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Solutions of Elliptic Boundary Value Problems

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1 Problem Description and Research Aims

Definition 1 (Multi-Index Notation)**.** *A multi-index is an n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ of nonnegative integers. Its length is defined as*

$$|\alpha| := \sum_{i=1}^{n} \alpha_i.$$

A multi-index is commonly used to quickly express derivatives. For a function $u : \mathbb{R}^n \to \mathbb{R}$, the multi-index derivative is written as

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} = (\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n})u.$$

It is also used occasionally as a map from a vector $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ to $\mathbb{R}$. We define

$$\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}.$$

For example,

$$(1,2,3)^{(1,2,3)} = 1^1 \cdot 2^2 \cdot 3^3 = 108.$$

Definition 2 (Linear Second Order Differential Operators)**.** *A second-order linear differential operator L in $\mathbb{R}^n$ is an operator of the form*

$$L : u \mapsto \sum_{|\alpha| \le 2} a_\alpha(x) D^\alpha u,$$

where the sum is over multi-indices $\alpha \in \mathbb{N}^n$, and acts on functions $u : \mathbb{R}^n \to \mathbb{R}$ where at least two partial derivatives in each direction exist. The coefficients a_α are functions $\mathbb{R}^n \to \mathbb{R}$. We say L is elliptic if, for all nonzero $\xi \in \mathbb{R}^n$,

$$\sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha > 0.$$

Further, L is uniformly elliptic if there exists $\lambda > 0$ such that

$$\sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha \ge \lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^n.$$

Remark. *Ellipticity is a similar property to positive-definiteness.*

The following problem is known as an elliptic boundary value problem (BVP) of Dirichlet kind, and is the subject of this report. Suppose U is an open and bounded subset of $\mathbb{R}^n$, let $f : U \to \mathbb{R}$, and $f_0 : \partial U \to \mathbb{R}$, and let L be an elliptic differential operator. Then what conditions are sufficient to guarantee that there exists a unique solution $u : \overline{U} \to \mathbb{R}$ to the following system?

$$\begin{aligned} Lu &= f, && \forall x \in U, \\ u &= f_0, && \forall x \in \partial U. \end{aligned}$$

Elliptic BVP are are eminent problems in dynamics because of their versatility in describing the long-term behaviour of physical systems. A notable example is Poisson's equation, where L is taken to be the Laplace operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, which describes the equilibrium temperature at every point in a region U if temperature is fixed to equal f_0 on ∂U, and heat is added at each point at a rate of $f(x)$.

$$\begin{aligned} \Delta u &= f, && \forall x \in U \\ u &= f_0, && \forall x \in \partial U \end{aligned}$$

The theory of elliptic PDE is well-described by Evans [1], which focuses on the study of elliptic BVPs where the boundary ∂U is sufficiently well-behaved. However, there is significantly less investigation into cases where ∂U may include cusps, slits, or removed points (punctures). The aim of this project was to understand why these domains may not admit solutions.

Definition 3 (Smooth-Boundary (C^1) Domains). *The boundary ∂U of a connected open subset (domain) $U \subseteq \mathbb{R}^n$ is classified as C^1 if it can be locally written as the graph of a smooth function. That is, for each $p \in \partial U$, there exists a small $\varepsilon > 0$, an index $j \in \{1, ..., n\}$, and a smooth function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $U \cap B(p, \varepsilon) = \{x_j > \gamma(x_1, ...x_{j-1}, x_{j+1}, ..., x_n)\} \cap B(p, \varepsilon)$ or $\{x_j < \gamma(x_1, ..., x_{j-1}, x_{j+1}, ..., x_n)\} \cap B(p, \varepsilon)$.*

Definition 4 (Lipschitz Function). *A function $f : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz if there exists a constant K such that for each distinct $x, y \in \mathbb{R}^n$,*

$$\frac{|f(x) - f(y)|}{|x - y|} \leq K.$$

Definition 5 (Lipschitz Domain). *The boundary ∂U of a domain $U \subseteq \mathbb{R}^n$ is Lipschitz if, for every $p \in \partial U$, there exists a hyperplane Π of dimension $n - 1$ through p with a unit normal ν, and a Lipschitz function $g : \Pi \to \mathbb{R}$ over the hyperplane, and a small $\varepsilon > 0$ such that*

$$U \cap B(p, \varepsilon) = \{x + t\nu \mid x \in B(p, \varepsilon) \cap \Pi, t > g(x)\} \cap B(p, \varepsilon)$$

Remark. *Lipschitz is a weaker condition than smooth, and allows for some disruptions such as corners.*

Remark. *The definition of C^1 boundaries uses only $\nu = \mathbf{e}_i$, as this is sufficient to capture all possible smooth boundaries. However, in the case where $p \in \partial U$ is sitting at an acute-angled corner, it is possible that $\partial U \cap B(p, \varepsilon)$ cannot be expressed as the graph of a function $x_i = \phi(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$, in other words, we need the ability to rotate the boundary to an arbitrary orientation, rather than just $90°$ rotations around coordinate axes.*

2 Counter-Example

The punctured unit disc $U = \{x \in \mathbb{R}^2 : |x| \in (0, 1)\}$ has a boundary $\partial U = \{|x| = 1\} \cup \{0\}$. There is no solution to the following boundary value problem problem.

$$\begin{aligned} \Delta u &= 0, && \forall x \in U \\ u &= 0, && \forall x \in \{|x| = 1\} \\ u(0) &= 1. && \end{aligned}$$

After changing to polar coordinates, the problem reduces by radial symmetry,

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = 0 \quad : \quad u(0) = 1, \quad u(1) = 0,$$

and in this form, the solution takes the form:

$$u(r) = C_1 + C_2 \ln(r).$$

This form admits no valid solutions to the boundary conditions since $C_2 \ln(r) \to \infty$ unless $C_2 = 0$, but if $C_2 = 0$ then $u(r) = C_1$ is constant and cannot satisfy the boundary conditions. However, if the domain is taken to be an annulus rather than a pierced disk, such that the boundary conditions can be expressed as

$$u(r) = C_1 + C_2 \ln(r), \quad u(\varepsilon) = 1, \quad u(1) = 0,$$

then there is a solution, $u(r) = \frac{\ln(r)}{\ln(\varepsilon)}$.

Interestingly, if the operator is instead taken to be the fourth-order bi-Laplacian $\Delta^2 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$ instead of the Laplacian $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, then there is a family of solutions to the same problem.

In polar coordinates, the bi-Laplacian reduces under radial symmetry to

$$\frac{\partial^4 u}{\partial r^4} + \frac{2}{r}\frac{\partial^3 u}{\partial r^4} - \frac{1}{r^2}\frac{\partial u^2}{\partial r^2} + \frac{1}{r^3}\frac{\partial u}{\partial r} = 0,$$

which is equivalent to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right)\right) = 0.$$

This equation possesses a general solution;

$$u(r) = C_1 r^2 \ln r + C_1 r^2 + C_3 \ln r + C_4 \text{ such that } \begin{cases} u(1) = 0 \\ u(0) = 1 \end{cases}.$$

In the case where $u(0) = 1$, it is necessary that $C_3 = 0$ since $\ln(0)$ is not defined. However, $x^2 \ln x \to 0$ as $x \to 0$, hence C_1 may be nonzero. Then, the general solution to the BVP is

$$u(r) = C_1 r^2 \ln r + 1 - r^2.$$

3 The Lax-Milgram Theorem

Definition 6 (Real Hilbert Space). *A real Hilbert space $\mathcal{H}$ is a vector space equipped with an inner product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ which has the following properties:*

- $\langle u, u \rangle \geq 0 \quad \forall u \in \mathcal{H}$,
- $\langle u, u \rangle = 0 \iff u = 0$,
- $\langle \alpha u + v, \beta w + x \rangle = \alpha\beta\langle u, w \rangle + \alpha\langle u, x \rangle + \beta\langle v, w \rangle + \langle v, x \rangle \quad \forall u, v, w, x \in \mathcal{H},\ \forall \alpha, \beta \in \mathbb{R}$,

- $\langle u, v\rangle = \langle v, u\rangle \quad \forall u, v \in \mathcal{H}$,
- *$\mathcal{H}$ is complete with respect to the norm induced by the inner product,* $\|x\| := \sqrt{\langle x, x\rangle}$.

Definition 7 (Bilinear Form)**.** *A bilinear form on a vector space V over $\mathbb{R}$ is a map $B : V \times V \to \mathbb{R}$ which is linear in both components. That is to say, for all $\alpha, \beta \in \mathbb{R}$ and $w, x, y, z \in \mathcal{H}$, it is true that*

$$B(\alpha w + x, \beta y + z) = \alpha\beta B(w, y) + \alpha B(w, z) + \beta B(x, y) + B(x, z).$$

Remark. *An inner product is a specific case of a bilinear form.*

If a boundary value problem is able to be re-stated in terms of bilinear forms on a Hilbert space, then the Lax-Milgram theorem provides clarity about when there is only one candidate for a solution. We prove this result first to justify later sections which will detail how the formulation may be done.

Definition 8 (Orthogonal Compliment)**.** *Let V be a subset of a Hilbert space $\mathcal{H}$. Its orthogonal compliment $V^\perp$ is the set of all $x \in \mathcal{H}$ such that $\langle x, v\rangle = 0 \quad \forall v \in V$.*

Theorem 1 (Orthogonal Decomposition Theorem)**.** *Let $\mathcal{M}$ be a closed subspace of a real Hilbert space $\mathcal{H}$, then $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, that is,*

$$\forall x \in \mathcal{H},\ \exists!\ y \in \mathcal{M}, z \in \mathcal{M}^\perp : x = y + z.$$

Remark. *This theorem extends the notion of vector projection to infinite-dimensional Hilbert spaces.*

Proof. First select an arbitrary $x \in \mathcal{H}$. It will be shown that $\min\{\|x - m\| : m \in \mathcal{M}\}$ exists and is achieved by a unique $u \in \mathcal{M}$. Let $\delta = \inf\{\|x - m\| : m \in \mathcal{M}\}$ and choose a sequence $\{u_n\}$ in $\mathcal{M}$ such that $\|x - u_n\|$ converges to δ. Such a sequence must exist since $\delta = \inf\{\|x - m\| : m \in \mathcal{M}\} \implies \forall\varepsilon > 0, \exists m \in \mathcal{M} : \|x - m\| \leq \delta + \varepsilon$. Recalling that $\frac{1}{2}(u_n + u_m) \in \mathcal{M}$, it follows from the parallelogram law that for any n, m,

$$\|u_m - u_n\|^2 = 2(\|x - u_m\|^2 + \|x - u_n\|^2) - 4\left\|x - \frac{1}{2}(u_n + u_m)\right\|^2 \leq 2(\|x - u_m\|^2 + \|x - u_n\|^2) - 4\delta^2.$$

Then, since $\|x - u_m\|^2 \to \delta^2$, $\lim_{m,n\to\infty} \|u_m - u_n\|^2 \leq 4\delta^2 - 4\delta^2 = 0$. Therefore, the sequence $\{u_n\}$ is Cauchy and the Hilbert space $\mathcal{H}$ is complete, so the sequence is convergent in $\mathcal{H}$. But $\mathcal{M}$ is closed, the sequence converges to a element $u \in \mathcal{M}$.

A similar argument shows u is unique. Suppose $u, v \in \mathcal{M}$ both satisfy $\|x - u\| = \|x - v\| = \inf\{\|x - m\| : m \in \mathcal{M}$. Then by the parallelogram law,

$$\|u - v\|^2 = 2\|u - x\|^2 + 2\|v - x\|^2 - 4\left\|\frac{u + v}{2} - x\right\|^2 \leq 2\delta^2 + 2\delta^2 - 4\delta^2 = 0,$$

$$\implies u = v.$$

To show $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, choose an arbitrary $x \in \mathcal{H}$ and use the argument above to generate the minimising $u \in \mathcal{M}$. Since $\mathcal{M}$ is a vector subspace,

$$
\begin{aligned}
u + \alpha m \in \mathcal{M} \quad \forall \alpha \in \mathbb{R}, m \in \mathcal{M} \\
\implies \|x - (u + \alpha m)\| \geq \|x - u\| \\
\implies \langle x - u, x - u\rangle \leq \langle x - u - \alpha m, x - u - \alpha m\rangle \\
\implies \langle \alpha m, \alpha m\rangle - 2\langle x - u, \alpha m\rangle \geq 0 \\
\implies \alpha^2\|m\|^2 - 2\alpha\langle x - u, m\rangle \geq 0 \\
\implies \min_{\alpha\in\mathbb{R}}\{\alpha^2\|m\|^2 - 2\alpha\langle x - u, m\rangle\} \geq 0 \\
\implies \frac{\langle x - u, m\rangle^2}{\|m\|^2} - 2\frac{\langle x - u, m\rangle^2}{\|m\|^2} \geq 0 \\
\implies -\langle x - u, m\rangle^2 \geq 0 \\
\implies \langle x - u, m\rangle = 0.
\end{aligned}
$$

Thus, $x - u \in \mathcal{M}^\perp$, and $x = u + (x - u)$ is a decomposition of x. To show uniqueness, suppose $x = v + (x - v)$ such that $v \in \mathcal{M}$ and $(x - v) \in \mathcal{M}^\perp$. Then $u + (x - u) = v + (x - v)$ so $u - v = (x - v) - (x - u)$. However, $u - v \in \mathcal{M}$ because $\mathcal{M}$ is a vector space, while $(x - v) - (x - u) \in \mathcal{M}^\perp$ because

$$\langle (x - v) - (x - u), m\rangle = \langle (x - v), m\rangle - \langle (x - u, m\rangle = 0 - 0 = 0.$$

Hence $u - v \in \mathcal{M} \cap \mathcal{M}^\perp$ and so $\langle u - v, u - v\rangle = 0$ which implies $u = v$. □

Theorem 2 (Riesz Representation Theorem)**.** *Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot\rangle$, and suppose $\varphi : \mathcal{H} \to \mathbb{R}$ is a bounded linear functional. Then there exists a unique $f_\varphi \in \mathcal{H}$, known as the Riesz representation of φ, such that*

$$\varphi(x) = \langle x, f_\varphi\rangle \quad \forall x \in \mathcal{H}.$$

Remark. *Note that in finite dimensional cases, for example if $\mathcal{H}$ is $\mathbb{R}^n$ with the conventional dot product, the Riesz representation theorem is equivalent to the statement that every bounded linear function $\varphi : \mathbb{R}^n \to \mathbb{R}$ can be expressed as $\mathbf{u} \mapsto \mathbf{u} \cdot \mathbf{v}$, which is trivially true. The value of this theorem is its validity in infinite-dimensional cases, where linear functionals take on more forms than performing sum-products on a vector's components.*

Proof. Let $K = \ker(\varphi) := \{v \in \mathcal{H} : \varphi(v) = 0\}$, which is a closed subspace of $\mathcal{H}$ because ϕ is linear and bounded, so it is continuous so the preimage of a closed set is closed. First, consider if $K = \mathcal{H}$ (so, φ is the zero map). Then $\langle x, f_\varphi\rangle = 0 \quad \forall x \in \mathcal{H}$. This is achieved if $f_\varphi = 0$ due to linearity of the inner product, and is unique since $f_\varphi \neq 0 \implies \langle f_\varphi, f_\varphi\rangle \neq 0$ which produces a contradiction.

Next, suppose $K \neq \mathcal{H}$. Since K is linear,

$$(\forall x, y \in \mathcal{H}) \quad 0 = \varphi(x)\varphi(y) - \varphi(y)\varphi(x) = \varphi(\varphi(x)y) + \varphi(-\varphi(y)x) = \varphi[\varphi(x)y - \varphi(y)x]. \tag{1}$$

From the orthogonal decomposition theorem, $\mathcal{H} = K \oplus K^{\perp}$ and since $K \neq \mathcal{H}$, $K^{\perp}$ is nontrivial. Thus, let $y \in K^{\perp} : y \neq 0$ then

$$\begin{aligned}(\forall x \in \mathcal{H}) \quad \varphi(x)y - \varphi(y)x \in K,\\ \implies \langle \varphi(x)y - \varphi(y)x, y\rangle = 0,\\ \implies \varphi(x)\langle y, y\rangle - \varphi(y)\langle x, y\rangle = 0,\\ \implies \varphi(x) = \varphi(y)\frac{\langle x, y\rangle}{\langle y, y\rangle} = \left\langle x, \frac{\varphi(y)}{\langle y, y\rangle} y\right\rangle.\end{aligned}$$

Hence, $f_\varphi = \frac{\varphi(y)}{\langle y,y\rangle} y$ is the Riesz representation of φ. To show it is unique, suppose $\exists f_\varphi, g_\varphi \in \mathcal{H} : (\forall x \in \mathcal{H}), \quad \varphi(x) = \langle x, f_\varphi\rangle = \langle x, g_\varphi\rangle$. Then $\langle f_\varphi, x\rangle - \langle g_\varphi, x\rangle = \langle f_\varphi - g_\varphi, x\rangle = 0$, which implies $\langle f_\varphi - g_\varphi, f_\varphi - g_\varphi\rangle = 0$ and hence, by positive-definiteness, $f_\varphi - g_\varphi = 0$ or $f_\varphi = g_\varphi$. □

Theorem 3 (Lax-Milgram Theorem)**.** *Let $\mathcal{H}$ be a Hilbert space over $\mathbb{R}$ and let $B : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a bilinear form on $\mathcal{H}$. Assume also that B is coercive and bounded, which is to say that $\exists m, M \in \mathbb{R}$ such that $\forall x, y \in \mathcal{H}$,*

- $B(x, y) \leq M\|x\|\|y\|$,
- $m\|x\|^2 \leq B(x, x)$.

Then for any bounded linear functional φ, there exists a unique element $b_\varphi \in \mathcal{H}$ such that $B(b_\varphi, v) = \varphi(v) \quad \forall v \in H$. In other words, φ has a representative element, $b_\varphi \in \mathcal{H}$, which transforms B into φ.

Proof. Begin by fixing $u \in H$. Then $v \mapsto B(u, v)$ is a bounded linear functional. Hence by the Riesz representation theorem, there exists a unique $w \in \mathcal{H}$ such that $B(u, v) = \langle w, v\rangle \quad \forall v \in \mathcal{H}$. It is possible to uniquely find such an element for any $u \in \mathcal{H}$, so define a function $S : \mathcal{H} \to \mathcal{H}$ which sends u to the Riesz representation of the functional $B(u, \cdot\,)$.

Separately, the Riesz representation theorem guarantees that for any bounded linear functional φ, there is a unique $f_\varphi \in \mathcal{H} : \varphi(u) = \langle u, f_\varphi\rangle \quad \forall u \in \mathcal{H}$. Combining these ideas, if S is bijective then it would have an inverse S^{-1} which, after applying to f_φ, would give the element in $\mathcal{H}$ with the needed property that $B(S^{-1}(f_\varphi), v) = \langle f_\varphi, v\rangle = \varphi(v) \quad \forall v \in \mathcal{H}$.

Linearity. Take an arbitrary $\alpha_1, \alpha_2 \in \mathbb{R}$ and $u_1, u_2 \in \mathcal{H}$. Then for any $v \in \mathcal{H}$,

$$\begin{aligned}\langle S(\alpha_1 u_1 + \alpha_2 u_2), v\rangle &= B[\alpha_1 u_1 + \alpha_2 u_2, v] \\ &= \alpha_1 B[u_1, v] + \alpha_2 B[u_2, v] \\ &= \alpha_1 \langle S u_1, v\rangle + \alpha_2 \langle S u_2, v\rangle \\ &= \langle \alpha_1 S(u_1) + \alpha_2 S(u_2), v\rangle\end{aligned}$$

$$\begin{aligned}&\implies \langle S(\alpha_1 u_1 + \alpha_2 u_2), v\rangle - \langle \alpha_1 S(u_1) + \alpha_2 S(u_2), v\rangle = 0, \\ &\implies \big\langle S(\alpha_1 u_1 + \alpha_2 u_2) - (\alpha_1 S(u_1) + \alpha_2 S(u_2)), S(\alpha_1 u_1 + \alpha_2 u_2) - (\alpha_1 S(u_1) + \alpha_2 S(u_2))\big\rangle = 0, \\ &\implies S(\alpha_1 u_1 + \alpha_2 u_2) - \alpha_1 S(u_1) - \alpha_2 S(u_2) = 0, \\ &\implies S(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 S(u_1) + \alpha_2 S(u_2).\end{aligned}$$

Thus S is linear.

Injectivity. The fact that B is elliptic implies S is injective. To see this, consider that, from the Cauchy-Schwartz inequality,

$$m\|u\|^2 \le B[u, u] \;= \langle S u, u\rangle \le \|S u\|\|u\|.$$

We will use this to show that $u_1 \neq u_2 \implies S(u_1) \neq S(u_2)$.

$$\begin{aligned}&u_1 \neq u_2, \\ &\implies \|u_1 - u_2\| > 0, \\ &\implies \|S(u_1 - u_2)\| \ge m\|u_1 - u_2\| \ge 0, \\ &\implies S(u_1 - u_2) \neq 0, \\ &\implies S(u_1) - S(u_2) \neq 0, \\ &\implies S(u_1) \neq S(u_2).\end{aligned}$$

Surjectivity.

Let $R := \{S(v) : v \in \mathcal{H}\}$ be the range of S and consider an element of its orthogonal compliment $p \in R^{\perp}$.

$$\begin{aligned}&\forall v \in \mathcal{H} \quad B[v, p] = \langle S(v), p\rangle = 0, \\ &\implies 0 = B[p, p] \ge m\|p\|^2, \\ &\implies p = 0, \\ &\implies R^{\perp} = \{0\}.\end{aligned}$$

By the orthogonal decomposition theorem, if R is a closed subspace of $\mathcal{H}$, which $\{0\}$ trivially is, then

$$\mathcal{H} = R \oplus R^{\perp} = R \oplus \{0\} = R.$$

.

Uniqueness. Suppose that $\exists w_\varphi, u_\varphi \in \mathcal{H}$ such that $B(w_\varphi, v) = B(u_\varphi, v) = \varphi(v) \quad \forall v \in \mathcal{H}$. Then

$$B(w_\varphi, v) - B(u_\varphi, v) = B(w_\varphi - u_\varphi, v) = 0,$$

$$\implies 0 = B(w_\varphi - u_\varphi, w_\varphi - u_\varphi) \geq m\|w_\varphi - u_\varphi\|,$$

but since $m > 0$, this implies that $\|w_\varphi - u_\varphi\| = 0$ and $w_\varphi = u_\varphi$. □

4 Weak Formulation

Definition 9 (Test Functions, $C_c^\infty(U)$)**.** *For an open set $U \subseteq \mathbb{R}^n$, a test function $\phi : U \to \mathbb{R}$ is a function which is infinitely differentiable and whose support is a a compact subset of U.*

Remark. *Since U is open, an immediate consequence of this is that $\phi = 0$ near ∂U.*

Remark. *The set of test functions on U is notated as $C_c^\infty(U)$.*

An example of a test function with support $B(\mathbf{0}, 1)$ is

$$\phi(x) = \begin{cases} \exp\left(\frac{1}{|x|^2-1}\right) & |x| < 1 \\ 0 & |x| \geq 1. \end{cases}$$

Suppose, for a particular second-order linear elliptic operator L on a bounded and open $U \subset \mathbb{R}^n$, that the function $u \in C^2(U)$ has the following properties:

$$Lu = f, \qquad \forall x \in U,$$
$$u = 0, \qquad \forall x \in \partial U.$$

Then it would also be true that for any test function $\phi \in C_c^\infty(U)$,

$$\int_U Lu(x)\phi(x)dx = \int_U f(x)\phi(x)dx.$$

Let $B(u, \phi) := \int_U Lu(x)\phi(x)dx$. It is apparent that up to a renaming of coefficients, it is possible to rewrite L as

$$Lu = \sum_{|\alpha|\leq 2} a_\alpha(x)D^\alpha u = \sum_{i=1}^n \sum_{j=1}^n \Big(-\partial_{x_i}(a_{i,j}(x)\partial_{x_j} u)\Big) + \sum_{j=1}^n \Big(b_j(x)\partial_{x_j} u\Big) + c(x)u.$$

At which point, B may be rewritten with integration by parts,

$$B(u, \phi) = \int_U Lu(x)\phi(x)dx$$
$$= \sum_{i=1}^n \sum_{j=1}^n \Big(\int_U -\partial_{x_i}(a_{(i,j)}(x)\partial_{x_j} u)\phi(x)dx\Big) + \sum_{j=1}^n \Big(\int_U b_j(x)(\partial_{x_j} u)\phi \; dx\Big) + \int_U c(x)u \; \phi \; dx =$$
$$\sum_{i=1}^n \sum_{j=1}^n \Big(\int_U a_{(i,j)}(x)\partial_{x_j} u \; \partial_{x_i}\phi(x)dx - \int_{\partial U} a_{(i,j)}(x)\partial_{x_j} u \; \phi(x) \; \nu_i(x) \; dx\Big) + \sum_{j=1}^n \Big(\int_U b_j(x)(\partial_{x_j} u) \; \phi \; dx\Big) + \int_U c(x)u \; \phi \; dx$$
$$= \int_U \Big(\sum_{i=1}^n \sum_{j=1}^n \Big(a_{(i,j)}(x)\partial_{x_j} u \; \partial_{x_i}\phi(x)\Big) + \sum_{j=1}^n \Big(b_j(x)(\partial_{x_j} u) \; \phi\Big) + c(x)u\phi\Big) \; dx,$$

where $\nu_i : \partial U \to \mathbb{R}$ denotes the ith component of the outward-facing unit normal vector at each point $x \in \partial U$.

Remark. *The application of Green's theorem, or integration by parts, assumes the existence of an outward-facing unit normal vector to ∂U which would require ∂U to be smooth. This does not matter, though, since* $\mathrm{spt}(\phi)$ *is compactly contained in U, there exists a set $W : \mathrm{spt}(\phi) \subset W \subset U$ such that W has smooth boundary.*

Definition 10 (Weak Formulation)**.** *The equation,*

$$B(u, \phi) = \int_U f(x)\phi(x)dx \quad \forall \phi \in C_c^\infty(U), \tag{2}$$

is known as the weak form of the equation

$$Lu = f \quad \forall x \in U.$$

Observe that B is a bilinear form because it is constructed with derivatives, which are linear maps; and $L^2(U)$ inner products $\langle u, v\rangle = \int_U uv\, dx$, which are bilinear. Also note that, arranged in this way, B may act on functions which only have one derivative, as opposed to at least two. The weak formulation is inspired by the fact that, within a Hilbert space $\mathcal{H}$, $\langle u, \phi\rangle = \langle v, \phi\rangle \ \forall \phi \in X : X$ dense in $\mathcal{H} \implies u = v$. If a suitable Hilbert space of once-differentiable functions on U can be found, and it could be shown that the Lax-Milgram Theorem applies to B within this Hilbert space, then there would be exactly one element satisfying (2).

5 The Sobolev Space $H^1(U)$

The Sobolev space $H^1(U)$ is a Hilbert space whose elements are, in a weak sense, once differentiable. To begin, suppose $u \in C^1(U) \cap L^2(U)$ and also $\partial_{x_i} u \in L^2(U)$, $\forall i \in \{1, ..., n\}$. Let us define the Sobolev norm $\|\cdot\|_{H^1(U)}$ as

$$\|u\|_{H^1(U)} = \sqrt{\int_U u^2 + \sum_{i=1}^n (\partial_{x_i} u)^2 \, dx} = \sqrt{\|u\|_{L^2(U)}^2 + \sum_{i=1}^n \|\partial_{x_i} u\|_{L^2(U)}^2}.$$

Then it follows from Pythagoras' theorem that $\|u\|_{L^2(U)} \le \|u\|_{H^1(U)}$ and $\|\partial_{x_i} u\|_{L^2(U)} \le \|u\|_{H^1(U)}$. Therefore,

$$\|u\|_{H^1(U)} := \sqrt{\|u\|_{L^2(U)}^2 + \sum_{i=1}^n \|\partial_{x_i} u\|_{L^2(U)}^2} < +\infty \implies (\partial_{x_i} u) \in L^2(U) \text{ and } u \in L^2(U).$$

This construction also allows for $\|\cdot\|_{H^1(U)}$ to be generated from an inner product. Define

$$\langle u, v\rangle_{H^1(U)} := \int_U u(x)v(x) + \sum_{i=1}^n (\partial_{x_i} u)(\partial_{x_i} v)\, dx = \langle u, v\rangle_{L^2(U)} + \sum_{i=1}^n \langle \partial_{x_i} u, \partial_{x_i} v\rangle_{L^2(U)}.$$

This is so far insufficient to create a Hilbert space, since $C^1(U) \cap L^2(U)$ is not complete with respect to $\|\cdot\|_{H^1(U)}$. To resolve this, it is necessary to weaken the definition of differentiability in a similar way that the weak formulation was constructed.

Definition 11 (Weak Derivative)**.** *If $f \in L^2(U)$, it is said to be weakly α-differentiable if there exists $g \in L^2(U)$ such that*

$$\int_U g(x)\phi(x)dx = (-1)^{|\alpha|} \int_U f(x) D^\alpha \phi(x)dx \quad \forall \phi \in C_c^\infty(U).$$

In which case, g is called the weak α-derivative of f, $g = D^\alpha f$.

Remark. *The notation D^α signifies a conventional (strong) derivative when it is applied to a test function, but otherwise refers to a weak derivative.*

Remark. *Weak derivatives satisfy the expected nice properties, in that they are unique if they exist, and agree with strong derivatives. If $u \in C^1(U) \cap L^2(U)$, then*

$$\int_U (\partial_{x_i} u)\phi dx = \int_{\partial U} u\phi\nu_i dx - \int_U u\ (\partial_{x_i}\phi) dx = -\int_U u\ (\partial_{x_i}\phi) dx.$$

A full explanation of these properties can be found in Evans (p.247) [1].

6 Trace

Since U is open, $u \in H^1(U)$ cannot be directly evaluated on ∂U. However, the structure imposed by the H^1 norm is sufficient for there to exist a natural extension to the boundary, known as the trace operator.

Definition 12. *For a bounded open subset U of $\mathbb{R}^n$, $C(\overline{U})$ denotes the set of functions $f : \overline{U} \to \mathbb{R}$ which are continuous on U.*

Remark. *There is a natural equivalence between uniformly continuous functions on U to continuous functions on $\overline{U}$. The closure $\overline{U}$ of U is a closed and bounded subset of $\mathbb{R}^n$ by assumption, therefore it is compact by the Heine-Borel theorem, therefore any function which is continuous on the domain $\overline{U}$ is uniformly continuous by the Heine-Cantor theorem. Further, consider $u : U \to \mathbb{R}$ uniformly continuous, any $x \in \partial U$, and any sequence $\{x_n\} \in U : x_n \to x$. Then $\forall \varepsilon > 0,\ \exists \delta > 0 : \forall x, y \in U, |x - y| < \delta \implies |u(x) - u(y)| < \varepsilon$. Let n, m be sufficiently large so that $|x - x_n|, |x - x_m| < \frac{\delta}{2}$. Then $|x_n - x_m| < \delta$ so $|u(x_n) - u(x_m)| < \varepsilon$. This shows that $\{u(x_n)\} \subset \mathbb{R}$ is Cauchy so converges to a unique limit because $\mathbb{R}$ is complete. Hence, there is a unique continuous function $\overline{u} : \overline{U} \to \mathbb{R}$ such that $\overline{u}(x) = u(x)\ \forall x \in U$.*

Theorem 4 (Continuous Extension)**.** *Let X be dense in Y as metric spaces and let $T : X \to Z$ be a continuous map. Then if Z is complete, there is a unique continuous map $\overline{T} : Y \to Z$ such that $\overline{T}x = Tx\ \ \forall x \in X$.*

Proof. Consider a sequence $\{x_n\} \in X : x_n \to y \in Y$. If $\overline{T}$ is continuous, then it must be true that $\overline{T}x_n \to \overline{T}y$ as $x_n \to y$. Since it is stipulated that $\overline{T}x_n = Tx_n$, $Tx_n \to \overline{T}y$ as $n \to \infty$. However, limits of sequences are unique in metric spaces, so there is a unique choice for Ty that satisfies the stipulations. ☐

Theorem 5 ($C(\overline{U}) \cap C^1(U)$ is dense in $H^1(U)$)**.** *Proof.*

Remark. *An incomplete proof is provided, showing convergence in the L^2 norm. A proof of density with respect to the H^1 norm can be found in Evans (p.252) [1].*

Extend $u \in L^2(U)$ to $L^2(\mathbb{R}^n)$ by defining $u(x) = 0\ \ \forall x \in \mathbb{R}^n \setminus U$. In this way, $\|u\|_{L^2(U)} = \|u\|_{L^2(\mathbb{R}^n)}$. It is well established that $C_c(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, and so for any $\varepsilon > 0$, choose a $v \in C_c(\mathbb{R}^n) : \|v - u\|_{L^2(\mathbb{R}^n)} < \varepsilon$.

The cost of extending to $\mathbb{R}^n$ is that v is not necessarily continuous (namely on ∂U). To resolve this, we introduce the mollifier;

$$\eta_\varepsilon(x) := \frac{\phi\left(\frac{x}{\varepsilon}\right)}{\varepsilon^n \int_{\mathbb{R}^n} \phi(x) dx},$$

where $\phi(x)$ is the example test function,

$$\phi(x) = \begin{cases} \exp\left(\frac{1}{|x|^2-1}\right) & |x| < 1 \\ 0 & |x| \geq 1. \end{cases}$$

It will be shown that $v_\varepsilon := v * \eta_\varepsilon \to v$ in the L^2 norm as $\varepsilon \to 0$. Firstly,

$$\|v_\varepsilon - v\|_{L^2} = \sqrt{\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} v(y)\eta_\varepsilon(x-y)dy - v(x)\right)^2 dx}$$

$$= \sqrt{\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} [v(x-y) - v(x)]\eta_\varepsilon(y)dy\right)^2 dx}.$$

Let $f(x,y) = [v(x-y) - v(x)]\eta_\varepsilon(y)$, then it is true that

$$= \sqrt{\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x,z)dz\right)\left(\int_{\mathbb{R}^n} f(x,y)dy\right) dx},$$

where y has been renamed to z in the first integral. This means we can apply linearity of the integral, treating the $f(x,z)$ as constant factor under y and vice versa.

$$= \sqrt{\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x,z)dz\right) f(x,y)dy\right) dx} = \sqrt{\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x,z)f(x,y)\ dz\right) dy\right) dx}.$$

Interchanging order of integration,

$$= \sqrt{\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x,z)f(x,y)\ dx\right) dy\right) dz}.$$

Cauchy-Shwartz may be applied to the innermost integral.

$$\leq \sqrt{\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \sqrt{\int_{\mathbb{R}^n} f(x,z)^2\ dx} \sqrt{\int_{\mathbb{R}^n} f(x,y)^2\ dx} \quad dy\right) dz}.$$

Observing that the first factor does not depend on y,

$$= \sqrt{\int_{\mathbb{R}^n} \sqrt{\int_{\mathbb{R}^n} f(x,z)^2\ dx} \ \int_{\mathbb{R}^n} \sqrt{\int_{\mathbb{R}^n} f(x,y)^2\ dx}.\ dy\ dz}$$

Then observing that the second factor does not depend on z,

$$= \sqrt{\left(\int_{\mathbb{R}^n} \sqrt{\int_{\mathbb{R}^n} f(x,z)^2\ dx} \quad dz\right)\left(\int_{\mathbb{R}^n} \sqrt{\int_{\mathbb{R}^n} f(x,y)^2\ dx}\ \ dy\right)}.$$

Then the two factors are equal,

$$= \sqrt{\left(\int_{\mathbb{R}^n} \sqrt{\int_{\mathbb{R}^n} f(x,y)^2\ dx}\ dy\right)^2} = \int_{\mathbb{R}^n} \sqrt{\int_{\mathbb{R}^n} f(x,y)^2\ dx}\ dy.$$

We return to the original problem by substituting $f(x,y) = [v(x-y) - v(x)]\eta_\varepsilon(y)$:

$$\|v_\varepsilon - v\|_{L^2} \leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} [v(x-y) - v(x)]^2 \eta_\varepsilon(y)^2\ dx\right)^{\frac{1}{2}} dy.$$

Next, since $\eta_\varepsilon(y)$ doesn't depend on x, it is brought out of the innermost integral:

$$\|v_\varepsilon - v\|_{L^2} \leq \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}[v(x-y)-v(x)]^2 dx\ \eta_\varepsilon(y)^2\right)^{\frac{1}{2}} dy = \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}[v(x-y)-v(x)]^2 dx\right)^{\frac{1}{2}} \eta_\varepsilon(y)\ dy.$$

Now let us show that translation is continuous with respect to the L^p norm for functions in $C_c(\mathbb{R}^n)$. Let $T_h : C_c(\mathbb{R}^n) \to C_c(\mathbb{R}^n)$ be the translation operator by h, given by $(T_h u)(x) := u(x-h)$. Then $\forall \varepsilon > 0,\ \exists \delta > 0 :$ $|h| < \delta \implies \|T_h u - u\|_{L^p(\mathbb{R}^n)}$

First consider that the support of $T_h u - u$ is compact since it is the union of two compact sets, and its measure is finite—bounded by twice the measure of U, which we shall name $\mu(U)$. Additionally, since $C_c(U)$ continuous and compactly supported, it is uniformly continuous on $\mathbb{R}^n$. Therefore, $\forall \varepsilon > 0,\ \exists \delta > 0 : |h| < \delta \implies$ $|u(x-h) - u(x)| < \varepsilon$. Hence, $\forall \frac{\varepsilon}{(2\mu(U))^{1/p}} > 0,\ \exists \delta > 0$:

$$\int_{\mathbb{R}^n} |u(x-h)-u(x)|^p dx = \int_{\text{spt}(u)\cup\text{spt}(T_h u)} |u(x-h)-u(x)|^p dx \leq \int_{\text{spt}(u)\cup\text{spt}(T_h u)} \frac{\varepsilon^p}{2\mu(U)}\ dx \leq \varepsilon^p,$$

$$\implies \left(\int_{\mathbb{R}^n} |u(x-h)-u(x)|^p dx\right)^{1/p} \leq \varepsilon.$$

Therefore, we may apply this this estimate to the previous working, then

$$\|v_\varepsilon - v\|_{L^2(\mathbb{R}^n)} \leq \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}[v(x-y)-v(x)]^2 dx\right)^{1/2} \eta_\varepsilon(y) dy \leq \int_{\mathbb{R}^n} \varepsilon \eta_\varepsilon(x) dx = \varepsilon.$$

Last, consider

$$\|u_\varepsilon - v_\varepsilon\|_{L^2(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(x-y)\eta_\varepsilon(y) dy - \int_{\mathbb{R}^n} v(x-y)\eta_\varepsilon(y) dy\ dx\right)^{1/2}.$$

Using the same working from before, an upper bound is found,

$$= \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}[u(x-y)-v(x-y)]\eta_\varepsilon(y) dy\right)^2 dx\right)^{1/2} \leq \int_{\mathbb{R}^n}\sqrt{\int_{\mathbb{R}^n}\Big([u(x-y)-v(x-y)]\eta_\varepsilon(y)\Big)^2 dx}\ dy$$

$$= \int_{\mathbb{R}^n}\sqrt{\int_{\mathbb{R}^n}\Big([u(x-y)-v(x-y)]\Big)^2 dx}\ \eta_\varepsilon(y)\ dy,$$

and since the inner most integral is translation-independent, it will simply evaluate to the L^2 norm.

$$\|u_\varepsilon - v_\varepsilon\|_{L^2(\mathbb{R}^n)} \leq \int_{\mathbb{R}^n} \|u - v\|_{L^2(\mathbb{R}^n)}\ \eta_\varepsilon(y)\ dy = \|u - v\|_{L^2(\mathbb{R}^n)} \leq \varepsilon.$$

Therefore, it follows from the triangle inequality that

$$\|u - u_\varepsilon\|_{L^2(U)} \leq \|u - u_\varepsilon\|_{L^2(\mathbb{R}^n)} \leq \|u - v\|_{L^2(\mathbb{R}^n)} + \|v - v_\varepsilon\|_{L^2(\mathbb{R}^n)} + \|v_\varepsilon - u_\varepsilon\|_{L^2(\mathbb{R}^n)} \leq 3\varepsilon,$$

and so $C^1(\overline{U})$ is dense in $L^2(U)$.

Theorem 6 (Trace Theorem)**.** *Assume U is a bounded Lipschitz domain. Then there is a bounded linear operator $T : H^1(U) \to L^2(\partial U)$, known as the trace, such that $Tu = u|_{\partial U}$ if $u \in H^1(U) \cap C(\bar{U})$ and $\|Tu\|_{L^2(\partial U)} \leq C\|u\|_{H^1(U)}$.*

Remark. *Since $C(\overline{U})$ is dense in $H^1(U)$, it follows from Theorem 4 that this operator is unique and there is only one natural choice for the trace of $u \in H^1(U)$.*

Proof. Firstly assume $u \in C(\overline{U})$. By assumption, for each $p \in \partial U$, there exists a hyperplane Π through p with a unit normal ν, and a Lipschitz function $g : \Pi \to \mathbb{R}$ over the hyperplane, and a small $\varepsilon > 0$ such that

$$U \cap B(p, \varepsilon) = \{\pi + t\nu \mid \pi \in B(p, \varepsilon) \cap \Pi, t > g(\pi)\} \cap B(p, \varepsilon).$$

Define $x' = (x_1, ..., x_{n-1}) \in \mathbb{R}^{n-1}$ and, without loss of generality, suppose this plane is $\{(x', x_n) \in \mathbb{R}^n : x_n = 0\}$. Define a change of coordinates Φ according to the following rule $\Phi : (x', x_n) \mapsto (x', x_n - g(x'))$. Its Jacobian is 1 since $\frac{\partial}{\partial x_n} g(x') = 0$.

Now, within a small ball $B(p, \varepsilon')$, for example with $\varepsilon' = \varepsilon/K$ where K is the Lipschitz constant of g, the boundary of the transformed set $\partial(\Phi U)$ coincides with $\{(x_1, ..., x_n) : x_n = 0\}$. Take yet another smaller ball inside this one, for example $B(p, \varepsilon'/2)$ and define the a cutoff function function χ which is exactly equal to 1 in $B(p, \varepsilon'/2)$, exactly equal to 0 outside of $B(p, \varepsilon')$, and smooth with range $[0, 1]$. Finally, for convenience write $y = \Phi x$. Then,

$$\int_{B(p,\varepsilon'/2)\cap\{y_n=0\}} |u(y)|^2 dy' \leq \int_{B(p,\varepsilon')\cap\{y_n=0\}} \chi|u(y)|^2 dy' = -\int_{B(p,\varepsilon')\cap\{y_n\geq 0\}} \partial_{y_n}(\chi|u(y)|^2) dy.$$

The third step was achieved by applying the fundamental theorem of calculus: $f(0) - f(a) = -\int_0^a f'(t)dt$ in the y_n direction. Then after a direct application of product rule,

$$\leq \left| \int_{B(p,\varepsilon')\cap\{y_n\geq 0\}} (\partial_{y_n}\chi)u(y)^2 + 2u(y)\chi(\partial_{y_n}u(y))dy \right|.$$

Since $\partial_{y_n}\chi$ is smooth with compact support it is bounded.

$$\leq \int_{B(p,\varepsilon')\cap\{y_n\geq 0\}} \left|(\partial_{y_n}\chi)u^2\right| + \left|2u\chi(\partial_{y_n}u)\right| dy$$

$$\leq \int_{B(p,\varepsilon')\cap\{y_n\geq 0\}} Cu^2 + \left(|u\chi|^2 + |\partial_{y_n}u|^2\right) dy \leq C \int_{B(p,\varepsilon')\cap\{y_n\geq 0\}} u^2 + \sum_{i=1}^{n} (\partial_{y_i}u)^2 dy.$$

$$\implies \left\| u|_{\partial U\cap B(p,\varepsilon')} \right\|^2_{L^2(\partial U\cap B(p,\varepsilon')} \leq \left\| u|_{U\cap B(p,\varepsilon')} \right\|_{H^1(U\cap B(p,\varepsilon')}$$

Since ∂U is a closed and bounded subset of $\mathbb{R}^n$ it is compact, meaning after taking a finite subcover of these balls $\{B(p_k, \varepsilon_k)\}_{k=1}^n$ we can sum the inequalities to have

$$\|u|_{\partial U}\|_{L^2(\partial U)} \leq \sum_{k=1}^{n} \|u|_{B(p_k,\varepsilon_k)\cap\partial U}\|_{L^2(B(p_k,\varepsilon_k)\cap\partial U)} \leq \sum_{k=1}^{n} \|u|_{B(p_k,\varepsilon_k)\cap U}\|_{H^1(B(p_k,\varepsilon_k)\cap U)} \leq \|u\|_{H^1(U)}$$

Therefore we have shown $\|Tu\|_{L^2(\partial U)} \leq C\|u\|_{H^1(U)}$. If we stipulate that T is linear, which is a sensible thing to do since it is a restriction operator, this implies T is continuous. □

7 Standard Existence Theorem

Definition 13. *The set* $H_0^1(U)$ *is the space of functions* $u \in H^1(U)$ *such that* $Tu = 0$*, where* T *is the trace operator from Theorem 6.*

Theorem 7 (Standard Existence Theorem (Evans))**.** *Let* $U \subset \mathbb{R}^n$ *be a Lipschitz, bounded and open domain and let* L *be a uniformly elliptic differential operator of the form*

$$Lu = \sum_{i=1}^{n}\sum_{j=1}^{n}\Big(-\partial_{x_i}(a_{i,j}(x)\partial_{x_j}u)\Big) + \sum_{j=1}^{n}\Big(b_j(x)\partial_{x_j}u\Big) + c(x).$$

Last, suppose $f \in L^2(U)$*. Then there exists a unique solution* $u \in H_0^1(U)$ *to the weak-form problem:*

$$\int_U \left(\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(a_{(i,j)}(x)\partial_{x_j}u\ \partial_{x_i}\phi(x)\Big) + \sum_{j=1}^{n}\Big(b_j(x)(\partial_{x_j}u)\ \phi\Big) + c(x)u\phi\right) dx = \int_U f(x)\phi(x)dx \quad \forall \phi \in C_c^\infty(U).$$

8 Unanswered Questions

1. Let $U = \{(x,y) \in \mathbb{R}^2 : y > x^{2/3} \ \wedge\ x^2 + y^2 < 1\}$, and $f \in L^2(U)$, $f_0 \in L^2(\partial U)$. Lastly, assume f_0 is continuous at $(0,0)$. Then does there exist a solution u in $H^1(U)$ to the problem?

$$\begin{aligned} \Delta u &= f, && \forall x \in U \\ u &= f_0, && \forall x \in \partial U \end{aligned}$$

2. Higher order elliptic boundary value problems often have boundary conditions that specify trace as well as normal derivatives, and these problems have been shown to be well-posed. Is it possible to also prove a well-posed formulation of boundary value problems which use punctures, as seen in Counter-Example?

3. Can the theory of slit domains and punctured domains be unified, in the same way that the theory of C^1 domains and Lipschitz domains has been unified?

References

[1] Lawrence C. Evans, *Partial Differential Equations.* Graduate Studies in Mathematics, American Mathematical Society, 1997.

[2] Rakesh *Math 836: Partial Differential Equations.* University of Delaware, 2023.

[3] David Gilbarg, Neil S. Trudinger, *Elliptic Partial Differential Equations of Second Order.* Classics in Mathematcs, American Mathematical Society, 2001. doi.org/10.1007/978-3-642-61798-0.

[4] Yoshito Ishiki *Quasi-symmetric invariant properties of Cantor metric spaces* Annales de l'Institut Fourier, 2019. arXiv:1710.08190v5.