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Slices in Schottky Space

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Contents

Abstract

Fractal curves on the plane can be obtained as limit sets of the so-called Schottky groups. These are groups of non-singular complex matrices acting as fractional linear transformations on the extended complex plane. The Hausdorff dimension of the associated fractal is intimately associated with the geometry and algebra of the corresponding Schottky group. Coarse properties of the space of all Schottky groups are well understood since the 1970s through work of Jorgensen, Marden and Maskit. Recent theoretical and computational advances allow a finer study of Schottky space, and helped understand the groups and limit sets associated with two 2-dimensional slices in the 6-dimensional space of space of all 2-generator Schottky groups. This report explicitly finds a 1-parameter family of 2-generator Schottky groups, and utilises an iterative formula to approximate the Hausdorff dimension of the limit set.

1 Introduction

Fractals are geometric shapes that have contains detail at any arbitrarily small length scale. Some examples include the Koch snowflake and the Mandelbrot Set. Moreover, objects in real life such as leaves and coastlines are approximately fractals, and display many fractal-like properties, such as irregularity with self-similarity.

One way to construct a fractal curve on the complex plane is through the so-called Schottky groups. These are groups of non-singular complex matrices acting as fractional linear transformations on the extended complex plane. These transformations allows certain disks in the complex plane to be nested inside other disks. This nesting process can happen arbitrarily many times, creating an infinite nesting of disks. The collection of the points that arise in the smallest disks create a fractal. The fractal created and its properties depend on the Schottky group. Specifically, the Hausdorff dimension of the associated fractal is intimately associated with the geometry and algebra of the corresponding Schottky group.

This report explicitly finds a 1-parameter family of 2-generator Schottky groups and and utilises an iterative formula to approximate the Hausdorff dimension of the limit set.

1.1 Statement of Authorship

The work here is heavily based from [1]. Both the motivation for the Schottky group in Section 3, and the iterative formula for approximating the Hausdorff dimension in Section 2.2.1 come from [1]. Calculations such as in Section 3.1 and Section 3.2 are my own work. Any results and figures from other sources have been cited.

2 Background

2.1 Möbius transformations

The *extended complex plane* $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is the complex plane together with the point at infinity. Intuitively, the point at infinity is the single point that is infinitely far away from the origin. The standard operations extend to $\overline{\mathbb{C}}$, with $z + \infty = \infty$ and $z \times \infty = \infty$, $\forall z \in \overline{\mathbb{C}}$ (with $\infty - \infty$ and $0 \times \infty$ left undefined). Furthermore, this allows the division by zero, namely $z/0 = \infty$, $z/\infty = 0$, $\forall z \in \overline{\mathbb{C}}$ (with $0/0$ and ∞/∞ left undefined).

The group of *Möbius transformations* under function composition is

$$\text{Aut}(\overline{\mathbb{C}}) = \{M \mid M(z) = \frac{az+b}{cz+d},\ a,b,c,d \in \mathbb{C},\ ad - bc \neq 0\} \tag{1}$$

It turns out that the automorphisms of $\overline{\mathbb{C}}$ is precisely the set of Möbius transformations. They can be represented by 2×2 complex matrices through the following map, which respects function composition via matrix multiplication

$$\frac{az+b}{cz+d} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2}$$

Observe that a Möbius transformation does not change under scalar multiplication of the corresponding matrix. Thus, the group of Möbius transformations is isomorphic to the *projective special linear group* $PSL(2,\mathbb{C}) = SL(2,\mathbb{C})/\pm I$, where the *special linear group* $SL(2,\mathbb{C})$ is the group of complex 2×2 matrices of determinant 1, under matrix multiplication. Therefore, it is possible to talk about the trace of a Möbius transformation (up to multiplication by ± 1) without ambiguity, by taking the trace of the corresponding, unique (up to multiplication by ± 1), determinant 1 matrix.

Möbius transformations have several interesting properties. They are orientation preserving, analytic (and thus conformal) maps (except at $z = -d/c$). Additionally, Möbius transformations map *generalised circles* to generalised circles, where a generalised circle is either a circle or line in $\overline{\mathbb{C}}$. For any two such generalised circles, there exists Möbius transformations that maps one to the other. Möbius transformations have either one or two fixed points, which can be explicitly calculated. For $M(z) = (az+b)/(cz+d)$, if $c \neq 0$, then

$$z = M(z) = \frac{az+b}{cz+d} \Rightarrow z = \frac{a - d \pm \sqrt{(a-d)^2 + 4bc}}{2c} = \frac{a - d \pm \sqrt{\text{tr}(M)^2 - 4}}{2c} \tag{3}$$

and if $c = 0$, then

$$z = M(z) = \frac{az+b}{d} \Rightarrow z = \frac{b}{d-a}, \infty \tag{4}$$

Lemma 1. *The fixed points of Möbius transformations correspond to the eigenvectors of the corresponding matrix.*

Proof. Consider some Möbius transformation $M(z) = (az+b)/(cz+d)$, and let $A \in PSL(2,\mathbb{C})$ be the corresponding matrix. Then given $z \in \mathbb{C}$ (not $\overline{\mathbb{C}}$),

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix} = (cz+d) \begin{pmatrix} \frac{az+b}{cz+d} \\ 1 \end{pmatrix} \tag{5}$$

Thus $\begin{pmatrix} z \\ 1 \end{pmatrix}$ being an eigenvector of A is equivalent to z being a fixed point of M. Note that if $cz + d = 0$, then either A has determinant zero, or the fixed point of M is $\infty \in \overline{\mathbb{C}} \setminus \mathbb{C}$, neither of which is true.

If $M(\infty) = \infty$, then by Equation (4), $c = 0$ and so $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of A. Similarly, if $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of A, then $c = 0$, and thus ∞ is a fixed point of M. □

There are three classifications of Möbius transformations. One way to distinguish the classification is by looking at the nature of a fixed point. Fixed points may behave as sinks or sources. Intuitively, a sink is a point which brings neighbouring points closer to the point under the transformation, whilst a source takes neighbouring points further from the point.

Parabolic maps have only one fixed point, which is both a sink and a source. These maps are conjugate to translations $T(z) = z + a$. From Equation (3) and Equation (4), it can be shown that $\mathrm{tr}(M) = \pm 2$.

Elliptic maps have two neutral fixed points (neither sink or source). These maps are conjugate to rotations $T(z) = \lambda z$, $|\lambda| = 1$, and satisfy $\mathrm{tr}(M) \in (-2, 2)$.

Loxodromic maps have two fixed points, one which is a sink and the other a source. These maps are conjugate to scaling $T(z) = \lambda z$, $|\lambda| > 1$, and satisfy $|\mathrm{tr}(M)| > 2$. When $\lambda \in \mathbb{R}$, then the map is called *hyperbolic*. Hyperbolic maps also have real trace.

The rest of this report will freely interchange between Möbius transformations and their corresponding matrices, unless explicitly stated.

2.2 Schottky Groups

Take $2g$ disjoint disks in the complex plane, $D_1, D_2, \ldots, D_{2g}$, and choose Möbius transformations M_i which map the exterior of D_i to the interior of D_{i+g}. The *Schottky group* is the group generated by the M_i. It turns out that these Möbius transformations are loxodromic. This report focuses specifically on the 2-generator Schottky groups case, $g = 2$.

Note that the above are *classical* Schottky groups $\mathcal{J}_0$, which is the focus of this report. *Generalised* Schottky groups $\mathcal{J}$ use disjoint Jordan curves (and are not restricted to circles).

Lemma 2 (Vogt, Fricke)**.** *Consider an arbitrary 2-generator Schottky group $\langle A, B\rangle$. The pair $(A, B) \in PSL(2, \mathbb{C})^2$ is uniquely determined by the triple $(tr(A), tr(B), tr(AB)) \in \mathbb{C}^3$, up to conjugation.*

Proof. Conjugating A and B is equivalent to a choice of a basis for $\mathbb{C}^2$. So, choose a basis $\mathcal{B}' = \{v_A, v_B\}$, such that v_A is some eigenvector of A with eigenvalue λ, and v_B is some eigenvector of B with eigenvalue μ. Note that there are $2 \times 2 = 4$ unique choices of this basis, (up to scaling of eigenvectors) since A and B are loxodromic maps. Then in this basis,

$$A_{\mathcal{B}'} = \begin{pmatrix} \lambda & a' \\ 0 & \lambda^{-1} \end{pmatrix}, \quad B_{\mathcal{B}'} = \begin{pmatrix} \mu & 0 \\ b' & \mu^{-1} \end{pmatrix} \tag{6}$$

where the inverse elements are determined by the determinant 1 condition, and $a', b' \in \mathbb{C}$. Note that $a', b' \neq 0$, since the fixed points of A and B lie inside the associated disks which are disjoint, so A and B do not share any eigenvectors. Now, it is possible to adjust the choice of basis to $\mathcal{B} = \{\alpha_1 v_A, \alpha_2 v_B\}$ (since the basis vectors still satisfy being eigenvectors of A and B respectively) such that

$$A_{\mathcal{B}} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad B_{\mathcal{B}} = \begin{pmatrix} \mu & 0 \\ b & \mu^{-1} \end{pmatrix} \tag{7}$$

with $b \in \mathbb{C} \setminus \{0\}$. This gives

$$\mathrm{tr}(A_{\mathcal{B}}) = \lambda + \lambda^{-1} \tag{8}$$

$$\mathrm{tr}(B_{\mathcal{B}}) = \mu + \mu^{-1} \tag{9}$$

$$\mathrm{tr}((AB)_{\mathcal{B}}) = \lambda\mu + \lambda^{-1}\mu^{-1} + b \tag{10}$$

Given $\mathrm{tr}(A)$ and $\mathrm{tr}(B)$, it is possible to solve Equation (8) and Equation (9) for $2 \times 2 = 4$ unique solutions for λ and μ, which can then be used to solve Equation (10) to get a unique b. The four solutions for $(\lambda, \mu, b) \in \mathbb{C}^3$ correspond precisely with the four choices of the basis $\mathcal{B}$ (again up to scaling of the basis vectors). Thus, $(A, B) \in PSL(2, \mathbb{C})^2$ is uniquely determined by the triple $(\mathrm{tr}(A), \mathrm{tr}(B), \mathrm{tr}(AB)) \in \mathbb{C}^3$. □

Although all 2-generator Schottky groups have a corresponding point in $\mathbb{C}^3$, not every point in $\mathbb{C}^3$ has a Schottky group. For example, $(2, 2, 2) \in \mathbb{C}^3$ corresponds to the identity matrices, which clearly do not form a Schottky group. It is of interest to see which subspaces of $\mathbb{C}^3$ correspond to 2-generator Schottky groups. Since $\mathbb{C}^3$ is a six real dimensional space which is a vast space to study, many people look at two dimensional slices in this space. Some famous slices include the diagonal slice and the Riley slice, which consists of triples $(z, z, z) \in \mathbb{C}^3$, and $(2, 2, 2 + z) \in \mathbb{C}^3$ respectively, for $z \in \mathbb{C}$. Although strictly speaking the Möbius transformations are loxodromic (i.e. $\mathrm{tr}A, \mathrm{tr}B > 2$), it is possible to extend the definition of the Schottky group by taking the limit of the traces approaching 2 to have parabolic maps, which is the case for the Riley slice. In fact, this corresponds to the disks being tangentially connected.

Given the setup with $2g$ disks, the *fundamental domain*

$$\mathcal{F} := \overline{\mathbb{C}} \setminus \bigcup_{i=0}^{2g} D_i \tag{11}$$

is the region of the extended complex plane excluding the disks. For the Schottky group $\Gamma = \langle M_1, \ldots, M_g \rangle$, the *limit set* is defined as the following

$$\Lambda(\Gamma) := \overline{\mathbb{C}} \setminus \Gamma(\mathcal{F}) \tag{12}$$

Intuitively, the Möbius transformations applied to certain disks create smaller nested disks. Thus, composing them creates further nesting of disks. Applying certain compositions of the Möbius transformations on the original disks, will result in smaller, nested disks, and in the limit of infinite compositions, the disk will become a singular point. The limit set is the collection of all of these points, which creates a fractal set.

2.2.1 Dimension of the limit set

The Hausdorff or fractal dimension is a measure of "roughness" of a fractal. The fractal dimension agrees with the standard, topological, integer dimension for standard, smooth shapes. For example, both the Hausdorff and topological dimension of a point is 0, of a line is 1, plane is 2, and so on. In general, the Hausdorff dimension is an (extended) real number.

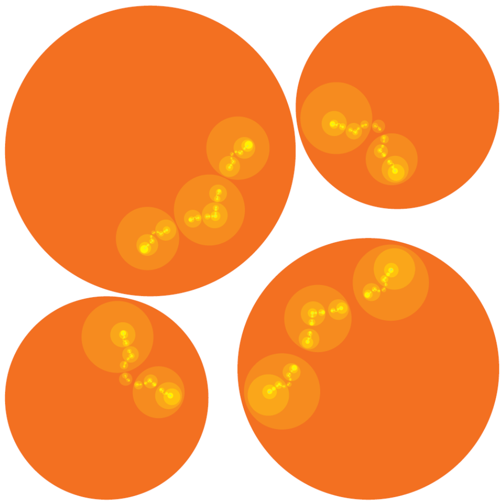

Figure 1: Example of nesting of disks from a 2-generator Schottky group, courtesy of [2]. The limit set is the collection of points inside the nested circles.

The dimension of the limit set can be explicitly calculated for Schottky groups. The following setup and formulas from this section is taken without proof from [1], and much more detail is given in that paper.

Definition 3 (Jenkinson, Pollicott)**.** Let A be a $g \times g$ matrix, with $A(i,j) = 1$ if (i,j) is *admissible*, and 0 otherwise. (i,j) is called admissible if there exists an associated $M_{ij} \in \{M_1, \ldots, M_g\} \subset \langle M_1, \ldots, M_g \rangle = \Gamma$ (which is unique) such that $M_{ij}(D_i) \subset \mathrm{int}(D_j)$. Construct M'_{ij} by restricting the domain of M_{ij} to D_i. An *admissible string* of length $n+1$ is a tuple $\underset{\sim}{i} = (i_1, \ldots, i_{n+1})$, such that $A(i_j, i_{j+1}) = 1$ for $j = 1, \ldots, n$. Call the map $M_{\underset{\sim}{i}} = M'_{i_{n+1}i_n} \circ M'_{i_n i_{n-1}} \circ \cdots \circ M'_{i_2 i_1}$ an *admissible n-fold composition.* Let T^n be the set of all admissible n-fold compositions. Then, define

$$a_n(s) := \frac{1}{n} \sum_{T^n z = z} \frac{|DT^n(z)|^{-s}}{\det(I - [DT^n(z)]^{-1})} \tag{13}$$

where $[DT^n(z)]$ denotes the Jacobian of T^n at z, and $|DT^n(z)|$ denotes the modulus of the real part of the derivative of T^n at z. Define

$$\Delta_N(s) := 1 + \sum_{n=1}^{N} \sum_{\substack{(n_1, \ldots, n_m) \\ n_1 + \cdots + n_m = n}} \frac{(-1)^m}{m!} a_{n_1}(s) \cdots a_{n_m}(s) \tag{14}$$

where the second summation is over all ordered m-tuples of positive integers whose sum is n. Then, the (Hausdorff) dimension of the limit set can be approximated by s_N, which is the largest zero of $\Delta_N(s)$, as

$$|\dim(\Lambda) - s_N| \leq C\delta^{N^{3/2}} \tag{15}$$

where $C > 0,\ 0 < \delta < 1$. Notice that the dimension converges super-exponentially in N, so practically, the dimension of the limit set can be well approximated even for low values of N.

3 McMullen's three circle family

Let $C_j = \{z \mid |z - z_j| = r\}$, where $z_j = ae^{\frac{2\pi i}{3}j}$, $a = \sec\frac{\theta}{2}$ and $r = \tan\frac{\theta}{2}$ for $0 < \theta < \frac{2\pi}{3}$ and $j \in \{0, 1, 2\}$. This gives $0 < r < \sqrt{3}$ and $1 < a < 2$. Let ρ_j be reflections in C_j, (which are in fact, orientation reversing). They take the form

$$\rho_j(z) = \frac{r^2}{\overline{z} - \overline{z_j}} + z_j \tag{16}$$

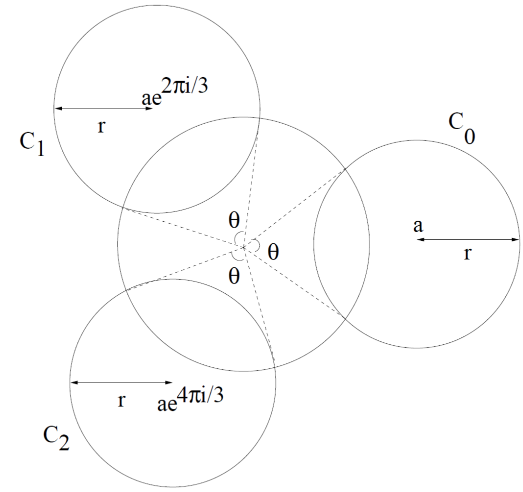

Figure 2: Diagram of three circle family, courtesy of [1]

3.1 Construction of a Schottky family

Consider $C_3 := \rho_0(\rho_1(C_1))$ and $C_4 := \rho_0(\rho_2(C_2))$, which creates two circles inside C_0. Denote with D_i the disk enclosed by C_i. Then, it can be shown that $A(z) = \rho_0\rho_1(z)$, $B(z) = \rho_0\rho_2(z)$ are Möbius transformations that map the exterior of D_1 to the interior of D_3 and the exterior of D_2 to the interior D_4 respectively. So let $\Gamma = \langle A, B\rangle$ be the Schottky group. The one parameter family of Schottky groups is created as a varies between 1 and 2. Explicitly computing A and B gives

$$A(a) = \frac{1}{a^2-1}\begin{pmatrix} 1 + a^2 e^{\frac{i\pi}{3}} & \sqrt{3}ae^{-\frac{i\pi}{6}} \\ \sqrt{3}ae^{\frac{i\pi}{6}} & 1 + a^2 e^{-\frac{i\pi}{3}} \end{pmatrix}, \quad B(a) = \frac{1}{a^2-1}\begin{pmatrix} 1 + a^2 e^{-\frac{i\pi}{3}} & \sqrt{3}ae^{\frac{i\pi}{6}} \\ \sqrt{3}ae^{-\frac{i\pi}{6}} & 1 + a^2 e^{\frac{i\pi}{3}} \end{pmatrix} \tag{17}$$

The fixed points of A are

$$z = \frac{a \pm \sqrt{3(4-a^2)} + i(\sqrt{3}a \mp \sqrt{4-a^2})}{4} \tag{18}$$

and the fixed points of B are

$$z = \frac{a \pm \sqrt{3(4-a^2)} - i(\sqrt{3}a \mp \sqrt{4-a^2})}{4} \tag{19}$$

and the corresponding triple in $\mathbb{C}^3$ (up to conjugation) is

$$(\mathrm{tr}A, \mathrm{tr}B, \mathrm{tr}AB) = \left(\frac{2+a^2}{a^2-1}, \frac{2+a^2}{a^2-1}, \frac{(2a^2+1)(a^2+2)}{(a^2-1)^2}\right) \tag{20}$$

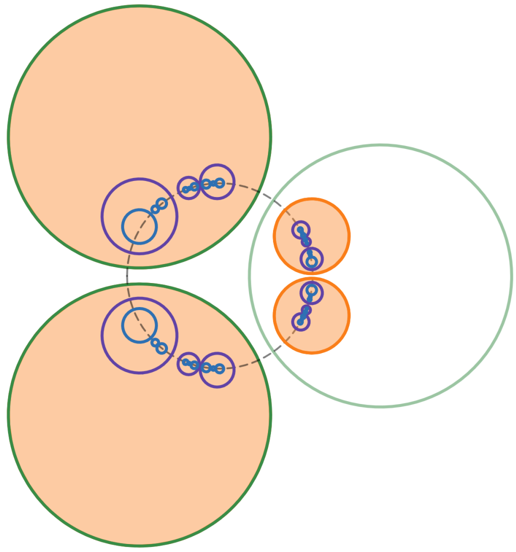

Figure 3: Schottky group for $a = 2$, with a few nestings of circles

It is also of interest to know what happens near the boundary of the family, namely as a approaches both 1 and 2.

As a approaches 2, all disks get larger and closer to each other, and when $a = 2$, the disks are tangent to each other. The trace triple approaches $(2, 2, 6)$, which corresponds to a point on the Riley slice. Thus, the maps indeed turn from loxodromic to parabolic maps.

As a approaches 1, all disks get smaller, and when $a = 1$, each disk become a point. Each component of trace triple grows without bound as a approaches 1, however not at the same rate. $\mathrm{tr}(AB)$ grows as $(a^2-1)^{-2}$, whilst $\mathrm{tr}A$ and $\mathrm{tr}B$ grow as $(a^2-1)^{-1}$.

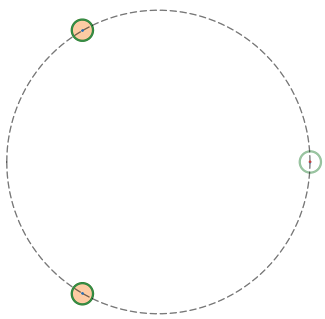

Figure 4: Schottky group as a approaches 1

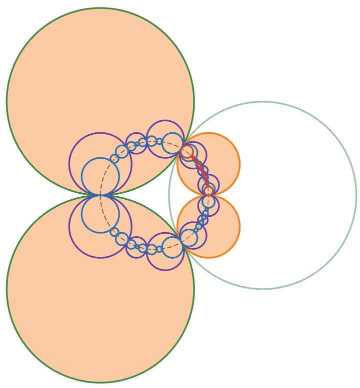

Figure 5: Schottky group for $a = 2$

3.2 Dimension of limit set

Notice that the reflections ρ_i fix the unit circle, and thus any compositions also fix the unit circle. Namely, since A and B are compositions of reflections, the limit set is a subset of the unit circle.

Lemma 4. *Recall that* $\Gamma = \langle A, B\rangle = \langle \rho_0\rho_1, \rho_0\rho_2\rangle$ *and* $\mathcal{F} = \overline{\mathbb{C}} \setminus \bigcup_{i=1}^4 D_i$. *Let* $\Gamma_- = \langle \rho_0, \rho_1, \rho_2\rangle$, *and* $\mathcal{F}_- = \overline{\mathbb{C}} \setminus \bigcup_{i=0}^2 D_i$. *Then*

$$\Lambda(\Gamma) = \Lambda(\Gamma_-) \tag{21}$$

Proof. Observe that $\Gamma_- = \Gamma \cup \rho_0\Gamma$. Then $\Gamma_-(\mathcal{F}_-) = \Gamma(\mathcal{F} \cup \rho_0\mathcal{F})$, and so

$$\Lambda(\Gamma_-) := \overline{\mathbb{C}} \setminus \Gamma_-(\mathcal{F}_-) = \overline{\mathbb{C}} \setminus \Gamma(\mathcal{F} \cup \rho_0\mathcal{F}) = \Lambda(\Gamma) \tag{22}$$

□

So to calculate the dimension of the limit set for the Schottky group, it is the same as calculating the dimension of the limit set for the reflection group Γ_-. Furthermore consider $f(z) = i(z-1)/(z+1)$, which is a bi-Lipschitz function that maps the unit circle to the real line. Then $T_i = f \circ \rho_i|_{S^1} \circ f^{-1}$ is a map from the real line to itself, and the dimension of the limit set generated by T_i is the same as the limit set generated by ρ_i (because the limit set of the reflection group is a subset of the unit circle, and f is bi-Lipschitz).

In fact, the iterative formula in Section 2.2.1 works exactly the same by replacing M_{ij} with T_{ij} (again, more details in [1]).

I personally calculated s_N for $N = 2, 3, 4$, utilising Desmos and Matlab. Detailed calculations are in Appendix A, with Matlab code in Appendix B. Note that b_i, c_i are merely intermediate calculations to get $a_i(s)$ and $\Delta_i(s)$.

$$c_2 = \frac{\left(a^2+2-a\sqrt{3(4-a^2)}\right)^2}{4(a^2-1)^2} \tag{23}$$

$$c_3 = \frac{\left(\sqrt{3}a(a+1)(a+2)-(a^2+2)(\sqrt{3}a+\sqrt{4a^2-1})\right)^2}{4(a^2-1)^3} \tag{24}$$

$$c_{41} = \frac{\left(\sqrt{3}a(a^2+2)(\sqrt{3}+\sqrt{4-a^2})+(a+1)(a^3-4a^2-4a-2)\right)^2}{4(a^2-1)^4} \tag{25}$$

$$c_{42} = \frac{\left(3a(2a^2+1+\sqrt{4a^4+a^2+4})-(a+1)(a+2)(2a^2+1)\right)^2}{4(a^2-1)^4} \tag{26}$$

$$b_2 = \left(1-(c_2)^2\right)^2 \tag{27}$$

$$b_3 = \left(1-(c_3)^2\right)^2 \tag{28}$$

$$b_{41} = \left(1-(c_{41})^2\right)^2 \tag{29}$$

$$b_{42} = \left(1-(c_{42})^2\right)^2 \tag{30}$$

$$a_2(s) = 3\frac{(c_2)^s}{b_2} \tag{31}$$

$$a_3(s) = 2\frac{(c_3)^s}{b_3} \tag{32}$$

$$a_4(s) = \frac{3}{2}\frac{(c_{41})^s}{b_{41}} + 3\frac{(c_{42})^s}{b_{42}} \tag{33}$$

$$\Delta_2(s) = 1 - a_2(s) \tag{34}$$

$$\Delta_3(s) = 1 - a_2(s) - a_3(s) \tag{35}$$

$$\Delta_4(s) = 1 - a_2(s) - a_3(s) - a_4(s) + \frac{1}{2}\left(a_2(s)\right)^2 \tag{36}$$

Since the limit set is a subset of the unit circle, the dimension should be a number between 0 and 1. By plotting the graph of the Δ_N above for $N = 2, 3, 4$, s_N seems to be well behaved for $0 < \theta \lesssim 1.8$. However, as θ approaches $2\pi/3$, s_N does not behave well (namely, s_N grows larger than 1 for s_2, s_3, and goes negative for s_4). So larger N is required for θ values closer to $2\pi/3$. On the other hand, as θ approaches 0, the dimension also approaches 0, which makes sense as the disks get smaller. The dimension of the limit set is a monotonic as a function of θ (i.e. the dimension grows monotonically from 0 to some upper bound (presumably 1) as θ ranges from 0 to $2\pi/3$).

Specifically for $a = \sec^{-1}(\pi/12)$ (corresponding to $\theta = \pi/6$),

$$s_2 = 0.146262063398418 \tag{37}$$

$$s_3 = 0.184185463230749 \tag{38}$$

$$s_4 = 0.183987414778089 \tag{39}$$

and so, the dimension of the limit set is approximately 0.18399. This generally agrees with the calculations done in [1].

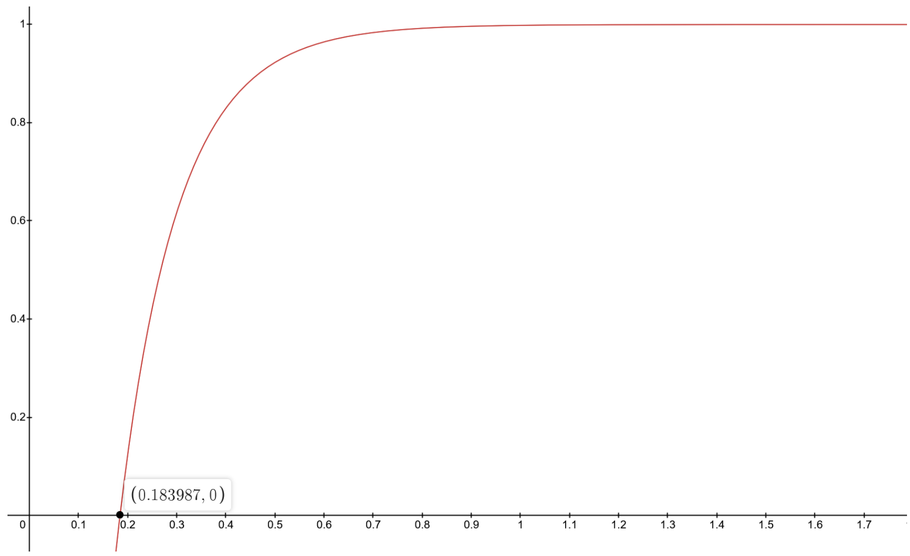

Figure 6: Plot of $\Delta_4(s)$ in desmos, along with s_4 for $\theta = \pi/6$

4 Discussion and Conclusion

In this report, we have constructed a one parameter family of 2-generator Schottky groups, motivated by the three circle reflection group. An iterative formula was then used to approximate the dimension of the corresponding limit set. Future work could involve a more detailed analysis on the limiting behaviours of the parameter a, specifically as a approaches both 1 and 2. Furthermore, the one parameter family could be turned into a two parameter family by making a complex (thus making it a 2-dimensional slice). This could involve taking the trace triple obtained in the one parameter case, extending the definition for a complex, and seeing which complex numbers make a Schottky group.

5 Acknowledgement

I would like to thank Professor Stephan Tillmann for the supervision of this project, and providing all the resources and references. I greatly appreciate all of the knowledge and help he has given me.

A Limit set calculation

Recall that

$$a_n(s) := \frac{1}{n} \sum_{T^n z = z} \frac{|DT^n(z)|^{-s}}{\det(I - [DT^n(z)]^{-1})} \tag{40}$$

$$\Delta_N(s) := 1 + \sum_{n=1}^{N} \sum_{\substack{(n_1,\ldots,n_m) \\ n_1+\cdots+n_m=n}} \frac{(-1)^m}{m!} a_{n_1}(s) \cdots a_{n_m}(s) \tag{41}$$

As outlined in Section 3.2, the transformations used will be compositions of the T_i, where $T_i = f \circ \rho_i|_{S^1} \circ f^{-1}$, $f(z) = i(z-1)/(z+1)$, $\rho_i(z) = \frac{r^2}{\overline{z}-\overline{z_i}} + z_i$. Computing gives

$$T_0(x) = \frac{a-1}{(a+1)x} = \begin{pmatrix} 0 & a-1 \\ a+1 & 0 \end{pmatrix} \tag{42}$$

$$T_1(x) = \frac{\sqrt{3}ax + (2+a)}{-(2-a)x - \sqrt{3}a} = \begin{pmatrix} \sqrt{3}a & 2+a \\ -(2-a) & -\sqrt{3}a \end{pmatrix} \tag{43}$$

$$T_2(x) = \frac{\sqrt{3}ax - (2+a)}{(2-a)x - \sqrt{3}a} = \begin{pmatrix} \sqrt{3}a & -(2+a) \\ 2-a & -\sqrt{3}a \end{pmatrix} \tag{44}$$

Note that T_i is a map from the real line to itself, but can be analytically extended to the extended complex plane. T_i are Möbius transformations, with fixed points which lie on the real line. It can be verified that the matrix from Section 2.2.1 which shows if a given pair is admissible is the following matrix

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{45}$$

Due to the diagonal elements being 0, the set of admissible 1-fold compositions is empty, so $a_1(s) = \Delta_1(s) = 0$.

For Möbius transformation $M(z) = (az+b)/(cz+d)$, the Jacobian matrix is

$$[DM(x+iy)] = \frac{ad-bc}{((cx+d)^2+(cy)^2)^2} \begin{pmatrix} (cx+d)^2-(cy)^2 & 2yc(cx+d) \\ -2yc(cx+d) & (cx+d)^2-(cy)^2 \end{pmatrix} \tag{46}$$

and so evaluating at any strictly real number gives

$$[DM(x)] = \frac{ad-bc}{(cx+d)^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{47}$$

$$\det(I - [DM(x)]^{-1}) = \left(1 - \left(\frac{(cx+d)^2}{ad-bc}\right)^2\right)^2 \tag{48}$$

Also,

$$DM(z) = \frac{ad-bc}{(cz+d)^2} \tag{49}$$

$$|DM(x)|^{-s} = \left|\frac{(cx+d)^2}{ad-bc}\right|^s \tag{50}$$

where $DM(z)$ is the real part of the derivative of M at z.

In the case of the three circle family, there are two symmetries that can be exploited. These symmetries are rotation by $2\pi/3$, and a reflection about the real line. What this means is that the derivative information above about fixed points for a transformation in one circle, correspond to derivative information for a similar transformation in a different circle. For example, $DT_0T_1(z_0) = DT_1T_2(z_1) = DT_2T_0(z_2)$ (rotational symmetry) and $DT_0T_1(z_0) = DT_0T_2(z_4)$ (reflection symmetry), where z_i are the relevant fixed points. Practically speaking, it is enough to do the above calculations for unique n-fold compositions starting in T_0, up to swapping T_1 and T_2. Note that by the admissibility restriction, T_i (and any compositions) only have one fixed point.

A.1 $N = 2$

There are six 2-fold compositions $(T_0T_1, T_0T_2, T_1T_0, T_1T_2, T_2T_0, T_2T_1)$, however utilising the symmetry, calculations only need to be done for one transformations, say T_0T_1. Let x_f denote the fixed point.

$$T_0T_1(z) = \begin{pmatrix} (a-1)(2-a) & \sqrt{3}a(a-1) \\ -\sqrt{3}a(a+1) & -(a+1)(2+a) \end{pmatrix} \tag{51}$$

$$x_f = \frac{-\sqrt{3} - \sqrt{4-a^2}}{a+1} \tag{52}$$

$$c_2 = \frac{(cx_f + d)^2}{ad - bc} = \frac{\left(a^2 + 2 - a\sqrt{3(4-a^2)}\right)^2}{4(a^2-1)^2} \tag{53}$$

$$b_2 = \left(1 - (c_2)^2\right)^2 \tag{54}$$

$$a_2(s) = \frac{1}{2}\left(6\frac{(c_2)^s}{b_2}\right) \tag{55}$$

$$\Delta_2(s) = 1 + \frac{(-1)^1}{1!}a_2(s) \tag{56}$$

A.2 $N = 3$

Again, there are six 3-fold compositions $(T_0T_1T_2, T_0T_2T_1, T_1T_0T_2, T_1T_2T_0, T_2T_0T_1, T_2T_1T_0)$, however utilising the symmetry, calculations only need to be done for one transformations, say $T_0T_1T_2$. Note that a function like $T_0T_1T_0$ is not a 3-fold composition, since $(1, 1)$ is not admissible (i.e. the reflection ρ_0 does not map D_0 to the interior of D_0). Let x_f denote the fixed point.

$$T_0T_1T_2(z) = \begin{pmatrix} \sqrt{3}a(a-1)(2-a) & -(a-1)(a^2+2) \\ -(a+1)(a^2+2) & \sqrt{3}a(a+1)(2+a) \end{pmatrix} \tag{57}$$

$$x_f = \frac{\sqrt{3}a + \sqrt{4a^2-1}}{a+1} \tag{58}$$

$$c_3 = \frac{(cx_f + d)^2}{ad - bc} = \frac{\left(\sqrt{3}a(a+1)(a+2) - (a^2+2)(\sqrt{3}a + \sqrt{4a^2-1})\right)^2}{4(a^2-1)^3} \tag{59}$$

$$b_3 = \left(1 - (c_3)^2\right)^2 \tag{60}$$

$$a_3(s) = \frac{1}{3}\left(6\frac{(c_3)^s}{b_3}\right) \tag{61}$$

$$\Delta_3(s) = 1 + \frac{(-1)^1}{1!}a_2(s) + \frac{(-1)^1}{1!}a_3(s) \tag{62}$$

A.3 $N = 4$

There are three 4-fold compositions up to symmetry. They are $T_0T_1T_0T_1$, $T_0T_1T_0T_2$, $T_0T_1T_2T_1$. In fact, it turns out that $T_0T_1T_0T_2$ and $T_0T_1T_2T_1$ give the same derivative information, since they are conjugate by a $2\pi/3$ rotation counterclockwise with the reflection ρ_0 in the original three circle setup. However, this proof is not shown here, and instead the calculations are simply done explicitly. Let x_f denote the fixed point.

$$T_0T_1T_0T_1(z) = \begin{pmatrix} (a-1)(a^3+4a^2-4a+2) & \sqrt{3}a(a-1)(a^2+2) \\ -\sqrt{3}a(a+1)(a^2+2) & (a+1)(a^3-4a^2-4a-2) \end{pmatrix} \tag{63}$$

$$x_f = \frac{-\sqrt{3}-\sqrt{4-a^2}}{a+1} \tag{64}$$

$$c_{41} = \frac{(cx_f+d)^2}{ad-bc} = \frac{\left(\sqrt{3}a(a^2+2)(\sqrt{3}+\sqrt{4-a^2}) + (a+1)(a^3-4a^2-4a-2)\right)^2}{4(a^2-1)^4} \tag{65}$$

$$b_{41} = \left(1-(c_{41})^2\right)^2 \tag{66}$$

$$T_0T_1T_0T_2(z) = \begin{pmatrix} (a-1)(2a-1)(a^2+2) & -3\sqrt{3}a^2(a-1) \\ -3\sqrt{3}a^2(a+1) & (a+1)(2a+1)(a^2+2) \end{pmatrix} \tag{67}$$

$$x_f = \frac{\sqrt{3}(a^2+2)+\sqrt{3}\sqrt{4a^4+a^2+4}}{3a(a+1)} \tag{68}$$

$$c_{42} = \frac{(cx_f+d)^2}{ad-bc} = \frac{\left(3a(2a^2+1+\sqrt{4a^4+a^2+4}) - (a+1)(a+2)(2a^2+1)\right)^2}{4(a^2-1)^4} \tag{69}$$

$$b_{42} = \left(1-(c_{42})^2\right)^2 \tag{70}$$

$$T_0T_1T_2T_1(z) = \begin{pmatrix} (a-1)(2-a)(2a^2+1) & -3\sqrt{3}a(a-1) \\ 3\sqrt{3}a(a+1) & -(a+1)(2+a)(2a^2+1) \end{pmatrix} \tag{71}$$

$$x_f = \frac{\sqrt{3}(2a^2+1)+\sqrt{3}\sqrt{4a^4+a^2+4}}{3(a+1)} \tag{72}$$

$$c_{43} = \frac{(cx_f+d)^2}{ad-bc} = c_{42} \tag{73}$$

$$b_{43} = b_{42} \tag{74}$$

$$a_4(s) = \frac{1}{4}\left(6\frac{(c_{41})^s}{b_{41}} + 12\frac{(c_{42})^s}{b_{42}}\right) \tag{75}$$

$$\Delta_4(s) = 1 + \frac{(-1)^1}{1!}a_2(s) + \frac{(-1)^1}{1!}a_3(s) + \frac{(-1)^1}{1!}a_4(s) + \frac{(-1)^2}{2!}(a_2(s))^2 \tag{76}$$

The dimension of the limit set can be found by plotting $\Delta_N(s)$ in some software, and finding the zero of the graph.

B Matlab code

```
theta = pi/6;
a = sec(theta/2);

syms s;

%a2
b2c1 = (a^2 + 2 -a*sqrt(3*(4-a^2)))/(2*(a^2-1));
b2d1 = (1-(b2c1)^4)^2;
```

```
    a2 = 3*(b2c1).^(2*s)/b2d1;

    %a3
    b3c1 = (sqrt(3)*a*(a+1)*(a+2)-(a^2+2)*(sqrt(3)*a+sqrt(4*a^2-1)))^2/(4*(a^2-1)^3);
    b3d1 = (1-(b3c1)^2)^2;
    a3 = 2*(b3c1).^(s)/b3d1;

    %a4
    b4c1 = (a*(a^2+2)*(3+sqrt(3*(4-a^2)))+(a+1)*(a^3-4*a^2-4*a-2))^2/(4*(a^2-1)^4);
    b4d1 = (1-(b4c1)^2)^2;
    b4c2 = (3*a*(2*a^2+1+sqrt(4*a^4+a^2+4))-(a+1)*(2+a)*(2*a^2+1))^2/(4*(a^2-1)^4);
    b4d2 = (1-(b4c2)^2)^2;
    a4 = (1/2)*(3*(b4c1).^(s)/b4d1 + 6*(b4c2).^(s)/b4d2);

    eqn = 1-a2-a3-a4+(a2).^(2)/2 == 0;

    %Intercept (dimension)
    format longg
    xInt = double(solve(eqn))
    yInt = zeros(size(xInt));

    %Plotting
    x=linspace(-0.05,1.3);

    a2 = 3*(b2c1).^(2*x)/b2d1;
    a3 = 2*(b3c1).^(x)/b3d1;
    a4 = (1/2)*(3*(b4c1).^(x)/b4d1 + 6*(b4c2).^(x)/b4d2);

    delta = 1-a2-a3-a4+(a2).^(2)/2;
    plot(x,delta,'k-')
    hold on
    plot(xInt,yInt, 'm*','MarkerSize', 10)
    yline(0)
    xline(0)
```

References

[1] Oliver Jenkinson and Mark Pollicott. Calculating Hausdorff dimensions of Julia sets and Kleinian limit sets. *Amer. J. Math.*, 124(3):495–545, 2002. 2, 6, 7, 9, 10

[2] David Mumford, Caroline Series, David James Wright, and David Wright. *Indra's pearls: The vision of Felix Klein*. Cambridge University Press, 2002. 6