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Vertex Operator Algebras and the Monster

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Abstract

The study of Vertex Operator Algebras is a relatively young field of mathematics which came to prominence due to its role in the solution of the Monstrous Moonshine conjectures. Vertex operators, initially studied by physicists working on the precursor to string theory known as dual resonance theory, first entered mathematics as part of the representation theory of infinite-dimensional Lie algebras known as affine Kac-Moody algebras. In this paper we introduce some of the basic techniques of vertex operator theory, ultimately working up to the twisted vertex operator realization of the simplest nontrivial affine Kac-Moody algebra $\mathfrak{sl}_2\mathbb{K}^{\sim}$.

Contents

1 Introduction

A crowning achievement of 20th century mathematics was the classification of finite simple groups, officially wrapped up in 2004 (see e.g. Aschbacher (2004)) as the culmination of over 10 thousand pages of mathematical proof. Although most of the finite simple groups can be constructed as the automorphisms of easily defined objects falling into regular infinite families, among these also lurk the much more mysterious sporadic groups, the largest of which is the so-called *Fischer-Griess Monster* or *Friendly Giant* $\mathbb{M}$ having order

$$|\mathbb{M}| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \tag{1.1}$$

with its smallest faithful representation on a complex vector space requiring $47 \cdot 59 \cdot 71 = 196{,}883$ dimensions. The Monster's identity is steeped in coïncidence and mystique: In the intervening time between the prediction of the group's existence and its construction, mathematicians had already accumulated an astonishing number of curious connections to seemingly disparate areas of mathematics, collectively termed *moonshine*.

The *modular group*

$$\Gamma = \mathrm{PSL}_2\,\mathbb{Z} = \mathrm{SL}_2\,\mathbb{Z}/\{\pm 1\} \tag{1.2}$$

acts on the upper-half complex plane $H = \{\tau \in \mathbb{C} : \Im(\tau) > 0\}$ via *Möbius transformations*

$$\gamma \odot \tau = \frac{a\tau + b}{c\tau + d}, \qquad \gamma = \pm \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma, \qquad \tau \in H \tag{1.3}$$

and a modular function is a meromorphic function on $H \cup \{i\infty\}$ which is invariant under Γ. It turns out that the set of all such functions is given by the field of rational functions of the so-called *J-invariant*

$$J(\tau) = \sum_{n\in\mathbb{Z}} a_n q^n = q^{-1} + 0 + 196884q + 21493760q^2 + \cdots \qquad q = \mathrm{e}^{2\pi i\tau} \tag{1.4}$$

It was McKay who first noticed the near coïncidence of a_1 with the minimal dimension of a faithful representation of $\mathbb{M}$, and a profusion of relations of this sort soon followed:

$$a_1 = d_0 + d_1 \tag{1.5}$$

$$a_2 = d_1 + d_2 + d_3 \tag{1.6}$$

$$a_3 = 2d_1 + 2d_2 + d_3 + d_4 \tag{1.7}$$

$$a_4 = 3d_1 + 3d_2 + d_3 + 2d_4 + d_5 \tag{1.8}$$

$$a_5 = 4d_1 + 6d_2 + 3d_3 + 2d_4 + d_5 + d_6 + d_7 \tag{1.9}$$

where d_n is the dimension of the nth smallest irreducible representation of $\mathbb{M}$ over $\mathbb{C}$. Many dismissed these concurrences as meaningless, after all any integer is expressible in terms of the dimensions of irreducible representations, since $d_0 = 1$! What was compelling were the small coëfficients involved in these expressions, which lead Thompson (1979) to conjecture the existence of a natural infinite-dimensional representation of $\mathbb{M}$

$$V^\natural = V_1 \oplus V_{-1} \oplus V_{-2} \oplus \cdots \tag{1.10}$$

such that $\dim V_{-n} = a_n$, whose rich structure should be responsible for the coïncidences of monstrous moonshine.

Up until this point, constructions of the Monster lacked the elegance and canonical character of the other sporadic groups. For example, the Mathieu groups arise naturally as the symmetries of the 24-bit Golay code, an exceptional error-correcting code; and the Conway groups arise as the symmetries of the Leech lattice, representing the best known sphere packing in 24 dimensions.

The eventual construction of the *moonshine module* $V^\natural$ involves vertex operators, bringing $\mathbb{M}$ into proximity with yet another area of mathematics: The theory of Lie algebras. A classificiation of finite-dimensional simple

Lie algebras over $\mathbb{C}$ was known as early as 1888, and later Kantor, Kac (1968), and Moody (1968) began studying infinite-dimensional generalizations of these, which came to be known as Kac-Moody algebras. An important class of these are the affine Kac-Moody algebras, the simplest nontrivial example of which is $\mathfrak{sl}_2\mathbb{K}^{\sim}$, the focus of this paper. Lepowsky and Wilson (1978) constructed the basic representation of $\mathfrak{sl}_2\mathbb{K}^{\sim}$ using apparently novel differential operators in infinitely many variables. It was Garland who first observed a similarity between these differential operators and the "vertex operators" physicists had been using in dual resonance theory, an early form of string theory, and the notions were shown to coïncide in Frenkel and Kac (1980). The concept of vertex operators was refined by Borcherds, who axiomatised vertex operator algebras an example being $V^\natural$.

In this paper, we lay out the foundations of the theory of vertex operators by studying them in their original mathematical context of the representation theory of $\mathfrak{sl}_2\mathbb{K}^{\sim}$. We introduce affine Lie algebras and the Heisenberg algebras which often arise as subalgebras, paying particular attention to the uniqueness of the so-called Heisenberg modules by an algebraic analogue to the Stone-Von Neumann theorem of quantum mechanics. We finish by constructing the twisted vertex operator realisation of $\mathfrak{sl}_2\mathbb{K}^{\sim}$ from an underlying Heisenberg module. This paper assumes basic knowledge from the theory of algebras over a field, in particular unital associative and Lie algebras, including the Poincaré-Birkhoff-Witt theorem, a treatment of which is found in Humphreys (1972). We also invoke some basic notions from category theory. We work over an algebraically closed field $\mathbb{K}$ of characteristic 0.

1.1 Statement of authorship

The majority of the historical background, definitions, and results are adapted or motivated from the introduction and first three chapters of Frenkel, Lepowsky, and Meurman (1988). Examples from physics are based on Schottenloher (2008). The proof of Dixmier's lemma in appendix A expands a sketch in Quillen (1969) and uses definitions from Roman (2008).

2 Formal calculus

Let V be a vector space over $\mathbb{K}$. We denote by $V\{z\}$ *formal sums* with exponents in $\mathbb{K}$ and coëfficients in V

$$\sum_{n\in\mathbb{K}} v_n z^n \in V\{z\} \tag{2.1}$$

which can be thought of as notation for a function $\mathbb{K} \to V$. Then $V\{z\}$ is a vector space, with the subspaces

- $V[\![z]\!]$ has exponents in $\mathbb{N}_0$ only, and is called *Taylor series* over V;
- $V[\![z, z^{-1}]\!]$ has exponents in $\mathbb{Z}$ only, and is called *Laurent series* over V;
- $V[z] = V \otimes \mathbb{K}[z]$ has exponents in $\mathbb{N}_0$ only and finitely many terms, and is called *polynomials* over V;
- $V[z, z^{-1}] = V \otimes \mathbb{K}[z, z^{-1}]$ has exponents in $\mathbb{Z}$ only and finitely many terms, and is called *Laurent polynomials* over V.

Given $v(z) = \sum_{n\in\mathbb{Z}} v_n z^n \in V[\![z, z^{-1}]\!]$ we define $v(\alpha z) = \sum_{n\in\mathbb{Z}} \alpha^n v_n z^n$ for $\alpha \in \mathbb{K}$, and if in addition $v(z) \in V[z, z^{-1}]$ we define evaluation similarly. There are well-defined bilinear multiplication maps defined in the obvious way:

$$V[z, z^{-1}] \times \mathbb{K}\{z\} \to V\{z\} \tag{2.2}$$

$$V[z, z^{-1}] \times \mathbb{K}[\![z, z^{-1}]\!] \to V[\![z, z^{-1}]\!] \tag{2.3}$$

For formal sums $\mathfrak{l}\{z\}$ over a Lie algebra $\mathfrak{l}$, we extend the bracket to a bilinear map

$$[-, -] : \mathfrak{l}\{z_1\} \times \mathfrak{l}\{z_2\} \to \mathfrak{l}\{z_1, z_2\} \tag{2.4}$$

so that given $x_m, y_n \in \mathfrak{l}$ for all $m, n \in \mathbb{K}$

$$\left[\sum_{m\in\mathbb{K}} x_m z_1^m, \sum_{n\in\mathbb{K}} y_n z_2^n\right] = \sum_{m,n\in\mathbb{K}} [x_m, y_n] z_1^m z_2^n \tag{2.5}$$

We fix the notation $D = z\frac{d}{dz}$ for the 'degree operator'[1] on $V\{z\}$

$$D : V\{z\} \to V\{z\} \tag{2.6}$$

$$vt^n \to nvt^n \tag{2.7}$$

for $n \in \mathbb{K}$, and use $D_j = D_{z_j} = z_j \frac{d}{dz_j}$ in case multiple variables are present. If V is a $\mathbb{K}$-graded vector space with degree operator $d \in \operatorname{End}_{\mathbb{K}} V$, and letting

$$v(z) = \sum_{n\in\mathbb{K}} v_n z^n \in V\{z\} \qquad\qquad X(z) = \sum_{n\in\mathbb{K}} x(n) z^{-n} \in (\operatorname{End} V)\{z\} \tag{2.8}$$

where $v_n \in V$ and $x(n) \in \operatorname{End} V$ for all $n \in \mathbb{K}$ we have

$$Dv(z) = dv(z) \iff (\forall n \in \mathbb{K})[\deg v_n = n] \tag{2.9}$$

$$-DX(z) = [d, X(z)] \iff (\forall n \in \mathbb{K})[\deg x(n) = n] \tag{2.10}$$

Often we will deal with formal sums $(\operatorname{End}_{\mathbb{K}} V)\{z\}$ over endomorphisms, in which case we can define algebraic operations under certain circumstances. We say a family $\{x_i\}_{i\in I} \subseteq \operatorname{End}_{\mathbb{K}} V$ of endomorphisms is *summable* iff for all $v \in V$ we have $x_i v = 0$ for all but finitely many $i \in I$, whence

$$\sum_{i\in I} x_i : V \to V \tag{2.11}$$

$$v \mapsto \sum_{i\in I} x_i v \tag{2.12}$$

The sum of a family $\{X_i(z)\}_{i\in I} \subseteq (\operatorname{End}_{\mathbb{K}} V)\{z\}$ with $X_i(z) = \sum_{n\in\mathbb{K}} x_i(n) z^n$ exists iff $\{x_i(n)\}_{i\in I}$ are summable for all $n \in \mathbb{K}$, whence

$$\sum_{i\in I} X_i(z) = \sum_{n\in\mathbb{K}} \left(\sum_{i\in I} x_i(n)\right) z^n \tag{2.13}$$

1. D cannot be a true degree operator as $V\{z\}$ is not $\mathbb{K}$-graded, however it is a degree operator on subspaces such as $V[z]$

The product of a finite sequence $(X_i(z))_{i=1}^r \subseteq (\mathrm{End}_{\mathbb{K}} V)\{z\}$ with $X_i(z) = \sum_{n\in\mathbb{K}} x_i(n)z^n$ exists iff for every $n \in \mathbb{K}$ the set

$$P_n = \left\{ \prod_{i=1}^r x_i(n_i) : \sum_{i=1}^r n_i = n \wedge \{n_i\}_{i=1}^r \subseteq \mathbb{K} \right\} \tag{2.14}$$

is summable, whence

$$\prod_{i=1}^r X_i(z) = \sum_{n\in\mathbb{K}} \left(\sum_{p\in P_n} p \right) z^n \tag{2.15}$$

Finally, let

$$X(z_1, z_2) = \sum_{m,n\in\mathbb{K}} x(m,n) z_1^m z_2^n \in (\mathrm{End}\, V)\{z_1, z_2\} \tag{2.16}$$

Then $\lim_{z_1\to z_2} X(z_1, z_2)$ exists iff for every $n \in \mathbb{K}$ the family $\{x(m, n-m)\}_{m\in\mathbb{K}}$ is summable, and is given by

$$\lim_{z_1\to z_2} \left(\sum_{m,n\in\mathbb{K}} x(m,n) z_1^m z_2^n \right) = \sum_{n\in\mathbb{K}} \left(\sum_{m\in\mathbb{K}} x(m, n-m) \right) z_2^n \tag{2.17}$$

We can mimic many techniques from calculus formally. An important object is the *formal exponential*

$$\exp(z) = \mathrm{e}^z = \sum_{n\in\mathbb{N}_0} \frac{z^n}{n!} \in \mathbb{K}[\![z]\!] \tag{2.18}$$

Theorem 2.1. *Let A be an associative algebra over $\mathbb{K}$ and Δ be a derivation on A, and $x \in A$ such that e^x exists and x commutes with $\Delta[x]$. Then*

$$\Delta[\mathrm{e}^x] = \Delta[x]\mathrm{e}^x \tag{2.19}$$

Proof. By the Leibniz rule, $\Delta[x^n] = nx^n\Delta[x]$. Thus

$$\Delta[\mathrm{e}^x] = \Delta\left[\sum_{n\in\mathbb{N}_0} \frac{x^n}{n!} \right] = \Delta[x] \sum_{n\in\mathbb{N}} \frac{x^{n-1}}{(n-1)!} = \Delta[x] \sum_{n\in\mathbb{N}_0} \frac{x^n}{n!} = \Delta[x]\mathrm{e}^x \tag{2.20}$$

as claimed. □

3 Some representation theory

Consider a unital associative algebra A over $\mathbb{K}$. By a *representation* of A on a vector space M over $\mathbb{K}$, we understand a unital algebra homomorphism $\Gamma : A \to \mathrm{End}_{\mathbb{K}}(M)$. It is useful to change perspective, and think of (M, π) as a single object. This unifying concept is the *A-module*. Recall that given a ring R, a (left) R-module M is an abelian group $(M, +)$ with a left R-action satisfying distributivity properties analogous to a vector space. A homomorphism $f : M \to N$ of R-modules M, N is a map satisfying $f(pu + qv) = pf(u) + qf(v)$ for any $p, q \in R$ and $u, v \in M$, and together with R-modules these form the category ${}_R\mathsf{Mod}$. A submodule $N \leq M$ is a subset which is an R-module under the same operations. Since $\mathbb{K}$ forms a subring of A, an A-module M is a vector space over $\mathbb{K}$ equipped with an appropriate left A-action, and an A-module homomorphism is $\mathbb{K}$-linear. We will typically denote the left-action by $(\odot)$.

Now by a *representation* of a Lie algebra $\mathfrak{g}$ on a vector space V, we understand a Lie algebra homomorphism $\Gamma : \mathfrak{g} \to \operatorname{End}_{\mathbb{K}}(M)$. By the universal property of the universal enveloping algebra, this induces a unique representation[2] of the unital associative algebra $U(\mathfrak{g})$. Hence we may identify a representation of a Lie algebra $\mathfrak{g}$ with a $U(\mathfrak{g})$-module. Hereinafter the terms $\mathfrak{g}$-module and $U(\mathfrak{g})$ module are used interchangeably.

Given a unital associative subalgebra $B \leq A$, we would like a way to canonically extend any B-module to an A-module. The A-module *induced* by the B-module V is a pair consisting of an A-module $\operatorname{Ind}_B^A V = A \otimes_B V$ and a B-module homomorphism $\iota : V \to \operatorname{Ind}_B^A V$ such that given any A-module W and B-module homomorphism $f : V \to W$, there exists a unique A-module homomorphism $\bar{f} : \operatorname{Ind}_B^A V \to W$ such that $f = \bar{f}\iota$.

$$\begin{array}{ccc} V & \xrightarrow{\;\iota\;} & \operatorname{Ind}_B^A V \\ & \searrow^{f} & \Big\downarrow {\scriptstyle \exists! \bar{f}} \\ & & W \end{array} \tag{3.1}$$

The induced module is easily constructed as the quotient vector space

$$A \otimes_B V = \frac{A \otimes_{\mathbb{K}} V}{\langle ab \otimes v - a \otimes b \cdot v : a \in A, b \in B, v \in V \rangle_{\leq_{\mathbb{K}} V}} \tag{3.2}$$

Remark 3.1. *The induced module is also defined for Lie algebras: If $\mathfrak{h} \leq \mathfrak{g}$ is a Lie subalgebra, and V is a $\mathfrak{h}$-module, then $\operatorname{Ind}_{\mathfrak{h}}^{\mathfrak{g}} V = U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V$.*

A module is called *simple* or *irreducible* iff it has no nontrivial proper submodules. We will need:

Theorem 3.2 (Schur's lemma)**.** *Let V, W be simple R-modules. Then any nonzero module homomorphism $f : V \to W$ is an isomorphism. In particular, the endomorphism ring $\operatorname{End}_R(V)$ of a simple module V is a division algebra.*

Proof. Since $\ker f \leq V$ and $\operatorname{im} f \leq W$ are submodules of simple modules, they must either be trivial or equal to V and W respectively. If $f \neq 0$ then $\ker f \neq V$ and $\operatorname{im} f \neq 0$, hence f is epic and monic and thus an R-module isomorphism □

Corollary 3.3. *If $\mathbb{K}$ is algebraically closed, A is a unital associative algebra over $\mathbb{K}$, and V is an A-module such that $\dim_{\mathbb{K}} V < |\mathbb{K}|$, then $\operatorname{End}_A V = \mathbb{K}$.*

Proof. See appendix A. □

4 Affine Lie algebras

Given a Lie algebra $\mathfrak{g}$ over $\mathbb{K}$, the *Loop algebra* is defined as the tensor product algebra[3]

$$\mathfrak{g}[t, t^{-1}] = \mathfrak{g} \otimes \mathbb{K}[t, t^{-1}] \tag{4.1}$$

2. As an abuse of notation we use the same symbol for the representation of $\mathfrak{g}$ and $U(\mathfrak{g})$.
3. This notation agrees with (2.5).

giving the Lie bracket structure

$$[xt^n, yt^m] = [x, y]t^{n+m} \tag{4.2}$$

for $x, y \in \mathfrak{g}$ and $n, m \in \mathbb{Z}$.

A bilinear form $\langle -, - \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ on a Lie algebra $\mathfrak{g}$ is said to be *invariant* or $\mathfrak{g}$-*invariant* iff the following identity holds

$$\langle x, [y, z]\rangle = \langle [x, y], z\rangle \tag{4.3}$$

A *quadratic Lie algebra* $\mathfrak{g}$ is a Lie algebra with a symmetric $\mathfrak{g}$-invariant bilinear form. The *affinization* $\hat{\mathfrak{g}}$ of a quadratic Lie algebra $\mathfrak{g}$ is a certain central extension of the loop algebra

$$0 \to \mathbb{K}c \hookrightarrow \hat{g} \twoheadrightarrow \mathfrak{g}[t, t^{-1}] \to 0 \tag{4.4}$$

Theorem 4.1. *Let $\mathfrak{g}$ be an algebra equipped with a bilinear form $\langle -, - \rangle$, with no additional assumed structure and define the vector space*

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{K}[t, t^{-1}] \oplus \mathbb{K}c \tag{4.5}$$

for nonzero c, with algebra structure given by the bilinear product $[-, -] : \hat{\mathfrak{g}} \times \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$

$$[c, \hat{\mathfrak{g}}] = [\hat{\mathfrak{g}}, c] = 0 \tag{4.6}$$

$$[xt^n, yt^m] = [x, y]t^{n+m} + \delta_{n+m} n\langle x, y\rangle c \tag{4.7}$$

where $\delta_n = \delta_n^0$ is the Kronecker delta. Then $\hat{\mathfrak{g}}$ is a Lie algebra iff $\mathfrak{g}$ and $\langle -, - \rangle$ form a quadratic Lie algebra.

Proof. First note the bracket on $\hat{\mathfrak{g}}$ is alternating iff that on $\mathfrak{g}$ is. Let $N = n + m + k$. Then the Jacobi identity on $\hat{\mathfrak{g}}$ is equivalent to

$$0 = [xt^n, [yt^m, zt^k]] + [yt^m, [zt^k, xt^n]] + [zt^k, [xt^n, yt^m]] \tag{4.8}$$

$$= [xt^n, [y, z]t^{m+k} + \cancel{C_1 c}] + [yt^m, [z, x]t^{k+n} + \cancel{C_2 c}] + [zt^k, [x, y]t^{n+m} + \cancel{C_3 c}] \tag{4.9}$$

$$= ([x, [y, z]] + [y, [z, x]] + [z, [x, y]])\, t^N + (\langle x, [y, z]\rangle n + \langle y, [z, x]\rangle m + \langle z, [x, y]\rangle k)\, \delta_{N,0} c \tag{4.10}$$

which holds iff the Jacobi identity holds for $\mathfrak{g}$ along with

$$\langle x, [y, z]\rangle n + \langle y, [z, x]\rangle m + \langle z, [x, y]\rangle k = 0 \tag{4.11}$$

for all n, m, k such that $n + m + k = 0$. The latter is equivalent to the bilinear map being symmetric and $\mathfrak{g}$-invariant, as can be shown by varying n, m, k. □

The *(untwisted) affine Lie algebra* $\hat{\mathfrak{g}}$ is the Lie algebra defined in thm. 4.1 with the $\mathbb{Z}$-grading $\hat{\mathfrak{g}} = \bigoplus_{n \in \mathbb{Z}} \hat{\mathfrak{g}}_n$ where

$$\hat{\mathfrak{g}}_n = \begin{cases} \mathfrak{g}t^0 \oplus \mathbb{K}c & n = 0 \\ \mathfrak{g}t^n & n \neq 0 \end{cases} \tag{4.12}$$

By adjoining the degree derivation, we form the *extended affine Lie algebra*

$$\tilde{\mathfrak{g}} = \hat{\mathfrak{g}} \rtimes \mathbb{K}d \tag{4.13}$$

where $\deg d = 0$. Note we have inclusions

$$\mathfrak{g} \cong \mathfrak{g}t^0 \overset{i}{\hookrightarrow} \hat{\mathfrak{g}} \overset{j}{\hookrightarrow} \tilde{\mathfrak{g}} \tag{4.14}$$

and we may extend $\hat{\cdot} : \mathsf{QLieAlg}_{\mathbb{K}} \to \mathsf{Gr}_{\mathbb{Z}}\mathsf{Lie}_{\mathbb{K}}$ and $\tilde{\cdot} : \mathsf{QLieAlg}_{\mathbb{K}} \to \mathsf{Gr}_{\mathbb{Z}}\mathsf{Lie}_{\mathbb{K}}$ so that

$$i : 1 \Rightarrow \hat{\cdot} : \mathsf{QLieAlg}_{\mathbb{K}} \to \mathsf{Lie}_{\mathbb{K}} \tag{4.15}$$

$$j : \hat{\cdot} \Rightarrow \tilde{\cdot} : \mathsf{Gr}_{\mathbb{Z}}\mathsf{Lie}_{\mathbb{K}} \to \mathsf{Gr}_{\mathbb{Z}}\mathsf{Lie}_{\mathbb{K}} \tag{4.16}$$

become natural transformations.[4]

We now consider a variant of the above construction called the *twisted affine lie algebra*: Let $\mathfrak{g}$ be a quadratic Lie algebra with bilinear form $\langle -, - \rangle$, and $\vartheta \in \mathrm{Aut}(\mathfrak{g})$ be an involutive isometry so that $\langle \vartheta x, \vartheta y \rangle = \langle x, y \rangle$ for $x, y \in \mathfrak{g}$. Let

$$\mathfrak{l} = \mathfrak{g}[t^{1/2}, t^{-1/2}] \oplus \mathbb{K}c \tag{4.17}$$

with the Lie bracket defined by

$$[c, \hat{\mathfrak{g}}] = [\hat{\mathfrak{g}}, c] = 0 \tag{4.18}$$

$$[xt^n, yt^m] = [x, y]t^{n+m} + \delta_{n+m} n \langle x, y \rangle \tag{4.19}$$

and the $\mathbb{Z}/2$-grading

$$\mathfrak{l}_n = \begin{cases} \mathfrak{g}t^0 + \mathbb{K}c & n = 0 \\ \mathfrak{g}t^n & n \neq 0 \end{cases} \tag{4.20}$$

Defining the following involution on $\mathbb{K}[t^{1/2}, t^{-1/2}]$

$$v : t^{1/2} \mapsto -t^{1/2} \tag{4.21}$$

$$t \mapsto t \tag{4.22}$$

and extending ϑ to $\mathfrak{l}$ so that for $x \in \mathfrak{g}$ and $f \in \mathbb{K}[t^{1/2}, t^{-1/2}]$

$$\vartheta : c \mapsto c \tag{4.23}$$

$$xf \mapsto (\vartheta x)(vf) \tag{4.24}$$

we define the twisted affine Lie algebra $\hat{\mathfrak{g}}[\vartheta]$ as the even subalgebra under ϑ

$$\hat{\mathfrak{g}}[\vartheta] = \{x \in \mathfrak{l} : \vartheta x = x\} \tag{4.25}$$

$$= \mathfrak{g}_{(0)} \otimes \mathbb{K}[t, t^{-1}] \oplus \mathfrak{g}_{(1)} \otimes t^{1/2}\mathbb{K}[t, t^{-1}] \oplus \mathbb{K}c \tag{4.26}$$

4. We omit obvious forgetful functors in expressions such as (4.15)

with the inherited $\mathbb{Z}/2$-grading. As in the untwisted case, we may for the *extended twisted affine Lie algebra*

$$\tilde{\mathfrak{g}}[\vartheta] = \hat{\mathfrak{g}}[\vartheta] \rtimes \mathbb{K}d \tag{4.27}$$

where $\deg d = 0$. The use of half-integer powers of t is a notational convention. Note this construction admits a natural generalization to automorphisms of any finite order.

We may rephrase the commutation relations of the untwisted affine Lie algebra $\tilde{\mathfrak{g}}$ by defining a 'generating function'

$$x(z) = x_{\mathbb{Z}}(z) = \sum_{n\in\mathbb{Z}} (x \otimes t^n) z^{-n} \in \tilde{\mathfrak{g}}[\![z, z^{-1}]\!] \tag{4.28}$$

for $x \in \mathfrak{g}$ whence

$$\begin{aligned}
[x(z_1), x(z_2)] &= [x, y](z_2)\delta(z_1/z_2) - \langle x, y\rangle (D\delta)(z_1/z_2)c && (4.29)\\
[c, x(z)] &= 0 && (4.30)\\
[d, x(z)] &= -Dx(z) && (4.31)\\
[c, d] &= 0 && (4.32)
\end{aligned}$$

For the twisted affine Lie algebra $\tilde{\mathfrak{g}}[\vartheta]$ we set

$$x(z) = x_{\mathbb{Z}+1/2}(z) = \sum_{n\in\mathbb{Z}} (x_{(n)} \otimes t^{n/2}) z^{-n/2} \in \tilde{\mathfrak{g}}[\vartheta][\![z^{1/2}, z^{-1/2}]\!] \tag{4.33}$$

for $x \in \mathfrak{g}$ where

$$x \mapsto x_{(i)} = \frac{1}{2}(x + (-1)^i \vartheta x) \tag{4.34}$$

denotes the appropriate projecion into the ϑ-eigenspace decomposition $\mathfrak{g} = \mathfrak{g}_{(0)} \oplus \mathfrak{g}_{(1)}$, whence

$$\begin{aligned}
[x(z_1), x(z_2)] &= \frac{1}{2}\sum_{i\in\mathbb{Z}_2} [\vartheta^i x, y](z_2)\delta((-1)^i z_1^{1/2}/z_2^{1/2}) - \frac{1}{2}\sum_{i\in\mathbb{Z}_2} \langle \vartheta^i x, y\rangle D_1\delta((-1)^i) z_1^{1/2}/z_2^{1/2})c && (4.35)\\
[c, x(z)] &= 0 && (4.36)\\
[d, x(z)] &= -Dx(z) && (4.37)\\
[c, d] &= 0 && (4.38)
\end{aligned}$$

5 Heisenberg Lie algebras

In order to study the representation theory of affine Lie algebras in the following sections, we first study the much simpler case of Lie algebras termed *Heisenberg*, which are familiar from quantum mechanics. A Heisenberg algebra $\mathfrak{l}$ is a Lie algebra with a 1-dimensional centre coïnciding with its commutator ideal, i.e.

$$\mathfrak{l}_0 = \mathfrak{z}(\mathfrak{l}) = \mathfrak{l}' = \mathbb{K}z \tag{5.1}$$

for some nonzero $z \in \mathfrak{l}$.

Example 5.1. *Consider the quantum-mechanical canonical commutation relations for a particle in one dimension*

$$[\hat{x}, \hat{p}] = i\hbar \tag{5.2}$$

where $\hat{x}$, $\hat{p}$ are the position and momentum operators respectively. Then $\mathrm{span}_{\mathbb{R}}\{\hat{x}, \hat{p}, i\hbar\}$ *is a Heisenberg algebra over $\mathbb{R}$. If we complexify and rotate so that*

$$\hat{a}_{\pm} = \frac{1}{\sqrt{2\hbar}} (\hat{x} \mp i\hat{p}) \tag{5.3}$$

we find

$$[\hat{a}_-, \hat{a}_+] = 1 \tag{5.4}$$

thus $\mathrm{span}_{\mathbb{C}}\{\hat{a}_+, \hat{a}_-, 1\}$ *forms a Heisenberg algebra over $\mathbb{C}$. In the context of the one-dimensional simple harmonic oscillator, $\hat{a}_+$ and $\hat{a}_-$ are called the* creation *and* annihilation *operators respectively, and are interpreted as adding or removing fixed quanta from the system.*

It is natural to demand that the centre of $\mathfrak{l}$ is represented centrally, so z acts as multiplication by some scalar $k \in \mathbb{K}$. Already in this very simple example of a Heisenberg algebra we find there can be no such representations of finite dimension: For the trace of the left hand side of (5.4) would be zero, while the right hand side would have nonzero trace.

In the more general case of a countable-dimensional Heisenberg algebra $\mathfrak{l}$, it is expedient to assume[5] a $\mathbb{Z}$-grading $\mathfrak{l} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{l}_i$ with $\mathfrak{l}_0$ central and $\dim \mathfrak{l}_i < \infty$ for all $i \in \mathbb{Z}$. Letting $\mathfrak{l}^{\pm} = \bigoplus_{i=1}^{\infty} \mathfrak{l}_i$, it follows that $\mathfrak{b}^{\pm} = \mathfrak{l}_0 \oplus \mathfrak{l}^{\pm}$ are maximal abelian subalgebras of $\mathfrak{l}$. One obtains an alternating bilinear form on $\mathfrak{l}$ by

$$[x, y] = (x, y)z \tag{5.5}$$

which is nondegenerate on $\mathfrak{l}_n \oplus \mathfrak{l}_{-n}$ for $n \in \mathbb{N}$, so one may form bases $(x_i)_{i\in I}$ of $\mathfrak{l}^+$ and $(y_i)_{i\in I}$ of $\mathfrak{l}^-$ such that $(x_i, y_j) = \delta_{ij}$, giving the *Heisenberg commutation relations*

$$[x_i, z] = [y_i, z] = [x_i, x_j] = [y_i, y_j] = 0 \qquad\qquad [x_i, y_j] = \delta_{ij} z \tag{5.6}$$

for $i, j \in I$. By the same argument as above, there is no finite-dimensional $\mathfrak{l}$-module V. Instead, we turn our attention to $\mathbb{Z}$-graded $\mathfrak{l}$-modules truncated from above, i.e. so that $V_n = 0$ for sufficiently large n. An $\mathfrak{l}$-module is said to satisfy $\mathfrak{S}_k$ iff it is truncated above and z acts as multiplication by $k \in \mathbb{K}$. This implies the existence of a *vacuum*: A nonzero vector $v \in V$ is called a *vacuum vector* iff $\mathfrak{l}^+ \odot v = 0$. The *vacuum space* Ω_V consists of all vacuum vectors and zero

$$\Omega_V = \{v \in V : A^+ \odot v = 0\} = \bigoplus_{n\in\mathbb{Z}} \Omega_{V_i} \tag{5.7}$$

5. In general it is possible to impose such a grading by taking the following construction in reverse, starting with bases for two maximal abelian subalgebras whose intersection is $\mathfrak{l}_0$.

and is a graded vector subspace, i.e. all vacuum vectors are linear combinations of homogeneous vacuum vectors.

The vacuum of a $\mathfrak{S}_k$ module thus forms a $\mathfrak{b}^+$-module. We may construct basic $\mathfrak{S}_k$ representations of $\mathfrak{l}$ by starting with a one-dimensional $\mathbb{Z}$-graded $\mathfrak{b}^+$-module $\mathbb{K}_k$ spanned by a vacuum vector 1 so that

$$z \odot 1 = k \qquad \mathfrak{b}^+ \odot 1 = 0 \qquad \deg 1 = 0 \tag{5.8}$$

defining the *Heisenberg module* $M(k)$ to be the induced module

$$M(k) = \mathrm{Ind}_{\mathfrak{b}^+}^{\mathfrak{l}} \mathbb{K}_k = U(\mathfrak{l}) \otimes_{U(\mathfrak{b}^+)} \mathbb{K}_k \tag{5.9}$$

It follows from the Poincaré-Birkhoff-Witt theorem that $M(k)$ is isomorphic as a vector space to the symmetric algebra $S^\bullet \mathfrak{l}^-$, which we interpret as the algebra of polynomials in indeterminates $\{y_i\}_{i \in I}$. Given such a polynomial $f \in S^\bullet \mathfrak{l}^-$, we have the following action for $i \in I$

$$z \odot f = kf \qquad y_i \odot f = y_i f \qquad x_i \odot f = k\frac{\partial f}{\partial y_i} \tag{5.10}$$

which is called the *canonical realization* of the Heisenberg commutation relations, and makes the irreducible nature of these representations clear. Assuming $\mathbb{K}$ is algebraically closed, it follows from cor. 3.3 that all $\mathfrak{l}$-module endomorphisms of $M(k)$ are homotheties. The significance of Heisenberg modules is revealed by the following theorem:

Theorem 5.2. *Let V be an $\mathfrak{l}$-module satisfying $\mathfrak{S}_k$, and $v \in \Omega_V$ be a vacuum vector. Then the $\mathfrak{l}$-module $\mathfrak{l} \odot v$ generated by v is isomorphic to $M(k)$. In particular $M(k)$ is the unique $\mathfrak{l}$-module satisfying $\mathfrak{S}_k$ up to isomorphism.*

Proof. Let $v \in \Omega_V$ be a vacuum vector, and $W = \mathfrak{l} \odot v$ be the irreducible $\mathfrak{l}$-module generated by v. Now $\mathbb{K}v \cong \mathbb{K}_v$ as $\mathfrak{b}^+$-modules, so $M(k) \cong \mathrm{Ind}_{\mathfrak{b}^+}^{\mathfrak{l}} \mathbb{K}v$. By the universal property of the induced module, there exists a unique $\mathfrak{l}$-module homomorphism $\varphi : M(k) \to W$ such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{K}v & \overset{\iota}{\hookrightarrow} & M(k) \cong \mathrm{Ind}_{\mathfrak{b}^+}^{\mathfrak{l}} \mathbb{K}v \\ & \searrow & \Big\downarrow {\scriptstyle \exists! \varphi} \\ & & W \end{array} \tag{5.11}$$

which by Schur's lemma is a $\mathfrak{l}$-module isomorphism. ☐

Thm. 5.2 may be viewed as a purely algebraic analogue of the Stone-Von Neumann theorem, which expresses the uniqueness of the canonical commutation relations of position and momentum operators in quantum mechanics. Indeed, constructing the appropriate Heisenberg module for ex. 5.1 gives the usual realisation of $\hat{p}$ and $\hat{x}$ as differential operators.

Of particular interest are Heisenberg algebras constructed via affinisation. Let $\mathfrak{h}$ be a non-degenerate finite-dimensional quadratic space, which we give the structure of an abelian Lie algebra. Let $\mathfrak{g}_{\mathbb{Z}} = \tilde{\mathfrak{h}}$ and $\mathfrak{g}_{\mathbb{Z}+\frac{1}{2}} = \tilde{\mathfrak{h}}[-1]$

denote untwisted and twisted affinisations of $\mathfrak{h}$ respectively. Then the commutator ideals of each of these form Heisenberg algebras

$$\hat{\mathfrak{h}}_{\mathbb{Z}} = \mathfrak{g}'_{\mathbb{Z}} =_{\mathsf{Vect}_{\mathbb{K}}} \mathbb{K}c \oplus \bigoplus_{n \in \mathbb{Z} \setminus \{0\}} \mathfrak{h} \otimes t^n \tag{5.12}$$

$$\hat{\mathfrak{h}}_{\mathbb{Z}+\frac{1}{2}} = \mathfrak{g}'_{\mathbb{Z}+\frac{1}{2}} =_{\mathsf{Vect}_{\mathbb{K}}} \mathbb{K}c \oplus \bigoplus_{n \in \mathbb{Z}+\frac{1}{2}} \mathfrak{h} \otimes t^n = \hat{\mathfrak{h}}[-1] \tag{5.13}$$

which we call the $\mathbb{Z}$- *and* $(\mathbb{Z}+\frac{1}{2})$*-natural Heisenberg algebras* of $\mathfrak{h}$ respectively.

Hereinafter $Z = \mathbb{Z}$ or $Z = \mathbb{Z} + \frac{1}{2}$. We have the commutation relations

$$[c, \hat{\mathfrak{h}}_Z] = 0 \tag{5.14}$$

$$[x \otimes t^m, y \otimes t^n] = \langle x, y \rangle m \delta_{m+n} c \tag{5.15}$$

for $x, y \in \mathfrak{h}$ and $m, n \in Z \setminus \{0\}$.

Example 5.3. *Let $\mathfrak{h} = \mathbb{C}\alpha$ so that $\langle \alpha, \alpha \rangle = 1$. Form the $\mathbb{Z}$-natural Heisenberg algebra $\mathfrak{h}_{\mathbb{Z}}$ and consider the corresponding Heisenberg module $M(1) = S^\bullet(\mathfrak{h}_{\mathbb{Z}}^-)$. For $x \in \mathfrak{h}$ and $n \in \mathbb{Z} \setminus \{0\}$ denote the action of xt^n on $M(1)$ as $x(n)$. $M(1)$ is the* Fock space *for an unspecified number of bosons, each of which may be in any eigenstate labelled by $n \in \mathbb{N}$. The state $1 \in M(1)$ represents the* vacuum, *the operator $\alpha(-n)$ is the* creation operator *for state n, and the operator $\alpha(n)$ is the* annihilation operator *for state n. We can express any Fock state by acting on the vacuum, e.g.*

$$\alpha(-3)\alpha(-2)^2\alpha(-1)1 \in M(1) \tag{5.16}$$

6 A twisted affinization of $\mathfrak{sl}_2\mathbb{K}$

The Lie algebra $\mathfrak{sl}_2\mathbb{K}$, hereinafter $\mathfrak{a}$, consists of traceless 2×2 matrices with the bracket given by the commutator. One (ordered) basis is $(\alpha_1, x_{\alpha_1}, x_{-\alpha_1})$ where

$$\alpha_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad x_{\alpha_1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad x_{-\alpha_1} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{6.1}$$

with the commutation relations

$$[\alpha_1, x_{\pm\alpha_1}] = \pm 2 x_{\pm\alpha_1} = \langle \alpha_1, \pm\alpha_1 \rangle x_{\pm\alpha_1} \tag{6.2}$$

$$[x_{\alpha_1}, x_{-\alpha_1}] = \alpha_1 \tag{6.3}$$

where we have the nondegenerate invariant symmetric bilinear form with Gram matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{6.4}$$

given by the trace form $\langle x, y\rangle = \operatorname{tr} xy$ of the fundamental representation, making $\mathfrak{a}$ a quadratic Lie algebra. Consider the suggestively named[6] involutive isometry

$$\sigma_1 : \alpha_1 \mapsto -\alpha_1 \tag{6.5}$$

$$x_{\pm\alpha_1} \mapsto x_{\mp\alpha_1} \tag{6.6}$$

Furthermore we let

$$x^+_{\alpha_1} = x_{\alpha_1} + x_{-\alpha_1} \tag{6.7}$$

$$x^-_{\alpha_1} = x_{\alpha_1} - x_{-\alpha_1} \tag{6.8}$$

We consider the twisted affine Lie algebra $\hat{\mathfrak{a}}_1 = \hat{\mathfrak{a}}[\sigma_1]$ which has the basis

$$\hat{\mathfrak{a}}_1 = \langle c, \alpha_1 \otimes t^{m+1/2}, x^+_{\alpha_1} \otimes t^m, x^-_{\alpha_1} \otimes t^{m+1/2} : m \in \mathbb{Z}\rangle \tag{6.9}$$

The 1-dimensional subalgebra $\mathfrak{h} = \mathbb{K}\alpha \leq \mathfrak{a}$ generates the natural Heisenberg algebra

$$\hat{\mathfrak{h}}_{\mathbb{Z}+\frac{1}{2}} = \mathbb{K}c \oplus \bigoplus_{n\in\mathbb{Z}+\frac{1}{2}} \alpha_1 \otimes t^n \leq \tilde{\mathfrak{h}}[-1] \leq \hat{\mathfrak{a}}_1 \tag{6.10}$$

as a Lie subalgebra and we have

$$[c, \hat{\mathfrak{a}}_1] = 0 \tag{6.11}$$

$$[\alpha_1 \otimes t^m, \alpha_1 \otimes t^n] = 2m\delta_{m+n}c \tag{6.12}$$

In terms of generating functions, we express the commutation relations of $\hat{\mathfrak{a}}_1$ by defining the formal sums

$$x_{\pm\alpha_1}(z) = \frac{1}{2}\sum_{n\in\mathbb{Z}}(x^+_{\alpha_1} \otimes t^n)z^{-n} \pm \frac{1}{2}\sum_{n\in\mathbb{Z}+\frac{1}{2}}(x^-_{\alpha_1} \otimes t^n)z^{-n} \in \hat{\mathfrak{a}}_1[\![z^{1/2}, z^{-1/2}]\!] \tag{6.13}$$

$$\alpha_1(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}}(\alpha_1 \otimes t^n)z^{-n} \in \hat{\mathfrak{a}}_1[\![z^{1/2}, z^{-1/2}]\!] \tag{6.14}$$

whence

$$[\alpha_1 \otimes t^m, x_{\pm\alpha_1}(z)] = \pm 2z^m x_{\pm\alpha_1}(z) = \langle \alpha_1, \pm\alpha_1\rangle z^m x_{\pm\alpha_1}(z) \tag{6.15}$$

$$[x_{\alpha_1}(z), x_{-\alpha_1}(z_2)] = \frac{1}{2}(\alpha_1(z_2) - cD_1)\delta(z_1^{1/2}/z_2^{1/2}) \tag{6.16}$$

and we also have

$$x_{-\alpha_1}(z) = \lim_{z^{1/2}\to -z^{1/2}} x_{\alpha_1}(z) \tag{6.17}$$

7 Twisted vertex operators

Let V denote the $\mathbb{Q}$-graded irreducible Heisenberg module $M(1) = S^\bullet(\mathfrak{h}^-_{\mathbb{Z}+\frac{1}{2}})$, which we extend to a $\tilde{\mathfrak{h}}[-1]$-module on which d acts as the degree operator. We denote the action of $\alpha \otimes t^n$ on V by $\alpha(n)$, which we realise

6. For $\mathbb{K} = \mathbb{C}$, we can conjugate by the Pauli matrix σ_1 for the same result.

by multiplication and partial differentiation operators on V, and we have the commutation relations

$$[\alpha(m), \beta(n)] = \langle \alpha, \beta \rangle m \delta_{m+n} \tag{7.1}$$

$$[d, \alpha(m)] = m\alpha(m) \tag{7.2}$$

for $\alpha, \beta \in \mathfrak{h}$ and $m, n \in \mathbb{Z} + \frac{1}{2}$. We wish to represent the rest of $\tilde{\mathfrak{a}}_1$ as operators on V, which amounts to finding

$$x_{\pm\alpha_1}(n) \in \operatorname{End}_{\mathbb{K}} V \tag{7.3}$$

for $n \in \mathbb{Z}/2$ so that the generating functions

$$X(\pm\alpha_1, z) = \sum_{n \in \mathbb{Z}/2} x_{\pm\alpha_1}(n) z^{-n} \in (\operatorname{End}_{\mathbb{K}} V)[\![z^{1/2}, z^{-1/2}]\!] \tag{7.4}$$

satisfy the commutation relations

$$[\beta(m), X(\pm\alpha_1, z)] = \langle \beta, \pm\alpha_1 \rangle z^m X(\pm\alpha_1, z) \tag{7.5}$$

$$[d, X(\pm\alpha_1, z)] = -DX(\pm\alpha_1, z) \tag{7.6}$$

for $\beta \in \mathfrak{h}$ and $m \in \mathbb{Z} + \frac{1}{2}$, by (6.15) and (2.10). Motivated by thm 2.1, Frenkel, Lepowsky, and Meurman (1988) define the following for $\alpha \in \mathfrak{h}$

$$E^{\pm}(\alpha, z) = \exp A^{\pm}(\alpha, z) \qquad A^{\pm}(\alpha, z) = \sum_{n \in \pm(\mathbb{N}_0 + \frac{1}{2})} \frac{\alpha(n)}{n} z^{-n} \tag{7.7}$$

Theorem 7.1. *$E^{\pm}(\alpha, z)$ is well-defined for $\alpha \in \mathfrak{h}$ and for $\alpha, \beta \in \mathfrak{h}$ we have:*

$$E^{\pm}(0, z) = 1 \tag{7.8}$$

$$E^{\pm}(\alpha + \beta, z) = E^{\pm}(\alpha, z) E^{\pm}(\beta, z) \tag{7.9}$$

$$[d, E^{\pm}(\alpha, z)] = -DE^{\pm}(\alpha, z) = \left(\sum_{n \in \pm(\mathbb{N}_0 + \frac{1}{2}} \alpha(z) z^{-n} \right) E^{\pm}(\alpha, z) \tag{7.10}$$

$$E^{\pm}(-\alpha, z) = \lim_{z^{1/2} \to -z^{1/2}} E^{\pm}(\alpha, z) \tag{7.11}$$

For $\alpha, \beta \in \mathfrak{h}$ and $m \in \mathbb{Z} + \frac{1}{2}$ we have:

$$[\beta(m), E^{+}(\alpha, z)] = \begin{cases} 0 & m > 0 \\ -\langle \beta, \alpha \rangle z^m E^{+}(\alpha, z) & m < 0 \end{cases} \tag{7.12}$$

$$[\beta(m), E^{-}(\alpha, z)] = \begin{cases} -\langle \beta, \alpha \rangle z^m E^{+}(\alpha, z) & m > 0 \\ 0 & m < 0 \end{cases} \tag{7.13}$$

Proof. If we expand $E^{\pm}(\alpha, z)$, we find the coëfficient of any power of z to be a finite linear combination of operator products, thus it is well-defined. (7.8) and (7.9) follow from basic properties of the exponential. Since

$$[d, A^{\pm}(\alpha, z)] = \sum_{n \in \pm(\mathbb{N}_0 + \frac{1}{2})} \alpha(n) z^{-n} = -DA^{\pm}(\alpha, z) \tag{7.14}$$

commutes with $A^{\pm}(\alpha, z)$, (7.10) follows from thm. (2.1). (7.11) follows from the analogous expression for $A^{\pm}(\alpha, z)$. Once again applying thm. 2.1 to the derivation $\operatorname{ad}_{\beta(m)}$ yields (7.12) and (7.13). ☐

The commutation relations (7.12) and (7.13) suggest a product of $E^+(\alpha, z)$ and $E^-(\alpha, z)$ as a solution to (7.5). The vacuum property of V means that if $E^+(\alpha, z)$ acts first, only finitely many coëfficients of z^{-n} will survive. Thus the following *twisted vertex operator* is well-defined in the sense of (2.15)

$$X(\pm\alpha_1, z) = \frac{1}{4} E^-(\mp\alpha_1, z) E^+(\mp\alpha_1, z) \tag{7.15}$$

Theorem 7.2. *$X(\pm\alpha_1, z)$ is the unique solution to the commutation relations (7.12) and (7.13) up to scaling.*

Proof. Suppose $W(z) \in (\operatorname{End}_{\mathbb{K}} V)\{z\}$ is a solution, and define

$$U(z) = \sum_{n \in \mathbb{K}} u(n) z^{-n} = E^-(\pm\alpha_1, z) W(z) E^+(\pm\alpha_1, z) \tag{7.16}$$

which again is well-defined by virtue of the vacuum property. Then for $\beta \in \mathfrak{h}$ and $m \in \mathbb{Z} + \frac{1}{2}$

$$[d, Z(z)] = -DU(z) \tag{7.17}$$

$$[\beta(m), U(z)] = 0 \tag{7.18}$$

whence $\deg u(n) = n$ and $[\beta(m), u(n)] = 0$, the latter implying $u(n)$ are Heisenberg module endomorphisms, which by cor. 3.3 are homotheties. Thus $U(z) = u(0)z^0 \in \mathbb{K}$, and solving for $W(z)$ using (7.8) and (7.9) gives $W(z) = 4u(0)X(\pm\alpha_1, z)$. □

8 Final comments

With a little extra work, and the introduction of a multiplication convention called *normal ordering*, one can show that the twisted vertex operators generate all the commutation relations of $\tilde{\mathfrak{a}}_1$, thus enabling a full representation of $\tilde{\mathfrak{a}}_1$ on V. Uniqueness of these $\tilde{\mathfrak{a}}_1$-modules as extensions of the underlying Heisenberg module then follows from thm. 7.2.

The untwisted case $\mathfrak{sl}_2\tilde{\mathbb{K}}$ has some more subtle features, and one has to modify the Heisenberg module $\mathfrak{h}_{\mathbb{Z}}$ to form an appropriate representation space. Frenkel, Lepowsky, and Meurman (1988) continues on from this point by introducing a uniform way of constructing the Lie algebras of type A_n, D_n, E_n from a root lattice, as well as creating vertex operator representations of the corresponding affine Lie algebras from this lattice directly. This same process is applied to the Leech lattice in the construction of the moonshine module $V^\natural$.

Another important ingredient, omitted from this discussion, is the Virasoro algebra $\mathfrak{v}$, which is the unique nontrival 1-dimensional central extension of the Witt algebra $\mathfrak{d}$ and plays an important role in conformal field theory. Heisenberg modules for the natural Heisenberg algebras carry natural representations of $\mathfrak{v}$, and thus so do the vertex operator representations of affine Lie algebras. The existence of such a representation is the main feature distinguishing *vertex operator algebras* from more general *vertex algebras*, and plays an important role in the conjectured uniqueness of the moonshine module, which if proven would be analogous to the constructions of the Mathieu and Conway groups as the symmetries of unique objects.

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A Dixmier's lemma

Let $\mathbb{K}$ be a field and A be a unital associative algebra over $\mathbb{K}$. An element $a \in A$ is called *algebraic* over $\mathbb{K}$ iff there exists a nonzero polynomial $p(x) \in \mathbb{K}[x]$ such that $p(a) = 0$, whence the solving monic polynomial of smallest degree $m_a(x) \in \mathbb{K}[x]$ is called the *minimal polynomial* of a. Otherwise a is called *transcendental*.

Theorem A.1. *Let $\mathbb{K}$ be an algebraically closed field and A be a division algebra such that every element $a \in A$ is algebraic. Then $A \cong \mathbb{K}$.*

Proof. Let $a \in A$ and $m_a(x) \in \mathbb{K}[x]$ be its minimal polynomial. Since A has no zero divisors, $m_a(x)$ must be an irreducible polynomial: For if $m_a(x) = p(x)q(x)$ then $p(a)q(a) = 0$ and hence either $p(a) = 0$ or $q(a) = 0$, a contradiction. Since $m_a(x)$ is irreducible it is linear, thus $m_a(x) = x - \lambda$ whence $a = \lambda \in \mathbb{K}$. □

Theorem A.2. *Let $\mathbb{K}$ be a field and $\mathbb{K}(x)$ be its field of rational functions. Then $\dim_{\mathbb{K}} \mathbb{K}(x) \geq |\mathbb{K}|$.*

Proof. We will show the following set to be linearly independent:

$$S = \left\{ \frac{1}{x - \lambda} : \lambda \in \mathbb{K} \right\}$$

Let

$$f_n(x) = \frac{p_n(x)}{q_n(x)} = \sum_{i=1}^{n} \frac{1}{x - \lambda_i}$$

where

$$q_n(x) = \prod_{i=1}^{n} (x - \lambda_i)$$

then

$$f_{n+1}(x) = \frac{p_n(x)}{q_n(x)} + \frac{1}{x - \lambda_{n+1}} = \frac{p_n(x)(x - \lambda_{n+1}) + q_n(x)}{q_n(x)(x - \lambda_{n+1})}$$

which is zero iff $q_n(x) = -(x - \lambda_{n+1})p_n(x)$. But this is impossible since $(x - \lambda_{n+1}) \nmid q_n(x)$. □

Theorem A.3 (Dixmier's lemma)**.** *Let A be a unital associative algebra over $\mathbb{K}$ and V be a simple A-module. If $|\mathbb{K}| > \dim_{\mathbb{K}} V$, then every A-module endomomorphism $\vartheta \in \mathrm{End}_A(V)$ is an algebraic element over $\mathbb{K}$.*

Proof. By Schur's lemma (thm. 3.2), $\mathrm{End}_A(V)$ is a division algebra over $\mathbb{K}$. Suppose $\vartheta \in \mathrm{End}_A(V)$ is transcendental over $\mathbb{K}$, i.e. $p(\vartheta) = 0$ iff $p = 0$ for $p(x) \in \mathbb{K}[x]$. The division algebra generated by ϑ is then

$$\mathbb{K}(\vartheta) = \left\{ \frac{p(\vartheta)}{q(\vartheta)} : p(x), q(x) \in \mathbb{K}[x], q \neq 0 \right\} \tag{A.1}$$

$$= \{ f(\vartheta) : f(x) \in \mathbb{K}(x) \} \tag{A.2}$$

where $\mathbb{K}(x)$ is the field of rational functions for $\mathbb{K}$, and we have a straightforward isomorphism of division $\mathbb{K}$-algebras $\mathbb{K}(x) \cong \mathbb{K}(\vartheta)$. By thm A.2 we have the inequality

$$|\mathbb{K}| \leq \dim_{\mathbb{K}} \mathbb{K}(x) = \dim_{\mathbb{K}} \mathbb{K}(\vartheta) \tag{A.3}$$

Since V is a vector space over $\mathbb{K}(\vartheta)$ with scalar multiplication given by the action of ϑ, we have

$$\dim_{\mathbb{K}} \mathbb{K}(\vartheta) \leq \dim_{\mathbb{K}} V \tag{A.4}$$

and thus

$$|\mathbb{K}| \leq \dim_{\mathbb{K}} V \tag{A.5}$$

a contradiction. □

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