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Open Boundary Stochastic Dualities

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1 Abstract

In this project we investigate dualilty observables which serve to relate the time-evolution of different instances of the many-particle Markov chain known as the asymmetric simple exclusion process (ASEP). Of particular interest is a rank-one duality observable which relates the open-boundary ASEP and the single-particle simple random walk on a finite integer lattice.

2 Introduction

The *multi-species charged asymmetric simple exlcusion process* (m-CASEP) is a continuous-time Markov chain of charged particles of different 'colours' hopping to the left and right on an integer lattice of size n with particle reservoirs at either boundary. The ASEP is a much studied integrable model which is well-understood in two distinct limits: (i) finite lattice size and finite time and (ii) infinite lattice size and finite time. A key outstanding problem is the case of finite lattice size and finite time which is needed for a proper understanding of the physics in a combined space-time scaling limit. Markov chain duality provides a promising tool for progress and analysis in this regime. We will begin by introducing the model and the relevant Markov chain systems. Then, following the approach of [1], we will demonstrate a duality for the open boundary ASEP, expressed in a different form. We conclude by discussing possible extensions of this work to dual systems with many particles.

3 Statement of authorship

The definitions and results in section 4.2 are adapted from [1]. The duality ansatz in section 5.1 is the work of Jan de Gier, inspired by [1]. The results in lemma 5.1 and theorem 5.2, as well as the discussion and observations in section 5.3 are my own work except where explicitly stated otherwise.

4 The setting

4.1 The asymmetric simple exclusion process

The asymmetric simple exclusion process is a continuous-time Markov chain and a well-studied integrable model. It consists of a series of particles of different charges hopping to the left and right on a subsection of the integer lattice, with uniform hopping rates except possibly at the boundary sites. In this paper, we consider an open-boundary m-CASEP which allows for particles to enter and exit from the left- and right-most sites, thus changing the number of particles within the system over time.

We label a configuration of the m-CASEP on the lattice $\Lambda_n = \mathbb{Z} \cap [1, n]$ by strings, $\mu = (\mu_1, \ldots, \mu_n)$, where $\mu_i \in \{-r, \ldots, -1, 0, 1, \ldots, r\}$. We refer to the interior of the lattice $\Lambda_n \setminus \{1, n\}$ as the 'bulk'. On the bulk, the

Markov transition rates are given by

$$(\ldots, \mu_i, \mu_{i+1}, \ldots) \mapsto (\ldots, \mu_{i+1}, \mu_i, \ldots) \quad \text{with rate } \begin{cases} 1 & \text{if } \mu_i < \mu_{i+1} \\ t & \text{if } \mu_i > \mu_{i+1} \end{cases}.$$

On the boundaries, we allow particles to change charge;

$$(-\mu_1, \ldots) \mapsto (+\mu_1, \ldots) \quad \text{with rate } \gamma,$$
$$(+\mu_1, \ldots) \mapsto (-\mu_1, \ldots) \quad \text{with rate } \alpha,$$

for $\mu_1 \in \{1, \ldots, r\}$ and likewise

$$(\ldots, +\mu_n) \mapsto (\ldots, -\mu_n) \quad \text{with rate } \delta,$$
$$(\ldots, -\mu_n) \mapsto (\ldots, +\mu_n) \quad \text{with rate } \beta,$$

for $\mu_n \in \{1, \ldots, r\}$. Unless stated otherwise, we assume all transition rates are strictly positive.

In this paper, we focus on the case of $r = 1$ so that $\mu_i \in \{-1, 0, 1\}$, and further simplify the system to have no netural particles (i.e. particles labeled 0), whereby we recover the standard ASEP. We may think of the states labeled by -1 as particles, whereas the states labeled by 1 are the 'holes'.

For the purpose of the embedding the transition rates into the generator, we also employ another common convention for multi-particle systems, which is built through the use of the Kronecker product. We first define the embedding map $\phi : \{-1, 1\} \to \mathbb{C}^2$ by mapping

$$-1 \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad 1 \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and then construct a unit vector $\eta \in \mathbb{C}^{2^n}$ from $\mu = (\mu_1, \ldots, \mu_n)$ by taking a tensor product over all site states, $\eta = \phi(\mu_1) \otimes \phi(\mu_2) \otimes \cdots \otimes \phi(\mu_n)$. With this convention, we can write the Markov generator as a sum over boundary operators and bond operators,

$$M = m_1 + \sum_{i=1}^{n-1} M_i + m_n$$

with boundary operators

$$m_1 = \begin{pmatrix} -\gamma & \gamma \\ \alpha & -\alpha \end{pmatrix} \otimes \mathbb{1}^{\otimes n-1}, \qquad m_n = \mathbb{1}^{\otimes n-1} \otimes \begin{pmatrix} -\beta & \beta \\ \delta & -\delta \end{pmatrix},$$

and bond operators

$$M_i = \mathbb{1}^{\otimes i-1} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & t & -t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \mathbb{1}^{\otimes n-1-i},$$

using $\mathbb{1}$ to denote the two-dimensional identity matrix. Having a concrete representation of this Markov generator allows us to precisely define duality in the context of the ASEP.

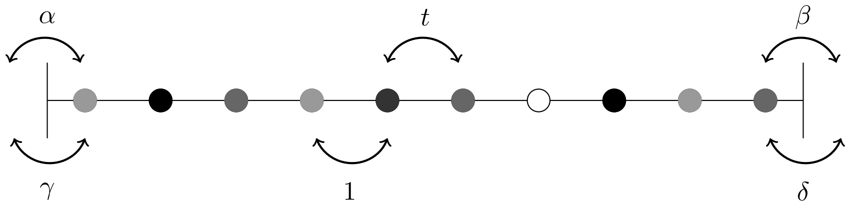

Figure 1: Rates of the multi-species asymmetric simple exclusion process with open boundary conditions

4.2 Duality and reverse duality in Markov processes

Throughout this paper, we refer to a continuous time Markov chain by the tuple $\{\eta(t),\ \Omega\}$, where $\eta(t)$ is the process indexed by the continuous variable t, and Ω is the state space of this process. We now state the definition of duality, which serves as the foundational concept behind this project.

Definition 4.1. Two Markov processes $\{\eta(t),\ \Omega\}$ and $\{\xi(t),\ \Xi\}$ with respective generators M and L are dual if there exits some observable $D : \Xi \times \Omega \to \mathbb{R}$ such that

$$LD = DM^T$$

or equivalently,

$$\sum_{\xi'} L_{\xi,\xi'} D(\xi', \eta) = \sum_{\eta'} D(\xi, \eta') M^T_{\eta',\eta}$$

for all $(\xi, \eta) \in \Xi \times \Omega$.

Note that duality is a reflexive property; if $\eta(t)$ is dual to $\xi(t)$ with duality observable D, then $\xi(t)$ is dual to $\eta(t)$ with observable D^T. This duality relationship is valuable in this context as it immediately grants the following result, which we will state without proof.

Lemma 4.2. *Let $\{\eta(t), \Omega\}$ and $\{\xi(t), \Xi\}$ be Markov processes with countable states spaces Ω, Ξ. If η and ξ are dual with respect to some bounded measurable observable $D : \Omega \times \Xi \to \mathbb{R}$, then for all $(\eta, \xi) \in \Omega \times \Xi$ and $t \geq 0$ we have*

$$\mathbb{E}_0 D(\eta(t), \xi) = \mathbb{E}_0 D(\eta, \xi(t)),$$

where $\mathbb{E}_0$ denotes an expectation taken with respect to an initial dirac measure correpsonding to $\eta = \eta(0)$ or $\xi = \xi(0)$ respectively.

We may also express interest in the notion of a reverse-dual process, as introduced in [1].

Definition 4.3. Let $\eta(t)$ and $\xi(t)$ be two Markov processes as above. We say $\eta(t)$ and $\xi(t)$ are *reverse dual* if there exists some observable $R : \Xi \times \Omega \to \mathbb{R}$ such that

$$RL = M^T R$$

Reverse duality leads to a somewhat altered time-evolution property.

Lemma 4.4. *Let $\{\eta(t), \Omega\}$ and $\{\xi(t), \Xi\}$ be Markov processes with countable states spaces Ω, Ξ, and let $\mathcal{F}_\Xi$ be a family of measures for the η process, indexed by Ξ. Suppose that η and ξ are reverse dual with respect to the duality function $R(\xi, \cdot) = \mu^\xi(\cdot) \in \mathcal{F}_\Xi$. Then for all $t \geq 0$ we have*

$$\mu_t^\xi(\cdot) = \sum_{\xi'} \mathbb{P}(\xi, t \mid \xi', 0)\mu_t^{\xi'}(\cdot)$$

Thus duality allows us to express the evolution of time-dependent expectation values for one process in terms of its dual process, whereas reverse duality allows us to relate the time-evolution of initial measures. Note that in the case of two reversible Markov chains, reverse duality is equivalent to a conventional duality for the time-reversed processes.

5 The duality observable

In the following section, we will demonstrate a duality observable between the ASEP $\eta(t)$ and a rank-one dual process, $\xi(t)$. Specifically, $\xi(t)$ will be a simple random walk in continuous time on an extended subsection of the integer lattice $\Lambda_n \cup \{0, n+1\}$, with holding rates derived from the parameters of the ASEP. The sites $0, n+1$ serve as *auxiliary sites*, which exhibit distinct behavoir to the bulk sites.

5.1 Duality observable

We now proceed to construct the duality observable. We begin by defining the left and right number operators,

$$\mathcal{N}_1(\xi, \eta) = \sum_{i=1}^{\xi-1} \mathbb{1}\{\eta_i = -1\}, \qquad \mathcal{N}_2(\xi, \eta) = \sum_{i=\xi+1}^{n} \mathbb{1}\{\eta_i = -1\},$$

which count the number of particles to the left or the right of a specified position ξ, respectively. Taking inspiration from [1], we make the following ansatz for a rank-one duality observable

$$D(\xi, \eta) = \begin{cases} b_1 t_2^{\mathcal{N}_2(0,\eta)}, & \xi = 0 \\ b^\xi t_1^{\mathcal{N}_1(\xi,\eta)} \times \mathbb{1}\{\eta_\xi = -1\} \times t_2^{\mathcal{N}_2(\xi,\eta)}, & 1 \leq \xi \leq n \\ b^n b_2 t_1^{\mathcal{N}_1(n+1,\eta)}, & \xi = n+1 \end{cases}$$

where we set $t_1 = -\gamma/\alpha$, $t_2 = -\beta/\delta$. This function exhibits a similar factorised form to the duality observable in [1], which is referred to as a factorised shock measure. We establish the following useful result for the duality observable.

Lemma 5.1. *The duality function $D(\xi, \eta)$ satisfies the following transfer relations on the bulk,*

$$\mathbb{1}\{\eta_{\xi+1} = -1\} \times bD(\xi, \eta) = \mathbb{1}\{\eta_\xi = -1\} \times tD(\xi+1, \eta),$$

while on the boundaries, it satisfies

$$\mathbb{1}\{\eta_n = -1\} \times \frac{1}{b_2} D(n+1, \eta) = t_1 D(n, \eta), \qquad \mathbb{1}\{\eta_1 = -1\} \times \frac{1}{b_1} D(0, \eta) = \frac{1}{t_2} D(1, \eta).$$

Proof. The proof is a straightforward algebraic exercise, which exploits the fact that with our choice of t_1 and t_2, we have $t_2/t_1 = t$. ☐

5.2 Proof of duality

The key contribution of this work is a proof of the following theorem which formalises the duality between the processes $\eta(t)$ and $\xi(t)$.

Theorem 5.2. *The duality observable $D(\xi, \eta)$ satisfies a dynamic relationship akin to that of a one-particle random walk,*

$$\sum_{\eta'} D(\xi, \eta') M^T_{\eta',\eta} = bD(\xi - 1, \eta) + t/bD(\xi + 1, \eta) - a_{\mathrm{b}} D(\xi, \eta), \quad 2 \le \xi \le n - 1, \tag{1}$$

$$\sum_{\eta'} D(1, \eta') M^T_{\eta',\eta} = b\alpha/b_1 D(0, \eta) + t/bD(2, \eta) - a_1 D(1, \eta) \tag{2}$$

$$\sum_{\eta'} D(n, \eta') M^T_{\eta',\eta} = bD(n - 1, \eta) + \delta/b_2 D(n + 1, \eta) - a_2 D(n, \eta). \tag{3}$$

when

$$a_{\mathrm{b}} = (1 + t + \alpha + \beta + \gamma + \delta), \quad a_1 = 1 + \beta + \delta + \gamma - \alpha t_2, \quad a_2 = t + \alpha + \gamma + \beta - \delta t_1.$$

and the transition rates are restricted to the parameter manifold defined by

$$\alpha\beta - t\gamma\delta = 0.$$

Remark 5.3. We note that this theorem describes a conventional duality between the $\eta(t)$ and $\xi(t)$ processes, and *not* a reverse duality, as is found in [1].

Proof. In what follows, we use the notation $\eta_1 \to \eta_2$ to describe a state η_2 which is accessible from η_1. We first consider the duality relationship for $2 \le \xi \le n - 1$ on the bulk. The left hand side of equation (1) reads

$$\sum_{\eta'} D(\xi, \eta') M^T_{\eta',\eta} = \sum_{\eta'} b^\xi t_1^{\mathcal{N}_1(\xi,\eta')} \times \mathbb{1}\{\eta'_\xi = -1\} \times t_2^{\mathcal{N}_2(\xi,\eta')} M_{\eta,\eta'},$$

and so for a fixed η, the only non-zero terms in the summation are given for η' such that $\eta'_\xi = -1$ and $\eta \to \eta'$. This gives three cases to consider:

1. Transition from the left: $(\eta_{\xi-1} = -1, \eta_\xi = 1) \to (\eta'_{\xi-1} = 1, \eta'_\xi = -1)$
2. Transition from the right: $(\eta_\xi = 1, \eta_{\xi+1} = -1) \to (\eta'_\xi = -1, \eta'_{\xi+1} = 1)$
3. Transition away from site ξ: $(\dots, \eta_\xi = -1, \dots) \to (\dots, \eta'_\xi = -1, \dots)$

We must also consider the case in which $\eta' = \eta$, where $M_{\eta,\eta}$ will be the holding rate of the Markov process. We begin with case 1, and impose the restriction by the use of an inidicator.

$$\sum_{\eta'} \mathbb{1}\{\eta_{\xi-1} = -1, \eta_\xi = 1, \eta'_{\xi-1} = 1, \eta'_\xi = -1\} b^\xi t_1^{\mathcal{N}_1(\xi,\eta')} \times \mathbb{1}\{\eta'_\xi = -1\} \times t_2^{\mathcal{N}_2(\xi,\eta')} M_{\eta,\eta'}$$

$$= b^\xi \mathbb{1}\{\eta_{\xi-1} = -1\} \sum_{\eta'} \mathbb{1}\{\eta_\xi = 1, \eta'_{\xi-1} = 1, \eta'_\xi = -1\} \times t_1^{\mathcal{N}_1(\xi,\eta')} t_2^{\mathcal{N}_2(\xi,\eta')} M_{\eta,\eta'} \tag{$*$}$$

Now if $\eta \to \eta'$ by the movement of a negative particle from site $\xi - 1$ to ξ, then $M_{\eta,\eta'} = 1$ and

$$\mathcal{N}_1(\xi, \eta') = \mathcal{N}_1(\xi - 1, \eta), \quad \mathcal{N}_2(\xi, \eta') = \mathcal{N}_2(\xi - 1, \eta),$$

which allows us to write

$$\begin{aligned}(*) &= b^\xi t_1^{\mathcal{N}_1(\xi-1,\eta)} \times \mathbb{1}\{\eta_{\xi-1} = -1\} \times t_2^{\mathcal{N}_2(\xi-1,\eta)} \sum_{\eta'} \mathbb{1}\{\eta_\xi = 1, \eta'_{\xi-1} = 1, \eta'_\xi = -1\} \\ &= bD(\xi - 1, \eta) \times \mathbb{1}\{\eta_\xi = 1\}.\end{aligned}$$

Similarly, for case 2, if $\eta \to \eta'$ by the movement of a negative particle from site $\xi + 1$ to ξ, then $M_{\eta,\eta'} = t$ and

$$\mathcal{N}_1(\xi, \eta') = \mathcal{N}_1(\xi + 1, \eta), \quad \mathcal{N}_2(\xi, \eta') = \mathcal{N}_2(\xi + 1, \eta).$$

Thus,

$$\begin{aligned}\sum_{\eta'} \mathbb{1}\{\eta_{\xi+1} = -1, \eta_\xi = 1,&\eta'_{\xi+1} = 1, \eta'_\xi = -1\} b^\xi t_1^{\mathcal{N}_1(\xi,\eta')} \times \mathbb{1}\{\eta'_\xi = -1\} \times t_2^{\mathcal{N}_2(\xi,\eta')} M_{\eta,\eta'} \\ &= b^\xi t_1^{\mathcal{N}_1(\xi+1,\eta)} \mathbb{1}\{\eta_{\xi+1} = -1\} t_2^{\mathcal{N}_2(\xi+1,\eta)} \sum_{\eta'} \mathbb{1}\{\eta_\xi = 1, \eta'_{\xi+1} = 1, \eta'_\xi = -1\} \times t \\ &= \frac{t}{b} D(\xi + 1, \eta) \times \mathbb{1}\{\eta_\xi = 1\}\end{aligned}$$

Now we consider the case in which $\eta \to \eta'$ by a transition away from site ξ. In this case, the number of negative particles may vary if the transition occurs on a boundary site, so for brevity we introduce the notation

$$\begin{aligned}\{1^+\} &= \{\text{particle gained at site 1}\}, \quad \{1^-\} = \{\text{particle lost at site 1}\} \\ \{n^+\} &= \{\text{particle gained at site n}\}, \quad \{n^-\} = \{\text{particle lost at site n}\}\end{aligned}$$

which allows us to write

$$\mathcal{N}_1(\xi, \eta') = \mathcal{N}_1(\xi, \eta) + \mathbb{1}\{1^+\} - \mathbb{1}\{1^-\}, \quad \mathcal{N}_2(\xi, \eta') = \mathcal{N}_2(\xi, \eta) + \mathbb{1}\{n^+\} - \mathbb{1}\{n^-\}.$$

Thus case 3 becomes

$$\begin{aligned}\sum_{\eta'} \mathbb{1}\{\eta_\xi = -1,&\eta'_\xi = -1\} b^\xi t_1^{\mathcal{N}_1(\xi,\eta')} \times \mathbb{1}\{\eta'_\xi = -1\} \times t_2^{\mathcal{N}_2(\xi,\eta')} M_{\eta,\eta'} \\ &= b^\xi \sum_{\eta'} \mathbb{1}\{\eta_\xi = -1, \eta'_\xi = -1\} t_1^{\mathcal{N}_1(\xi,\eta)+\mathbb{1}\{1^+\}-\mathbb{1}\{1^-\}} \times \mathbb{1}\{\eta'_\xi = -1\} \times t_2^{\mathcal{N}_2(\xi,\eta)+\mathbb{1}\{n^+\}-\mathbb{1}\{n^-\}} M_{\eta,\eta'} \\ &= b^\xi t_1^{\mathcal{N}_1(\xi,\eta)} \times \mathbb{1}\{\eta_\xi = -1\} \times t_2^{\mathcal{N}_2(\xi,\eta)} \sum_{\eta'} \mathbb{1}\{\eta'_\xi = -1\} t_1^{\mathbb{1}\{1^+\}-\mathbb{1}\{1^-\}} t_2^{\mathbb{1}\{n^+\}-\mathbb{1}\{n^-\}} M_{\eta,\eta'} \\ &= D(\xi, \eta) \sum_{\eta'} \mathbb{1}\{\eta'_\xi = -1\} t_1^{\mathbb{1}\{1^+\}-\mathbb{1}\{1^-\}} t_2^{\mathbb{1}\{n^+\}-\mathbb{1}\{n^-\}} M_{\eta,\eta'}\end{aligned}$$

Now it remains to evaluate the sum. For a given configuration η, we introduce further notation

$$\begin{aligned}N_L(\eta) &= \#\text{ left transitions which preserve particle number} \\ N_R(\eta) &= \#\text{ right transitions which preserve particle number}\end{aligned}$$

Note that, for fixed η, there is only a transtion $\eta \to \eta'$ corresponding to the event $\{1^+\}$ if $\eta_1 = 1$, and similarly for the other boundary transition events. Thus,

$$\sum_{\eta'} = \mathbb{1}\{\eta_1 = 1\}\alpha t_1 + \mathbb{1}\{\eta_1 = -1\}\gamma/t_1 + \mathbb{1}\{\eta_n = 1\}\delta t_2 + \mathbb{1}\{\eta_n = -1\}\beta/t_2 + (N_L(\eta) - \mathbb{1}\{\eta_{\xi-1} = 1\})t + (N_R(\eta) - \mathbb{1}\{\eta_{\xi+1} = 1\})$$
$$= -\mathbb{1}\{\eta_1 = 1\}\gamma - \mathbb{1}\{\eta_1 = -1\}\alpha - \mathbb{1}\{\eta_n = 1\}\beta - \mathbb{1}\{\eta_n = -1\}\delta + (N_L(\eta) - \mathbb{1}\{\eta_{\xi-1} = 1\})t + (N_R(\eta) - \mathbb{1}\{\eta_{\xi+1} = 1\})$$

Now the final term to consider is for $\eta' = \eta$. In this case, by definition of the holding time for a continous time Markov chain we have

$$D(\xi,\eta)M_{\eta,\eta} = -D(\xi,\eta)\sum_{\eta'} M_{\eta,\eta'}$$
$$= -D(\xi,\eta)(\mathbb{1}\{\eta_1 = 1\}\alpha + \mathbb{1}\{\eta_1 = -1\}\gamma + \mathbb{1}\{\eta_n = 1\}\delta + \mathbb{1}\{\eta_n = -1\}\beta + N_L(\eta)t + N_R(\eta))$$

Collecting these terms with case 3 gives

$$-D(\xi,\eta)\left(\alpha + \beta + \gamma + \delta + \mathbb{1}\{\eta_{\xi-1} = 1\}t + \mathbb{1}\{\eta_{\xi+1} = 1\}\right),$$

and finally summing over contributions from all cases gives;

$$\sum_{\eta'} D(\xi,\eta')M^T_{\eta',\eta} = \frac{t}{b}D(\xi+1,\eta) \times \mathbb{1}\{\eta_\xi = 1\} + bD(\xi-1,\eta) \times \mathbb{1}\{\eta_\xi = 1\}$$
$$- D(\xi,\eta)\left(\alpha + \beta + \gamma + \delta + \mathbb{1}\{\eta_{\xi-1} = 1\}t + \mathbb{1}\{\eta_{\xi+1} = 1\}\right).$$

This is identical to our desired result. To make this explicit, we use the result of lemma 5.1,

$$\frac{t}{b}D(\xi+1,\eta) \times \mathbb{1}\{\eta_\xi = 1\} + bD(\xi-1,\eta) \times \mathbb{1}\{\eta_\xi = 1\} - D(\xi,\eta)\left(\alpha + \beta + \gamma + \delta + \mathbb{1}\{\eta_{\xi-1} = 1\}t + \mathbb{1}\{\eta_{\xi+1} = 1\}\right)$$
$$= (1 - \mathbb{1}\{\eta_\xi = -1\})\left(\frac{t}{b}D(\xi+1,\eta) + bD(\xi-1,\eta)\right) - D(\xi,\eta)\left(\alpha + \beta + \gamma + \delta + \mathbb{1}\{\eta_{\xi-1} = 1\}t + \mathbb{1}\{\eta_{\xi+1} = 1\}\right)$$
$$= \frac{t}{b}D(\xi+1,\eta) + bD(\xi-1,\eta) - \mathbb{1}\{\eta_\xi = -1\}\left(\frac{t}{b}D(\xi+1,\eta) + bD(\xi-1,\eta)\right)$$
$$- (\alpha + \beta + \gamma + \delta + \mathbb{1}\{\eta_{\xi-1} = 1\}t + \mathbb{1}\{\eta_{\xi+1} = 1\})D(\xi,\eta)$$
$$= \frac{t}{b}D(\xi+1,\eta) + bD(\xi-1,\eta) - \mathbb{1}\{\eta_{\xi+1} = -1\}D(\xi,\eta) - \mathbb{1}\{\eta_{\xi-1} = -1\}tD(\xi,\eta)$$
$$- (\alpha + \beta + \gamma + \delta + \mathbb{1}\{\eta_{\xi-1} = 1\}t + \mathbb{1}\{\eta_{\xi+1} = 1\})D(\xi,\eta)$$
$$= \frac{t}{b}D(\xi+1,\eta) + bD(\xi-1,\eta) - a_bD(\xi,\eta)$$

This concludes the proof on the bulk of the lattice.

Right boundary

Now we approach the result for $\xi = n$. The proof follows the same general outline as on the bulk, but we will repeat key steps for clarity. The left hand side of equation (3) reads

$$\sum_{\eta'} D(n,\eta')M^T_{\eta',\eta} = \sum_{\eta'} b^n t_1^{\mathcal{N}_1(n,\eta')} \times \mathbb{1}\{\eta'_n = -1\} \times t_2^{\mathcal{N}(n,\eta')} M_{\eta,\eta'}$$

So we must only consider η' such that $\eta'_n = -1$ and $\eta \to \eta'$. As above, we consider three distinct cases;

1. Transition from the left: $(\eta_{n-1} = -1,\, \eta_n = 1) \to (\eta'_{n-1} = 1,\, \eta'_n = -1)$
2. Particle gained at the right boundary: $(\eta_n = 1) \to (\eta'_n = -1)$
3. Transition away from site n: $(\dots, \eta_n = -1) \to (\dots, \eta_n = -1)$

We begin with case 1. If $\eta \to \eta'$ by the movement of a negative particle from $n-1$ to n, then $M_{\eta,\eta'} = 1$ and thus

$$\begin{aligned}\sum_{\eta'} \mathbb{1}\{\eta_{n-1} = -1, \eta_n = 1, \eta'_{n-1} = 1, \eta'_n = -1\} \times b^n t_1^{\mathcal{N}_1(n,\eta')} \times \mathbb{1}\{\eta'_n = -1\} \times t_2^{\mathcal{N}(n,\eta')} M_{\eta,\eta'} \\ = b^n t_1^{\mathcal{N}_1(n-1,\eta)} \times \mathbb{1}\{\eta_{n-1} = -1\} \times t_2^{\mathcal{N}(n-1,\eta)} \sum_{\eta'} \mathbb{1}\{\eta_n = 1, \eta'_{n-1} = 1, \eta'_n = -1\} \\ = bD(n-1,\eta) \times \mathbb{1}\{\eta_n = 1\}\end{aligned}$$

Now for case 2, we suppose that a negative particle hops from the boundary to site n. Hence $M_{\eta,\eta'} = \delta$ and so

$$\begin{aligned}\sum_{\eta'} \mathbb{1}\{\eta_n = 1, \eta'_n = -1\} \times b^n t_1^{\mathcal{N}_1(n,\eta')} \times \mathbb{1}\{\eta'_n = -1\} \times t_2^{\mathcal{N}(n,\eta')} M_{\eta,\eta'} = b^n t_1^{\mathcal{N}_1(n+1,\eta)} \sum_{\eta'} \mathbb{1}\{\eta_n = 1\}\delta \\ = \frac{\delta}{b_2} D(n+1,\eta) \times \mathbb{1}\{\eta_n = 1\}.\end{aligned}$$

For case 3, we suppose that $\eta_n = -1$ and that the transition occurs away from site n. We use the same notation as above for boundary transition events to write

$$\begin{aligned}\sum_{\eta'} \mathbb{1}\{\eta_n = -1, \eta'_n = -1\} b^n t_1^{\mathcal{N}_1(n,\eta')} \times \mathbb{1}\{\eta'_n = -1\} \times t_2^{\mathcal{N}(n,\eta')} M_{\eta,\eta'} \\ = b^n t_1^{\mathcal{N}_1(n,\eta)} \times \mathbb{1}\{\eta_n = -1\} \times t_2^{\mathcal{N}(n,\eta)} \sum_{\eta'} \mathbb{1}\{\eta'_n = -1\} t_1^{\mathbb{1}\{1^+\} - \mathbb{1}\{1^-\}} M_{\eta,\eta'} \\ = D(n,\eta)\left(\mathbb{1}\{\eta_1 = 1\}\alpha t_1 + \mathbb{1}\{\eta_1 = -1\}\gamma/t_1 + N_L(\eta)(1 - \mathbb{1}\{\eta_{n-1} = 1\})t + N_R(\eta)\right).\end{aligned}$$

For $\eta' = \eta$, we calculate the holding rate to be

$$\begin{aligned}D(n,\eta) M_{\eta,\eta} = -D(n,\eta) \sum_{\eta'} M_{\eta,\eta'} \\ = -D(n,\eta)(\mathbb{1}\{\eta_1 = -1\}\gamma + \mathbb{1}\{\eta_1 = 1\}\alpha + \beta + N_L(\eta) + N_R(\eta)),\end{aligned}$$

and so combining the above with case 3 we have

$$-D(n,\eta)(\alpha + \gamma + \beta + \mathbb{1}\{\eta_{n-1} = 1\}t).$$

Combining all cases yields the result

$$\sum_{\eta'} D(n,\eta') M_{\eta,\eta'} = bD(n-1,\eta) \times \mathbb{1}\{\eta_n = 1\} + \delta/b_2 D(n+1,\eta) \times \mathbb{1}\{\eta_n = 1\} - D(n,\eta)(\alpha + \gamma + \beta + \mathbb{1}\{\eta_{n-1} = 1\}t).$$

We once again use the result of lemma 5.1 to transform this result into the desired identity.

$$
\begin{aligned}
&= (1-\mathbb{1}\{\eta_n=-1\})(bD(n-1,\eta)+\delta/b_2D(n+1,\eta)) - D(n,\eta)(\alpha+\gamma+\beta+\mathbb{1}\{\eta_{n-1}=1\}t)\\
&= bD(n-1,\eta)+\delta/b_2D(n+1,\eta)-\mathbb{1}\{\eta_n=-1\}bD(n-1,\eta)-\mathbb{1}\{\eta_n=-1\}\delta/b_2D(n+1,\eta)\\
&\qquad - D(n,\eta)(\alpha+\gamma+\beta+\mathbb{1}\{\eta_{n-1}=1\}t)\\
&= bD(n-1,\eta)+\delta/b_2D(n+1,\eta)-\mathbb{1}\{\eta_{n-1}=-1\}tD(n,\eta)-\delta t_1D(n,\eta)-D(n,\eta)(\alpha+\gamma+\beta+\mathbb{1}\{\eta_{n-1}=1\}t)\\
&= bD(n-1,\eta)+\delta/b_2D(n+1,\eta)-D(n,\eta)(\alpha+\gamma+\beta+t+\delta t_1)\\
&= bD(n-1,\eta)+\delta/b_2D(n+1,\eta)-a_2D(n,\eta)
\end{aligned}
$$

Left boundary

The proof for the duality on the left boundary site, $\xi = 1$, extends by analogy from the proof above, for the right boundary site. The result reads

$$\sum_{\eta'} D(1,\eta')M^T_{\eta',\eta} = b\alpha/b_1D(0,\eta)+t/bD(2,\eta)-a_1D(1,\eta),$$

with $a_1 = 1+\beta+\delta+\gamma+\alpha t_2$.

□

5.3 Towards other dualities

Extending this result to dual processes of higher rank would enable us to exploit this duality on different rate parameter manifolds. We conjecture that dualities should extend naturally from the form of the above rank-one duality observable, with a similar factorised shock profile. For a general rank-k dual process, we first introduce the counting operator

$$\mathcal{N}(x,y;\eta) = \sum_{i=x+1}^{y-1} \mathbb{1}\{\eta_i=-1\}$$

which counts the number of -1 particles between specified positions x and y. For $\xi_1 < \xi_2 < \cdots < \xi_k$ and auxiliary particles $\xi_0 = 0, \xi_{k+1} = n+1$, we consider a rank-$k$ duality observable of the form

$$
D(\xi_1,\ldots,\xi_k;\eta) = \begin{cases}
b^{\sum_{i=1}^k \xi_i} \prod_{i=2}^k \mathbb{1}\{\eta_{\xi_i}=-1\} \prod_{i=2}^{k+1} t_i^{\mathcal{N}(\xi_{i-1},\xi_i;\eta)}, & \xi_1=0,\ \xi_k\neq n+1\\
b^{\sum_{i=1}^k \xi_i} \prod_{i=1}^k \mathbb{1}\{\eta_{\xi_i}=-1\} \prod_{i=1}^{k+1} t_i^{\mathcal{N}(\xi_{i-1},\xi_i;\eta)}, & 1\le \xi_1,\ \xi_k\le n\\
b^{n+\sum_{i=1}^{k-1} \xi_i} \prod_{i=1}^{k-1} \mathbb{1}\{\eta_{\xi_i}=-1\} \prod_{i=1}^{k} t_i^{\mathcal{N}(\xi_{i-1},\xi_i;\eta)}, & \xi_1\neq 0,\ \xi_k=n+1\\
b^{n+\sum_{i=1}^{k-1} \xi_i} \prod_{i=2}^{k-1} \mathbb{1}\{\eta_{\xi_i}=-1\} \prod_{i=2}^{k} t_i^{\mathcal{N}(\xi_{i-1},\xi_i;\eta)}, & \xi_1=0,\ \xi_k=n+1
\end{cases}
$$

In assessing the viability of this duality observable, We note that to achieve a similar result to lemma 5.1 in the rank-1 case, we require $t_1 = -\gamma/\alpha$, $t_{k+1} = -\beta/\delta$, and for $1 \le i \le k$,

$$t_{i+1}/t_i = t$$

which we contextualise as a condition analoguous to the *microscopic shock stability condition* in [1]. Multiplying through, we note that this implies the condition

$$t_{k+1}/t_1 = t^k$$

which is analogous to the *macroscopic shock stability condition* in [1]. Importantly, this condition specifies our rate parameter manifold

$$\alpha\beta - t^k\gamma\delta = 0.$$

We leave our discussion of higher-rank dualities at these key observations, noting that further work must be done to achieve a concrete result.

6 Conclusion

In this report we have introduced variations of the multispecies charged asymmetric simple exclusion process (m-CASEP) alongside the concept of duality and reverse duality in Markov processes. We constructed a factorised duality observable for a rank-one process and demonstrated a direct proof of the duality relationship. We have followed this work with a discussion of possible extensions to dual models of higher rank, drawing links to existing work on the problem found in [1]. There remains more work to be done in order to finalise these dualities.

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