# Amplitude Equations for Modelling Electromagnetically Induced Flows

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## Abstract

Electromagnetically driven flows in free liquid films have gained significant attention due to their theoretical and practical relevance in magnetohydrodynamics, microfluidics, and industrial processes. This report investigates the stability and bifurcation behavior of such flows, focusing on a thin electrolyte film subjected to a uniform magnetic field and radial electric current. The governing equations are derived under the lubrication approximation, leading to a weakly nonlinear analysis that produces an amplitude equation describing flow instabilities. Successive perturbation orders provide corrections to the amplitude equation, accounting for nonlinear effects and saturation mechanisms. A linear stability analysis identifies critical conditions for instability and bifurcation, revealing the emergence of non-trivial fixed points. The results highlight the occurrence of a transcritical bifurcation, where the trivial and non-trivial fixed points exchange stability, governing the transition between different flow states.


## Introduction

Electromagnetically driven flows in free liquid films have attracted much attention due to their importance in magnetohydrodynamic networks (Bau et al., 2003), and electromagnetic stirring (Bau, Zhong and Yi, 2001). These systems are of both theoretical interest and practical significance, with applications in microfluidics (Shang, Cheng and Zhao, 2017) and liquid bridges (Eksevora et al., 2018). The dynamics and stability of these films are influenced by various factors, including electromagnetic forces, fluid viscosity, surface tension, and boundary conditions, making them an ideal platform for exploring hydrodynamic instabilities and flow bifurcations. For instance, in microfluidics electromagnetic forcing provides a non-invasive method for controlling small fluid volumes, enabling advancements in lab-on-a-chip technologies, drug delivery systems, and diagnostics. Similarly, in industrial processes, such as froth flotation and wastewater treatment, the stability and behavior of thin films are critical for efficiency and effectiveness.

## 1 Problem Formulation

The experimental setup for studying these flows involves a horizontal free electrolyte film stretched between two coaxial cylindrical electrodes with radii $r_1$ and $r_2$, $r_2 > r_1$. A uniform vertical magnetic field $\mathbf{B} = (0, 0, B)$ is applied, and an electric current passes radially through the film. The interaction between the current and the magnetic field generates Lorentz force that drives fluid motion in the azimuthal direction. Key assumptions include: the fluid is incompressible and has constant density $\rho$, dynamic viscosity $\mu$, surface tension

$\gamma$, and electrical conductivity $\sigma$; the electrolyte film contains dissolved salts, ensuring electrical conductivity; the surface of the electrodes is assumed to be chemically inert and impenetrable to the salts dissolved in the film; small magnetic Reynolds number ensures that the induced magnetic field is negligible; it is defined as $Re_m = \mu_0 \sigma \sqrt{\frac{\gamma \langle h \rangle}{\rho}} Q$, where $\mu_0$ is the magnetic permeability, $Q$ is the Lorentz force parameter which will be discussed in section (2) and $h$ is the thickness of the film, which is small compared to the distance between the electrodes justifying the lubrication approximation. The fluid motion is described by the Navier-Stokes equations augmented by the Lorentz force term (Müller and Bühler, 2001),

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} + \frac{1}{\rho}\mathbf{j} \times \mathbf{B}\,, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{1.2}$$

where $\mathbf{u} = (u, v)$ represent the horizontal velocity components in the radial and azimuthal directions respectively, $t$ is the time, $p$ is the pressure, $\rho$ is the fluid density, $\nu = \frac{\mu}{\rho}$ represents kinematic viscosity, where $\mu$ is the dynamic viscosity. Current density is represented by $\mathbf{j}$ while the $\mathbf{B}$ represents magnetic flux density. Viscous dissipation is accounted by $\nabla^2 \mathbf{u}$. The Lorentz force per unit volume is represented by $\mathbf{j} \times \mathbf{B}$. Equation (1.1) is the momentum equation while equation (1.2) states the conservation of mass for incompressible fluids, meaning that the total mass remains constant within a closed system, even as the fluid flows and undergoes various changes.

Boundary conditions play a crucial role in defining the system's dynamics. The normal stress at the free surfaces ensures that the pressure is balanced by surface tension. The tangential stress accounts for surface tension gradients, which arise due to variations in surfactant concentration due to the soluto-Marangoni effect balancing the tangential viscous stresses. The kinematic boundary condition ensures that the normal velocity at the free surface is equal to the rate of change of the film height. The contact line dynamics describe how the contact angle deviates from its static value, as governed by the molecular kinetic model (Ruckenstein and Dunn, 1977; Slattery, 1991; Blake and Haynes, 1969; Blake, 2006) and relate this deviation to the velocity of the contact line. The no-slip condition ensures that both the tangential and normal velocities vanish on the electrode surfaces, except at the contact line, where partial slip conditions may apply depending on the wetting-line friction coefficient.

## 2 Governing Equations

Driven by Lorentz force, the system maintains a steady azimuthal base flow. However, when the applied voltage exceeds a critical value, this steady state destabilises. The reduced hydrodynamic model derived under the lubrication approximation, simplifies the Navier-Stokes equations for azimuthally invariant flow fields (Pototsky and Suslov, 2024). The governing equations that describe the destabilised fluid flow are,

$$\begin{aligned} -\frac{2\mathrm{Ca}}{h}\frac{\partial h}{\partial r}\left(\frac{u}{r} + 2\frac{\partial u}{\partial r}\right) + \mathrm{CaHa}^2 u - 4\mathrm{Ca}\left(-\frac{u}{r^2} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \\ -\frac{\partial^3 h}{\partial r^3} - \frac{1}{r}\frac{\partial^2 h}{\partial r^2} + \frac{1}{r^2}\frac{\partial h}{\partial r} + u\frac{\partial u}{\partial r} - \frac{v^2}{r} + \frac{\partial u}{\partial t} = 0\,, \end{aligned} \tag{2.1}$$

$$\begin{aligned} \frac{\mathrm{CaKQ}}{hr} - \frac{\mathrm{Ca}}{h}\frac{\partial h}{\partial r}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) - \mathrm{Ca}\left(-\frac{v}{r^2} + \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}\right) \\ +\frac{uv}{r} + u\frac{\partial v}{\partial r} + \frac{\partial v}{\partial t} = 0\,, \end{aligned} \tag{2.2}$$

$$\frac{hu}{r} + u\frac{\partial h}{\partial r} + h\frac{\partial u}{\partial r} + \frac{\partial h}{\partial t} = 0\,, \tag{2.3}$$

where constant $K$ is determined by requiring that the potential difference between the electrodes is equal to the (scaled) applied voltage. These equations describe complex interplay between various forces acting on the fluid. Incorporating both shear and extensional contributions to the stresses, the viscous terms account for the dissipation of fluid flow. Damping terms are introduced to capture the dissipative effects of friction, particularly near the contact line, where wetting-line friction influences the dynamics. Advection terms represent the non-linear transport of momentum due to the motion of the fluid itself contributing to the convective acceleration in both radial and azimuthal direction. The equations also incorporate several parameters that quantify the relative importance of different forces in the system. These parameters have been derived using the Buckingham Pi procedure and are given below

$$\mathrm{Ca} = \frac{\mu}{\sqrt{\rho\gamma\langle h\rangle}}\,, \quad \mathrm{Ha}^2 = \frac{B^2(r_2-r_1)^2\sigma}{\mu}\,, \quad \mathrm{Q} = \frac{\sigma BV(r_2-r_1)}{U\mu}\,.$$

The capillary number Ca characterises the ratio of viscous to surface tension forces. The Hartmann number Ha measures the strength of the Lorentz force relative to the viscous drag in the bulk of the fluid. The Lorentz force parameter Q quantifies the influence of electromagnetic force, which arise from the interaction of the radial electric current and the vertical magnetic field. Together, these terms and parameters define a system of coupled nonlinear partial differential equations that govern the evolution of the fluid velocity and the film thickness capturing the intricate dynamics of electromagnetically driven flows.

# 3 Weakly non-linear analysis

Given the non-linear nature, establishing analytical solutions of equations (2.2), (2.3), and (2.3) is a laborious task. Thus, to explore the dynamics of the unstable fluid flow a weakly nonlinear analysis is employed. Variables $h$, $u$, and $v$ are expanded in asymptotic series as

$$\begin{aligned} h &= h_{00} + \epsilon\left(Ah_{11} + \bar{A}\bar{h}_{11}\right) + \epsilon^2\left(A\bar{A}h_{20} + A^2h_{22} + \bar{A}^2\bar{h}_{22}\right) \\ &\quad + \epsilon^3\left(A^2\bar{A}h_{31} + A\bar{A}^2\bar{h}_{31} + A^3h_{33} + \bar{A}^3\bar{h}_{33}\right) + \ldots\,, \\ u &= u_{00} + \epsilon\left(Au_{11} + \bar{A}\bar{u}_{11}\right) + \epsilon^2\left(A\bar{A}u_{20} + A^2u_{22} + \bar{A}^2\bar{u}_{22}\right) \\ &\quad + \epsilon^3\left(A^2\bar{A}u_{31} + A\bar{A}^2\bar{u}_{31} + A^3u_{33} + \bar{A}^3\bar{u}_{33}\right) + \ldots\,, \\ v &= v_{00} + \epsilon\left(Av_{11} + \bar{A}\bar{v}_{11}\right) + \epsilon^2\left(A\bar{A}v_{20} + A^2v_{22} + \bar{A}^2\bar{v}_{22}\right) \\ &\quad + \epsilon^3\left(A^2\bar{A}v_{31} + A\bar{A}^2\bar{v}_{31} + A^3v_{33} + \bar{A}^3\bar{v}_{33}\right) + \ldots\,, \end{aligned}$$

where $\epsilon$ is a formal small parameter introduced to conveniently represent the size of small complex amplitude $A$, and the bar represents complex conjugates. The terms $h_{00}$, $u_{00}$, and $v_{00}$ correspond to the base flow while the higher-order terms $h_{ij}$, $u_{ij}$, $v_{ij}$, $i = 1, 2, 3$, $j = 0, 1, 2, 3$, represent the perturbed flow at respective successive orders of $\epsilon$. These terms describe the spatial distributions of the perturbations at each order. These expansions are substituted into the governing equations (2.2)–(2.3) which yields several systems of equations at successive orders of the perturbation parameter $\epsilon$.

At the zeroth-order, $\epsilon^0$, the system of equations derived are,

$$u_{00}h_{00}' + h_{00}u_{00}' + \frac{h_{00}u_{00}}{r} = 0\,, \tag{3.1}$$

$$\begin{aligned} -2\mathrm{Ca}\frac{h_{00}'}{h_{00}}\left(2u_{00}' + \frac{u_{00}}{r}\right) + \mathrm{Ca}\mathrm{Ha}^2u_{00} - 4\mathrm{Ca}\left(-\frac{u_{00}}{r^2} + u_{00}'' + \frac{u_{00}'}{r}\right) - h_{00}'''(r) \\ -\frac{h_{00}''}{r} + \frac{h_{00}'}{r^2} + u_{00}u_{00}' - \frac{v_{00}^2}{r} = 0\,, \end{aligned} \tag{3.2}$$

$$\begin{aligned} -\frac{\mathrm{Ca}h_{00}'}{h_{00}}\left(v_{00}' - \frac{v_{00}}{r}\right) + \frac{\mathrm{Ca}KQ}{rh_{00}} + \mathrm{Ca}\left(\frac{v_{00}}{r^2} - v_{00}'' - \frac{v_{00}'}{r}\right) \\ +u_{00}v_{00}' + \frac{u_{00}v_{00}}{r} = 0\,. \end{aligned} \tag{3.3}$$

Once the numerical solutions of $h_{00}$, $u_{00}$, and $v_{00}$ are obtained, the equations (2.2) – (2.3), are linearised about the base flow.

# 4 First-Order Analysis ($\mathcal{O}(\epsilon)$)

At the order $\epsilon$, the system of equations take the form of an eigenvalue problem which is then analysed to determine the growth rate of amplitude,

$$\mathcal{L}_\lambda \mathbf{w}_{11} = (\mathcal{A} + \lambda\mathcal{B})\, \mathbf{w}_{11} = \mathbf{0}\,, \tag{4.1}$$

where $\mathcal{L}_\lambda$, $\mathcal{A}$ and $\mathcal{B}$ are matrix-differential and matrix operators, respectively, arising from the linearised perturbation equations and boundary conditions. Explicitly, $\mathcal{B}$ is a matrix of the form,

$$\mathcal{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\,,$$

and $\mathcal{A}$ is a matrix-differential operator of the form,

$$\mathcal{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\,,$$

where the elements $\mathbf{a_{nm}}$ are

$$\begin{aligned}
a_{11} &= Du_{00} + u'_{00} + \frac{u_{00}}{r},\\
a_{12} &= Dh_{00} + h'_{00} + \frac{h_{00}}{r},\\
a_{13} &= 0,\\
a_{21} &= \text{Ca}\left(-4D^2 - \frac{4D}{r} - \frac{2h'_{00}}{rh_{00}} + \text{Ha}^2 + \frac{4}{r^2}\right) + D\left(h_{00} + h'_{00}\right),\\
a_{22} &= \text{Ca}\left(-\frac{4D}{h_{00}}\left(u'_{00} + \frac{u_{00}}{2r}\right) + \frac{2h'_{00}}{h_{00}^2}\left(2u'_{00} + \frac{u_{00}}{r}\right)\right)\\
&\quad - D^3 - \frac{D^2}{r} + \frac{D}{r^2},\\
a_{23} &= -\frac{2v_{00}}{r},\\
a_{31} &= \text{Ca}\left(-\frac{D}{h_{00}}\left(v'_{00} - \frac{v_{00}}{r}\right) + \frac{h'_{00}}{h_{00}^2}\left(v'_{00} - \frac{v_{00}}{r}\right) - \frac{KQ}{rh_{00}^2}\right),\\
a_{32} &= v'_{00} + \frac{v_{00}}{r},\\
a_{33} &= \text{Ca}\left(-D^2 - \frac{D}{r} + \frac{1}{r^2}\right) + \text{Ca}\frac{h'_{00}}{h_{00}}\left(-D + \frac{1}{r}\right) + Du_{00} + \frac{u_{00}}{r}.
\end{aligned}$$

$\mathbf{w}_{nm} = [h_{nm}, u_{nm}, v_{nm}]^{\mathbf{T}}$ is the vector of terms that represent the perturbed flow at higher orders of $\epsilon$ for $n = 1, 2, 3$ and $m \in$. Clearly any vector of the form $\mathbf{a}\mathbf{w}_{11}$, where $\mathbf{a}$ is some constant, is also a solution. If we solve equation (4.2), we obtain

$$\frac{dA}{dt} = \lambda A\,, \tag{4.2}$$

where $\lambda$ represents the growth rate of amplitude. This means that the solution of the linearised problem is $A \sim e^{\lambda t}$. If $\lambda > 0$, the solution will grow exponentially in time and most importantly, it would grow infinitely. From an energy point of view, this is nonphysical. Realistically what we expect is that the perturbations grow quickly initially but once their amplitude becomes large enough its growth slows down and eventually becomes zero when the amplitude saturates. From observation, it can be concluded that equation (4.2) does not take the saturation of the amplitude into account. We must derive an amplitude equation which describes takes into account of amplitude saturation. Let us analyse equations at highers orders to achieve the solution we are looking for.

# 5 Second-Order Analysis ($\mathcal{O}(\epsilon^2)$)

At the second order $\epsilon^2$, we obtain

$$\mathcal{L}_{2\lambda}\mathbf{w}_{22} + K_{22}\mathbf{w}_{11} = (\mathcal{A} + 2\lambda\mathcal{B})\,\mathbf{w}_{22} + K_{22}\mathbf{w}_{11} = \mathbf{f}_{22},, \tag{5.1}$$

$$\mathcal{L}_{\lambda+\bar{\lambda}}\mathbf{w}_{20} + K_{20}\mathbf{w}_{11} = \left(\mathcal{A} + (\lambda+\bar{\lambda})\mathcal{B}\right)\mathbf{w}_{20} + K_{20}\mathbf{w}_{11} = \mathbf{f}_{20}\,, \tag{5.2}$$

where the vector $\mathbf{f}_{nm} = [\mathbf{f}^{(1)}_{nm}, \mathbf{f}^{(2)}_{nm}, \mathbf{f}^{(3)}_{nm}]^{\mathbf{T}}$, represents the right-hand side of the matrix equation for $n = 1, 2, 3$ and $m \in$. The elements of vector $\mathbf{f}_{22}$ are defined as

$$\begin{aligned}
\mathbf{f}^{(1)}_{22} &= -h_{11}\left(u'_{11} + K_{22} + \frac{u_{11}}{r}\right) - u_{11}h'_{11}\,,\\
\mathbf{f}^{(2)}_{22} &= \mathrm{Ca}\left[\frac{h^2_{11}h'_{00}}{h^3_{00}}\left(4u'_{00} + \frac{2u_{00}}{r}\right) - \frac{h_{11}h'_{00}}{h^2_{00}}\left(4u'_{11} + \frac{2u_{11}}{r}\right) - \frac{h_{11}h'_{11}}{h^2_{00}}\left(4u'_{00} + \frac{2u_{00}}{r}\right)\right.\\
&\quad\left.+\frac{h'_{11}}{h_{00}}\left(4u'_{11} + \frac{2u_{11}}{r}\right)\right] - K_{22}u_{11} - u_{11}u'_{11} + \frac{v^2_{11}}{r},\\
\mathbf{f}^{(3)}_{22} &= \mathrm{Ca}\left[\frac{h^2_{11}h'_{00}}{h^3_{00}}\left(v'_{00} - \frac{v_{00}}{r}\right) - \frac{h_{11}h'_{00}}{h^2_{00}}\left(v'_{11} - \frac{v_{11}}{r}\right) - \frac{h_{11}h'_{11}}{h^2_{00}}\left(v'_{00} - \frac{v_{00}}{r}\right)\right.\\
&\quad\left.+\frac{h'_{11}}{h_{00}}\left(v'_{11} - \frac{v_{11}}{r}\right) - \frac{\mathrm{KQ}h^2_{11}}{rh^3_{00}}\right] - K_{22}v_{11} - u_{11}v'_{11} - \frac{u_{11}v_{11}}{r}.
\end{aligned}$$

The elements of the vector $\mathbf{f}_{20}$ are defined as,

$$\begin{aligned}
\mathbf{f}^{(1)}_{20} &= -h_{11}\left(\bar{u}'_{11} + K_{20} + \frac{\bar{u}_{11}}{r}\right) - \bar{h}_{11}\left(u'_{11} + K_{\bar{2}0} + \frac{u_{11}}{r}\right) - \bar{u}_{11}h'_{11} - u_{11}\bar{h}'_{11},\\
\mathbf{f}^{(2)}_{20} &= \mathrm{Ca}\left[\frac{h_{11}\bar{h}_{11}h'_{00}}{h^3_{00}}\left(8u'_{00} + \frac{4u_{00}}{r}\right) - \frac{h'_{00}}{h^2_{00}}\left(4h_{11}\bar{u}'_{11} + \frac{2h_{11}\bar{u}_{11}}{r}\right)\right.\\
&\quad\left.-\frac{\bar{h}_{11}h'_{11}u'_{00}}{h^2_{00}}\left(4 + \frac{2u_{00}}{r}\right) + \frac{\bar{h}'_{11}}{h_{00}}\left(4u'_{11} + \frac{2u_{11}}{r}\right)\right]\\
&\quad - K_{20}u_{11} - K_{\bar{2}0}\bar{u}_{11} + \frac{2v_{11}\bar{v}_{11}}{r},\\
\mathbf{f}^{(3)}_{20} &= \mathrm{Ca}\left[\frac{h_{11}\bar{h}_{11}h'_{00}}{h^3_{00}}\left(v'_{00} - \frac{v_{00}}{r}\right) - \frac{h'_{00}}{h^2_{00}}\left(h_{11}\bar{v}'_{11} - \frac{h_{11}\bar{v}_{11}}{r} + \bar{h}_{11}v'_{11} - \frac{\bar{h}_{11}v_{11}}{r}\right)\right.\\
&\quad\left.-\frac{\bar{h}_{11}h'_{11}v'_{00}}{h^2_{00}}\left(1 - \frac{v_{00}}{r}\right) - \frac{h_{11}\bar{h}'_{11}v'_{00}}{h^2_{00}}\left(1 - \frac{v_{00}}{r}\right) + \frac{h'_{11}}{h_{00}}\left(\bar{v}'_{11} - \frac{\bar{v}_{11}}{r}\right) + \frac{\bar{h}'_{11}}{h_{00}}\left(v'_{11} - \frac{v_{11}}{r}\right)\right]\\
&\quad - \frac{2\mathrm{CaKQ}h_{11}\bar{h}_{11}}{rh^3_{00}} - K_{20}v_{11} - K_{\bar{2}0}\bar{v}_{11} - u_{11}\bar{v}'_{11} - \frac{u_{11}\bar{v}_{11}}{r} - \bar{u}_{11}v'_{11} - \frac{\bar{u}_{11}v_{11}}{r}.
\end{aligned}$$

In equations (5.2) and (5.2), $\mathbf{w}_{22}$, $K_{22}$, $\mathbf{w}_{20}$, $K_{20}$ are unknown. In order to solve the equations, we apply orthogonality conditions,

$$\begin{aligned}
\langle \mathbf{w}_{22}, \mathcal{B}\mathbf{w}_{11}\rangle &= 0\,,\\
\langle \mathbf{w}_{20}, \mathcal{B}\mathbf{w}_{11}\rangle &= 0.
\end{aligned}$$

The orthogonality condition is applied to solve the second-order equations because the system of equations at this order is underdetermined. Specifically, the equations for $\mathbf{w}_{22}$ and $\mathbf{w}_{20}$ are coupled with the first-order solutions $\mathbf{w}_{11}$, and without additional constraints, the system would have infinitely many solutions. This condition eliminates the components of the second-order solutions that are linearly dependent on the first-order solutions. The amplitude equation must also be corrected at this stage to include quadratic terms proportional to $A^2$. This is because the second-order analysis introduces nonlinear interactions that were not captured in the first-order analysis. The corrected amplitude equation takes the form,

$$\frac{dA}{dt} = \lambda A + (K_{22}A^2 + K_{20}A\bar{A}).$$

The corrections introduce quadratic terms into the amplitude equation, ensuring that perturbations do not only grow exponentially but also saturate. To achieve higher accuracy for the asymptotic series expansions, we shall analyse the system of equations at $\epsilon^3$.

# 6 Third-Order Analysis ($\mathcal{O}(\epsilon^3)$)

At $\epsilon^3$, we further obtain,

$$\mathcal{L}_{3\lambda}\mathbf{w}_{33} + K_{33}\mathbf{w}_{11} = (\mathcal{A} + 3\lambda\mathcal{B})\,\mathbf{w}_{33} + K_{33}\mathbf{w}_{11} \quad = \quad \mathbf{f}_{33},, \tag{6.1}$$

$$\mathcal{L}_{2\lambda+\bar{\lambda}}\mathbf{w}_{31} + K_{31}\mathbf{w}_{11} = \left(\mathcal{A} + 2\lambda + \bar{\lambda}\mathcal{B}\right)\mathbf{w}_{31} + K_{31}\mathbf{w}_{11} \quad = \quad \mathbf{f}_{31}\,, \tag{6.2}$$

where the elements of vector $\mathbf{f}_{33}$ are defined as,

$$\mathbf{f}_{33}^{(1)} = -h_{11}\left(u'_{22} + \frac{u_{22}}{r} + K_{33}\right) - u_{22}h'_{11} - h_{22}\left(u'_{11} + \frac{u_{11}}{r} + 2K_{22}\right) - u_{11}h'_{22},$$

$$\begin{aligned}
\mathbf{f}_{33}^{(2)} = \mathrm{Ca}\Bigg[&\frac{4h_{11}h'_{00}}{h_{00}^3}\left(2h_{22}u'_{00} + \frac{h_{22}u_{00}}{r} - h_{11}^2u'_{00} - \frac{h_{11}^2u_{00}}{2r}\right) \\
&+ \frac{2h_{11}h'_{00}}{h_{00}^3}\left(2h_{11}u'_{11} + \frac{h_{11}u_{11}}{r} - 2u'_{22} - \frac{u_{22}}{r}\right) \\
&- \frac{2h'_{00}}{h_{00}^2}\left(2h_{22}u'_{11} + \frac{h_{22}u_{11}}{r} + 2h_{11}u'_{22} + \frac{h_{11}u_{22}}{r}\right) \\
&+ \frac{4h'_{11}}{h_{00}^2}\left(h_{11}u'_{00} + \frac{u_{00}h_{11}}{2r} - h_{22}u'_{00} - \frac{u_{00}h_{22}}{r}\right) \\
&+ \frac{4}{h_{00}}\left(h'_{11}u'_{22} + h'_{22}u'_{11}\right)\Bigg] \\
&- K_{33}u_{11} - 2K_{22}u_{22} - u_{11}u'_{22} - u_{22}u'_{11} + \frac{2v_{11}v_{22}}{r},
\end{aligned}$$

$$\begin{aligned}
\mathbf{f}_{33}^{(3)} = \mathrm{Ca}\Bigg[&\frac{h_{11}h'_{00}}{h_{00}^3}\left(2h_{22}v'_{00} - \frac{2h_{22}v_{00}}{r} - h_{11}^2v'_{00} + \frac{h_{11}^2v_{00}}{r}\right) \\
&+ \frac{h'_{00}}{h_{00}^3}\left(h_{11}^2v'_{11} - \frac{h_{11}^2v_{11}}{r}\right) - \frac{h'_{00}}{h_{00}^2}\left(h_{11}v'_{22} - \frac{h_{11}v_{22}}{r} + h_{22}v'_{11} - \frac{h_{22}v_{11}}{r}\right) \\
&+ \frac{h'_{11}}{h_{00}^3}\left(h_{11}^2v'_{00} - \frac{h_{11}^2v_{00}}{r}\right) - \frac{h'_{11}}{h_{00}^2}\left(h_{11}v'_{11} - \frac{h_{11}v_{11}}{r}\right) \\
&+ \frac{1}{h_{00}}\left(h'_{11}v'_{22} - \frac{h'_{11}v_{22}}{r} + h'_{22}v'_{11} - \frac{h'_{22}v_{11}}{r}\right) - \frac{2\mathrm{KQ}}{rh_{00}^3}\left(h_{11}h_{22} - \frac{h_{11}^3}{2h_{00}}\right)\Bigg] \\
&- K_{33}v_{11} - 2K_{22}v_{22} - u_{11}v'_{22} - \frac{u_{11}v_{22}}{r} - u_{22}v'_{11} - \frac{u_{22}v_{11}}{r}\,,
\end{aligned}$$

and the elements of vector $\mathbf{f}_{31}$ are defined as,

$$
\begin{aligned}
\mathbf{f}_{31}^{(1)} =& -\Big(h_{11}(u_{20}' + \frac{u_{20}}{r} + K_{31}) + u_{20}h_{11}'\Big) \\
& - \Big(\bar{h}_{11}(u_{22}' + \frac{u_{22}}{r}) + u_{22}\bar{h}_{11}'\Big) \\
& - \Big(h_{20}(u_{11}' + \frac{u_{11}}{r} + K_{\bar{20}} + K_{22}) + u_{11}h_{20}'\Big) \\
& - \Big(h_{22}(\bar{u}_{11}' + \frac{\bar{u}_{11}}{r} + 2K_{20}) + \bar{u}_{11}h_{22}'\Big), \\
\mathbf{f}_{31}^{(2)} =& \, 2\mathrm{Ca}\Bigg[\frac{h_{11}^2}{rh_{00}^3}\Big(\bar{u}_{11}h_{00}' + u_{00}\bar{h}_{11}'\Big) + \frac{2h_{11}^2 h_{00}'}{h_{00}^3}\Big(\bar{h}_{11}'u_{00}' + \bar{u}_{11}'\Big) \\
& - \frac{3h_{11}^2 h_{00}'}{rh_{00}^4}\Big(\bar{h}_{11}u_{00} + 2h_{11}u_{00}'\Big) \\
& + \frac{2h_{11}}{rh_{00}^3}\Big(h_{20}u_{00}h_{00}' + \bar{h}_{11}u_{11}h_{00}' + \bar{h}_{11}u_{00}h_{11}'\Big) \\
& + \frac{4h_{11}}{h_{00}^3}\Big(h_{20}h_{00}'u_{00}' + \bar{h}_{11}h_{11}'u_{00}' + \bar{h}_{11}h_{00}'u_{11}'\Big) \\
& - \frac{2h_{11}}{h_{00}^2}\Big(h_{20}'u_{00}' + \bar{h}_{11}'u_{11}' + h_{11}'\bar{u}_{11}' + h_{00}'u_{20}'\Big) \\
& - \frac{h_{11}}{rh_{00}^2}\Big(u_{20}h_{00}' + \bar{u}_{11}h_{11}' + u_{11}\bar{h}_{11}' + u_{00}h_{20}'\Big) \\
& - K_{31}u_{11} - \Big(K_{\bar{20}} + K_{22}\Big)u_{20} - 2K_{20}u_{22} + \frac{2}{r}\Big(v_{11}v_{20} + \bar{v}_{11}v_{22}\Big) \\
& + \frac{2}{rh_{00}^3}\Big(\bar{h}_{11}h_{22}u_{00}h_{00}' + u_{20}h_{11}' + u_{22}\bar{h}_{11}' + u_{11}h_{20}' + \bar{u}_{11}h_{22}'\Big) \\
& + \frac{2}{h_{00}}\Big(h_{20}'u_{11}' + h_{22}'\bar{u}_{11}' + h_{11}'u_{20}' + \bar{h}_{11}'u_{22}'\Big) \\
& - \frac{2}{h_{00}^2}\Big(h_{20}h_{11}'u_{00}' + h_{22}\bar{h}_{11}'u_{00}' + \bar{h}_{11}h_{22}'u_{00}' + h_{20}h_{00}'u_{11}' \\
& \quad + \bar{h}_{11}h_{11}'u_{11}' + h_{22}h_{00}'\bar{u}_{11}' + \bar{h}_{11}h_{00}'u_{22}'\Big) \\
& - \frac{1}{rh_{00}^2}\Big(h_{20}u_{11}h_{00}' + h_{22}\bar{u}_{11}h_{00}' + \bar{h}_{11}u_{22}h_{00}' + h_{20}u_{00}h_{11}' \\
& \quad + \bar{h}_{11}u_{11}h_{11}' + h_{22}u_{00}\bar{h}_{11}' + \bar{h}_{11}u_{00}h_{22}'\Big)\Bigg], \\
\mathbf{f}_{31}^{(3)} =& \frac{\mathrm{Ca}h_{11}^2}{rh_{00}^3}\Bigg[\frac{3\mathrm{KQ}\bar{h}_{11}}{h_{00}} + \frac{3\bar{h}_{11}v_{00}h_{00}'}{h_{00}} + \frac{\bar{h}_{11}'v_{00}'}{h_{11}} + \frac{h_{00}'\bar{v}_{11}'}{h_{11}} - \frac{\bar{v}_{11}h_{00}'}{h_{00}} - \frac{v_{00}\bar{h}_{11}'}{h_{00}}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{3\bar{h}_{11}h'_{00}v'_{00}}{h_{00}^2}\Bigg] + \frac{\text{Ca}h_{11}}{rh_{00}^2}\Bigg[v_{20}h'_{00} + \bar{v}_{11}h'_{11} + v_{11}\bar{h}'_{11} + v_{00}h'_{20} - \frac{2\text{KQ}h_{20}}{h_{00}} \\
&- \frac{2h_{20}v_{00}h'_{00}}{h_{00}} - \frac{2\bar{h}_{11}v_{11}h'_{00}}{h_{00}} - \frac{2\bar{h}_{11}v_{00}h'_{11}}{h_{00}} + h_{20}v_{11}h'_{00} + h_{22}\bar{v}_{11}h'_{00} \\
&+ \bar{h}_{11}v_{22}h'_{00} + h_{20}v_{00}h'_{11} + \bar{h}_{11}v_{11}h'_{11} + h_{22}v_{00}\bar{h}'_{11} + \bar{h}_{11}v_{00}h'_{22}\Bigg] \\
&+ \frac{\text{Ca}}{h_{00}^2}\Bigg[2h_{20}h'_{00}v'_{00}h_{11} + 2\bar{h}_{11}h'_{11}v'_{00}h_{11} + 2\bar{h}_{11}h'_{00}v'_{11}h_{11} - h'_{20}v'_{00}h_{11} \\
&- \bar{h}'_{11}v'_{11}h_{11} - h'_{11}\bar{v}'_{11}h_{11} - h'_{00}v'_{20}h_{11} - h_{20}h'_{11}v'_{00} - h_{22}\bar{h}'_{11}v'_{00} \\
&- \bar{h}_{11}h'_{22}v'_{00} - h_{20}h'_{00}v'_{11} - \bar{h}_{11}h'_{11}v'_{11} - h_{22}h'_{00}\bar{v}'_{11} - \bar{h}_{11}h'_{00}v'_{22}\Bigg] \\
&- \frac{\text{Ca}}{rh_{00}}\Big[v_{20}h'_{11} + v_{22}\bar{h}'_{11} + v_{11}h'_{20} + \bar{v}_{11}h'_{22}\Big] - \frac{2\text{CaKQ}}{rh_{00}^3}\Big[\bar{h}_{11}h_{22} + \bar{h}_{11}h_{22}v_{00}h'_{00}\Big] \\
&- \frac{1}{r}\Big[u_{20}v_{11} + u_{22}\bar{v}_{11} + u_{11}v_{20} + \bar{u}_{11}v_{22}\Big] - u_{20}v'_{11} - u_{22}\bar{v}'_{11} - u_{11}v'_{20} \\
&- \bar{u}_{11}v'_{22} + \frac{\text{Ca}}{h_{00}}\Big[h'_{20}v'_{11} + h'_{22}\bar{v}'_{11} + h'_{11}v'_{20} + \bar{h}'_{11}v'_{22}\Big].
\end{aligned}
$$

In equations (6.2) and (6.2), $\mathbf{w}_{33}$, $K_{33}$, $\mathbf{w}_{31}$, $K_{31}$ are unknown. Thus, we apply the orthogonality conditions,

$$
\begin{aligned}
\langle \mathbf{w}_{33}, \mathcal{B}\mathbf{w}_{11} \rangle &= 0\,, \\
\langle \mathbf{w}_{31}, \mathcal{B}\mathbf{w}_{11} \rangle &= 0.
\end{aligned}
$$

Again, we should correct our amplitude equation so that there are proportional terms at the order of $A^3$. These terms arise from the third-order nonlinear interactions and are necessary to capture the higher-order saturation effects that were not accounted for in the second-order analysis. Further correcting the amplitude equation we finally obtain,

$$
\frac{dA}{dt} = \lambda A + (K_{22}A^2 + K_{20}A\bar{A}) + (K_{33}A^3 + K_{31}A^2\bar{A}). \tag{6.3}
$$

The inclusion of these cubic terms ensures that the amplitude equation describes the saturation of perturbations more accurately which is essential for predicting the long-term behavior of the system.

Now that the electromagnetically driven flow has been modeled using an amplitude equation, a linear stability analysis is carried in order to determine when the fluid flow destabilises.

# 7 Linear Stability Analysis

It was stated in section (3) that the amplitude, $A$ is a complex quantity. Let us consider the amplitude $A = |A|e^{i\theta}$, where $|A|$ is the magnitude of the amplitude.

For mathematical simplicity, let the amplitude $A = |A|$. This simplifies equation (6.3) to,

$$
\frac{dA}{dt} = A\left(\lambda + A(K_{22} + K_{20}) + A^2(K_{33} + K_{31})\right).
$$

This equation is further simplified by considering $K_{22} + K_{20} = K_2$ and $K_{33} + K_{31} = K_3$. Thus we have,

$$\frac{dA}{dt} = \lambda A + K_2 A^2 + K_3 A^3 \,, \tag{7.1}$$

where $A = A(t)$, with parameters, $\lambda$ and K$_i$.

The fixed points are determined by considering that $\frac{dA}{dt} = 0$. Thus,

$$\Rightarrow \lambda A + K_2 A^2 + K_3 A^3 = 0,$$

$$\Rightarrow A\left(\lambda + K_2 A + K_3 A^2\right) = 0.$$

A trivial fixed point exists if $A = 0$. On the other hand, non-trivial fixed points exist if the quadratic equation $\lambda + K_2 A + K_3 A^2 = 0$ has real solutions for,

$$K_2^2 - 4\lambda K_3 \geq 0.$$

Let us consider $A = a$, where $a$ is a small quantity representing a small perturbation. This means the terms $a^2$ and $a^3$ are smaller and we can neglect $K_2 a^2$ and $K_3 a^3$. Thus,

$$\Rightarrow \frac{da}{dt} = \lambda a \Rightarrow a(t) = a_0 e^{\lambda t} \,, \tag{7.2}$$

where $a_0$ is the initial perturbation. If $\lambda < 0$, the solution will decay to zero. In other words we will approach the fixed point $A = 0$ even though initially we were away from it at $a_0$. The initial perturbation of a fixed point $a_0$ will decay in time. Such a fixed point is called stable. If $\lambda > 0$ then the initial perturbation $a_0$ will grow exponentially quickly and we will depart further and further away from the fixed point. In such a case, the fixed point is unstable. In this simple analysis we neglected nonlinear terms assuming that they are negligible. This procedure is called linearisation which always leads to a linear ordinary differential equation that has analytical exponential solution regardless of how complicated the original problem is. In this particular problem, in the case of an unstable fixed point, the amplitude becomes so large that our assumption of non-linear terms being negligible is not valid anymore since from an energy point of view it is unrealistic. Since non-linear effects do become important, let us investigate the stability of the non-trivial fixed points.

Let, $A = \bar{A} + a$, where $\bar{A}$ is a fixed point $a$ is a small quantity. Substituting this in equation (7.1) we obtain,

$$\frac{d\bar{A}}{dt} + \frac{da}{dt} = \lambda \bar{A} + \lambda a + K_2 \bar{A}^2 + 2K_2 \bar{A} a + K_3 \bar{A}^3 + 3K_3 \bar{A}^2 a + ...$$

Note that $\frac{d\bar{A}}{dt} = 0$ and again since $a$ is a small quantity, the terms $a^2$, $a^3$ are even smaller. Thus,

$$\bar{A} = 0 \,,$$

or

$$\bar{A} = \frac{-K_2 \pm \sqrt{K_2^2 - 4\lambda K_3}}{2K_3} \,.$$

Since $\frac{d\bar{A}}{dt}$, we are left with,

$$\frac{da}{dt} = a\left(\lambda + 2K_2 \bar{A} + 3K_3 \bar{A}^2\right) \,.$$

Let $\mu = \lambda + 2K_2 \bar{A} + 3K_3 \bar{A}^2$. Thus,

$$\frac{da}{dt} = \mu a \Rightarrow a(t) = a_0 e^{\mu t} \,,$$

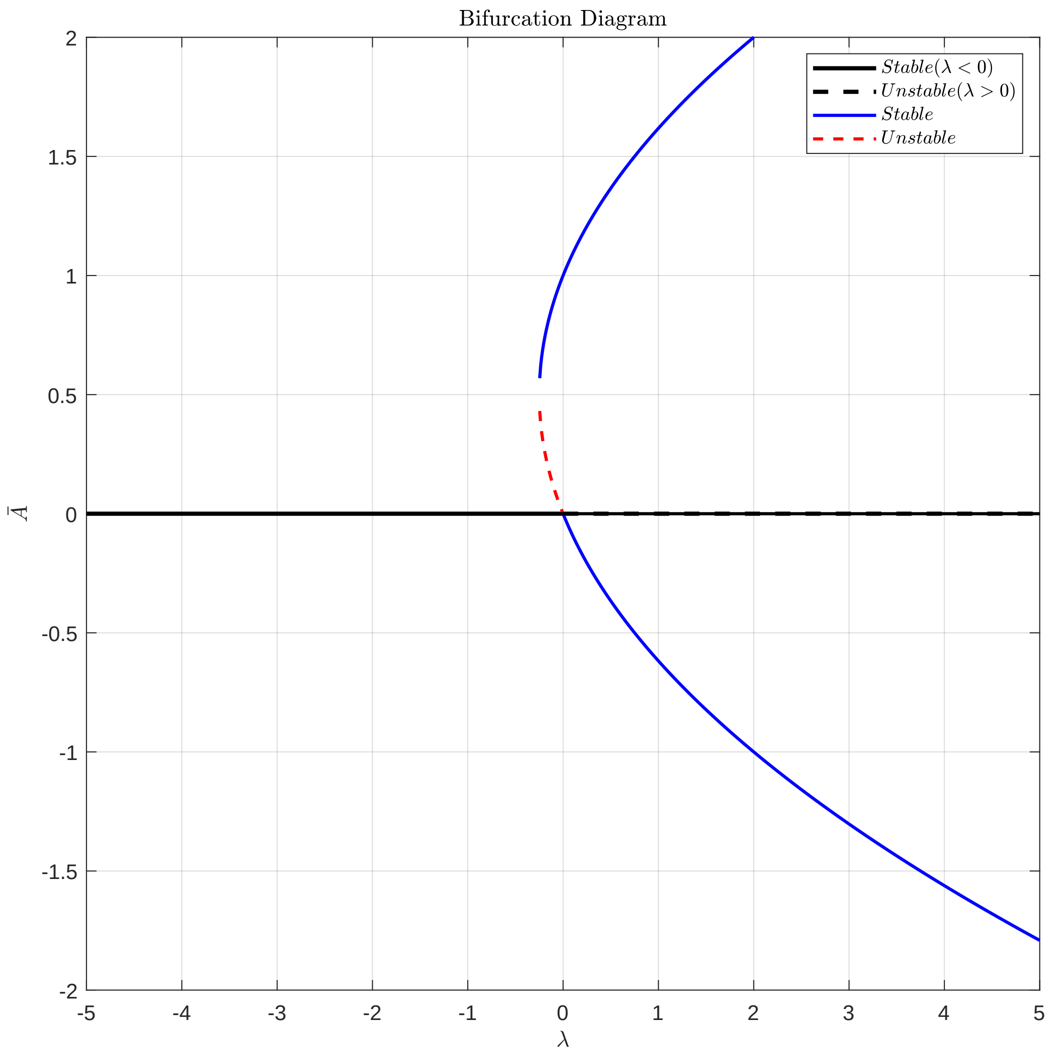


**Figure 1:** Case: $K_2 = -1$ and $K_3 = 1$

If $\bar{A} = 0$, then $\mu = \lambda$ and we again obtain equation (7.2). However, if $\bar{A} = \frac{-K_2 \pm \sqrt{K_2^2 - 4\lambda K_3}}{2K_3}$, then $\mu$ simplifies to

$$\mu = K_2\bar{A} + 2K_3\bar{A}^2.$$

If $\mu < 0$, $a_0$ will approach the fixed point $\bar{A}$. In this case, $\bar{A}$ is considered a stable fixed point. On the other hand, for $\mu > 0$ $a_0$ will move further away from the fixed point $\bar{A}$ and it is considered unstable.

Though similar conclusions have been reached for both the trivial and non-trivial fixed points, the sign of $\mu$ does depend on the values of $K_2$ and $K_3$. Figure 1 presents the bifurcation diagram for the amplitude $A$ as a function of the bifurcation parameter $\lambda$, using the numerical values $K_2 = -1$ and $K_3 = 1$. For $\lambda < 0$, the trivial fixed point at $\bar{A} = 0$ is stable. As $\lambda$ decreases further, two additional non-trivial fixed points emerge at $\bar{A} = \pm 0.5$. One of these fixed points move away from $\bar{A} = 0$ and remains stable, while the other initially approaches $\bar{A} = 0$ and is unstable before becoming stable again as it moves further away.

At $\lambda = 0$, the trivial fixed point $\bar{A} = 0$ exchanges stability with one of the non-trivial fixed point. Specifically, for $\lambda > 0$, the trivial fixed point becomes unstable, while the previously unstable non-trivial fixed point becomes stable as it moves away from $\bar{A} = 0$.

This stability exchange is characteristic of a transcritical bifurcation, where two fixed points intersect and swap their stability properties. The bifurcation diagram effectively captures this nonlinear behavior, demonstrating how the system transitions from a single stable equilibrium at $A = 0$ to a regime where multiple fixed points coexist and exchange stability.

# Conclusion

This report has analysed the stability of electromagnetically induced flows in a free electrolyte film. Through weakly nonlinear analysis, an amplitude equation was derived to capture the growth and saturation of perturbations. Higher-order corrections demonstrated the role of nonlinear interactions in stabilizing the system. Linear stability analysis revealed that when the bifurcation parameter exceeds a critical value, the trivial steady-state solution destabilizes, leading to the emergence of stable non-trivial fixed points. The bifurcation diagram confirmed the presence of a transcritical bifurcation, where the stability is exchanged between fixed points. Although the values of the coefficients $K_2$ and $K_3$ have been assumed to be -1 and 1 respectively for this report, they have to be computed through MATLAB, which the author of this report will aim to achieve for future prospects of this project.

# References

Bau, H.H., Zhong, J. and Yi, M., 2001. A minute magneto hydro dynamic (MHD) mixer. Sensors and Actuators B: Chemical, 79(2-3), pp.207-215.

Bau, H.H., Zhu, J., Qian, S. and Xiang, Y., 2003. A magneto-hydrodynamically controlled fluidic network. Sensors and actuators B: Chemical, 88(2), pp.205-216.

Blake, T.D. and Haynes, J.M., 1969. Kinetics of liquidliquid displacement. Journal of colloid and interface science, 30(3), pp.421-423.

Blake, T.D., 2006. The physics of moving wetting lines. Journal of colloid and interface science, 299(1), pp.1-13.

Ekserova, D. R., Exerowa, D., Gochev, G., Platikanov, D., Liggieri, L. and Miller, R. eds., 2018. Foam films and foams: Fundamentals and applications. CRC Press.

Müller, U. and Bühler, L., 2001. Magnetofluiddynamics in channels and containers. Springer Science & Business Media.

Pototsky, A. and Suslov, S.A., 2024. Effect of the dynamic contact angle on electromagnetically driven flows in free liquid films. Physics of Fluids, 36(7).

Ruckenstein, E. and Dunn, C.S., 1977. Slip velocity during wetting of solids. Journal of Colloid and Interface Science, 59(1), pp.135-138.

Shang, L., Cheng, Y. and Zhao, Y., 2017. Emerging droplet microfluidics. Chemical reviews, 117(12), pp.7964-8040.

Slattery, J.C., 1991. Moving common lines, thin films, and dynamic contact angles. In Modelling Macroscopic Phenomena at Liquid Boundaries (pp. 217-288). Vienna: Springer Vienna.