi(;x,x+3;) + \varphi(x) \tag{5.27}$$

$$\Lambda\varphi(x) = t\varphi(;x;) + \psi(;x-1,x+1;) + 2\psi(;x,x+2) + \psi(;x+1,x+3;) \tag{5.28}$$

for $1 < x < L - 3$. We again make a Bethe ansatz, for this larger value of n this looks like

$$\psi(;x_1,x_2;) = f(x_1,x_2) = \sum_{\pi\in S_2}\sum_{\sigma} A^{\sigma_1\sigma_2}_{\pi_1\pi_2} z^{\sigma_1 x_1}_{\pi_1} z^{\sigma_2 x_2}_{\pi_2}, \tag{5.29}$$

where S_2 is the symmetric group of 2 integers and σ is the set of possible signs, i.e $\sigma_1 = \pm 1$ and $\sigma_2 = \pm 1$. As previously, Equation 5.26 gives us the form of our eigenvalue,

$$\Lambda = \lambda(z_1) + \lambda(z_2). \tag{5.30}$$

We can then apply this to Equation 5.27 to get

$$\psi(x) = f(x, x+1) + f(x+1, x+2). \tag{5.31}$$

Plugging this into Equation 5.28, one can show that this equation is satisfied if

$$\frac{A^{\sigma_1\sigma_2}_{\pi_1\pi_2}}{A^{\sigma_2\sigma_1}_{\pi_2\pi_1}} = \frac{S(z^{\sigma_2}_{\pi_2}, z^{\sigma_1}_{\pi_1})}{S(z^{\sigma_1}_{\pi_1}, z^{\sigma_1}_{\pi_2})}. \tag{5.32}$$

So it is these nested loop equations which introduce the $S(z, w)$ function into the general solutions detailed below.

5.3 General Results

With a bit of work, which is done in [2], one can extend the previous idea to a trial eigenvector consisting of vectors with at most n excited sites. This result says that for $n < L$, H_L has eigenvalues described by:

- if n is even then

$$\Lambda = \sum_{j=1}^{n/2} \lambda(z_j) \quad \text{with} \quad z_i^{2L} = \frac{K_+(z_i)K_-(z_i)}{K_+(z_i^{-1})K_-(z_i^{-1})} \prod_{\substack{j=1 \\ j\neq i}}^{n/2} \frac{S(z_i^{-1}, z_j)S(z_j, z_i)}{S(z_j, z_i^{-1})S(z_i, z_j)}, \tag{5.33}$$

  and

$$\Lambda = a_+s_+ + a_-s_- + \sum_{j=1}^{n/2-1} \lambda(z_j) \quad \text{with} \quad z_i^{2L} = \frac{\tilde{K}_+(z_i)\tilde{K}_-(z_i)}{\tilde{K}_+(z_i^{-1})\tilde{K}_-(z_i^{-1})} \prod_{\substack{j=1 \\ j\neq i}}^{n/2-1} \frac{S(z_i^{-1}, z_j)S(z_j, z_i)}{S(z_j, z_i^{-1})S(z_i, z_j)}, \tag{5.34}$$

- if n is odd then

$$\Lambda = a_+s_+ + \sum_{j=1}^{(n-1)/2} \lambda(z_j) \quad \text{with} \quad z_i^{2L} = \frac{\tilde{K}_+(z_i)K_-(z_i)}{\tilde{K}_+(z_i^{-1})K_-(z_i^{-1})} \prod_{\substack{j=1 \\ j\neq i}}^{(n-1)/2} \frac{S(z_i^{-1}, z_j)S(z_j, z_i)}{S(z_j, z_i^{-1})S(z_i, z_j)}, \tag{5.35}$$

  and

$$\Lambda = a_-s_- + \sum_{j=1}^{(n-1)/2} \lambda(z_j) \quad \text{with} \quad z_i^{2L} = \frac{K_+(z_i)\tilde{K}_-(z_i)}{K_+(z_i^{-1})\tilde{K}_-(z_i^{-1})} \prod_{\substack{j=1 \\ j\neq i}}^{(n-1)/2} \frac{S(z_i^{-1}, z_j)S(z_j, z_i)}{S(z_j, z_i^{-1})S(z_i, z_j)}. \tag{5.36}$$

Under the constraint $b = 0$ the above result extends to $n = L$ and describes the entire spectrum of H_L. To summarise what has been done, for $b = 0$ we have been able to write down all the eigenvalues of H_L given we can solve these sets of polynomial equations. Note that this is a big improvement as unlike the characteristic equation these polynomial equations behave very nicely for large L and have a nice factorised form in terms of simple quadratic or linear factors. They are thus relatively easy for computers to solve and there exists a whole theory around numerically approximating the solutions to these kind of equations.

5.4 Extending the $L = 2$ case to alternative constraints

The previous result tells us that when $L = 2$ we have as eigenvalues

$$\Lambda = 0,\ a_- s_-,\ a_+ s_+ \tag{5.37}$$

and, when $b = 0$ the remaining eigenvalues are described by

$$\Lambda = a_+ s_+ + a_- s_-$$

and

$$\Lambda = \lambda(z) \quad \text{with} \quad z^2 = \frac{K_+(z)K_-(z)}{K_+(z^{-1})K_-(z^{-1})}. \tag{5.38}$$

So for $b = 0$, the linear term $\Lambda - a_+ s_+ - a_- s_-$ factorises out of the characteristic equation and, the roots of what is left are described by the above Bethe equation. In [1] it was found that each of the constraints

$$b = s_-,\ s_+,\ s_+ + s_- - s_+ s_- t, \tag{5.39}$$

cause a linear term to factorise out of the characteristic equation, similar to the behavior at $b = 0$. We find that the remaining eigenvalues for these constraints are described by the following Bethe equations:

- for $b = s_-$

  $$\Lambda = a_+ s_+$$

  and

  $$\Lambda = a_- s_- + \lambda(z) \quad \text{with} \quad z^2 = \frac{K_+(z)\tilde{K}_-(z)}{K_+(z^{-1})\tilde{K}_-(z^{-1})}, \tag{5.40}$$

- for $b = s_+$

  $$\Lambda = a_- s_-$$

  and

  $$\Lambda = a_+ s_+ + \lambda(z) \quad \text{with} \quad z^2 = \frac{\tilde{K}_+(z)K_-(z)}{\tilde{K}_+(z^{-1})K_-(z^{-1})}, \tag{5.41}$$

- for $b = s_+ s_- - s_+ s_- t$

  $$\Lambda = 0$$

  and

  $$\Lambda = a_+ s_+ + a_- s_- + \lambda(z) \quad \text{with} \quad z^2 = \frac{\tilde{K}_+(z)\tilde{K}_-(z)}{\tilde{K}_+(z^{-1})\tilde{K}_-(z^{-1})}. \tag{5.42}$$

Observe that the form of these Bethe equations mirrors the Bethe equations we saw in Equation 5.33 - 5.36, with the same K and $\tilde{K}$ terms appearing alongside the same forms of Λ.

5.5 Extending the $L = 2$ to no constraint

We make the transformation

$$K_{\pm}(z) \to K_{\pm}(z) \pm X(z), \qquad \tilde{K}_{\pm}(z) \to \tilde{K}_{\pm}(z) \pm X(z). \tag{5.43}$$

Under this transformation we find that we can write the eigenvalues for general b using the exact same equations we have just seen under specific choices of the *shift term* $X(z)$.

- Setting
$$X(z) = \frac{a_+ a_- b(a_+ s_+ + a_- s_- - 2\Lambda)}{(a_- s_- - a_+ s_+)(a_+ s_+ + a_- s_- - \Lambda)} \tag{5.44}$$
recovers Equation 5.38, the solution when $b = 0$,

- setting
$$X(z) = \frac{a_+ a_- (b - s_-)(a_+ s_+ + a_- s_- - 2\Lambda)}{(-a_- s_- - a_+ s_+)(a_+ s_+ - \Lambda)} \tag{5.45}$$
recovers Equation 5.40, the solution when $b = s_-$,

- setting
$$X(z) = \frac{a_+ a_- (b - s_+)(a_+ s_+ + a_- s_- - 2\Lambda)}{(a_- s_- + a_+ s_+)(a_- s_- - \Lambda)} \tag{5.46}$$
recovers Equation 5.41, the solution when $b = s_+$,

- setting
$$X(z) = \frac{a_+ a_- (b - s_+ - s_- + s_+ s_- t)(a_+ s_+ + a_- s_- - 2\Lambda)}{(-a_- s_- + a_+ s_+)(-\Lambda)} \tag{5.47}$$
recovers Equation 5.42, the solution when $b = s_+ + s_- - s_+ s_- t$.

So we have four distinct ways of writing down the eigenvalues for general b, one for each of our previously examined constraints. For the curious reader, we are not just plucking these shift terms out of thin air. They are actually the result of taking the characteristic equation for generic b modulus the characteristic equation under one of our constrains on b. We then divide this result by the linear term corresponding to that choice of constraint, and then solve for the remaining constant factor.

6 Conclusion

In this report we introduce the Temperley-Lieb algebra and it's loop representation, in which we phrased an eigenvalue problem which is closely related to many models in mathematical physics. We saw how one could use a Bethe ansatz method to derive a simple description of the spectrum of H_L under the constraint $b = 0$. For $L = 2$ we found similar discriptions of the spectrum for a set of alternative constraints on b and where then able to extended these solutions to no constraints on b. The next step would be finding a similar description of the spectrum for all L and generic b. This task is not an easy one as for $b = 0$, the $L = 2$ case has only one Bethe equation and one form of the eigenvalues, whereas for larger L there are two Bethe equations and two forms of the eigenvalues as we saw in subsection 5.3. So it is not immediately clear how one would generalise our methods from $L = 2$ to the somewhat different behaviour of larger L. More work will need to be done.

7 Acknowledgments

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