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Presentations for the full-domain partition monoid and its singular ideal

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Abstract

We prove novel presentations for the full-domain partition monoid $\mathcal{P}_n^{\mathrm{fd}}$ and its singular ideal $\mathcal{P}_n^{\mathrm{fd}}\backslash\mathcal{S}_n$. The monoid $\mathcal{P}_n^{\mathrm{fd}}$ consists of all partition diagrams with full domain. It is a right restriction submonoid of the partition monoid $\mathcal{P}_n$, having factorisation $\mathcal{P}_n^{\mathrm{fd}} = \mathcal{T}_n\mathcal{F}_n$ where $\mathcal{T}_n$ is (an isomorphic copy of) the full-transformation monoid on $\mathbf{n} = \{1, 2, \ldots, n\}$ and $\mathcal{F}_n$ is (an isomorphic copy of) $\mathfrak{Eq}_{\mathbf{n}}$, the semilattice of equivalences on $\mathbf{n}$. This paper obtains presentations for $\mathcal{P}_n^{\mathrm{fd}}$ through the use of existing presentations for the symmetric group $\mathcal{S}_n$, $\mathcal{T}_n\backslash\mathcal{S}_n$, and $\mathcal{F}_n$. These are proven both directly and via generalised techniques recently developed for restriction monoids.

1 Introduction

Diagram monoids are fundamental algebraic structures with origins and applications in many different mathematical and scientific disciplines, such as knot theory and theoretical physics. The recent paper [5] uncovered a so-called *Ehresmann structure* on the partition monoid $\mathcal{P}_n$, opening up a new categorical way to understand diagram monoids. Such structures involve an underlying semilattice (monoid of commuting idempotents). The semilattice in this case is $\mathcal{F}_n$, the set of all quotient identities of $\mathbf{n} = \{1, 2, \ldots, n\}$. This is isomorphic to the join semilattice $\mathfrak{Eq}_n$ of equivalence relations of $\mathbf{n}$ which is studied in [6].

A consequence of the Ehresmann structure was the discovery of a new monoid, the *full-domain partition monoid* $\mathcal{P}_n^{\mathrm{fd}} = \{\rho \in \mathcal{P}_n : \mathrm{dom}(\rho) = \mathbf{n}\}$, which is a right restriction submonoid of $\mathcal{P}_n$ [5]. At a similar time, this monoid was also discovered independently through the study of constellations [9]. The monoid $\mathcal{P}_n$ contains (an isomorphic copy of) the full-transformation monoid $\mathcal{T}_n$, the set of all mappings $\mathbf{n} \to \mathbf{n}$ under composition, and general Ehresmann theory leads to the product decomposition $\mathcal{P}_n^{\mathrm{fd}} = \mathcal{T}_n\mathcal{F}_n$. In this way, $\mathcal{P}_n^{\mathrm{fd}}$ can be thought of as a categorical dual to the partial transformation monoid $\mathcal{PT}_n$, the set of all mappings $A \to \mathbf{n}$ for each $A \subseteq \mathbf{n}$ under composition, which is a left restriction monoid with respect to the semilattice $\mathcal{E}_n$ of partial identities, and decomposes as $\mathcal{PT}_n = \mathcal{E}_n\mathcal{T}_n$.

This paper initiates the study of the monoid $\mathcal{P}_n^{\mathrm{fd}}$ by obtaining presentations for it and its singular ideal $\mathcal{P}_n^{\mathrm{fd}}\backslash\mathcal{S}_n$, where $\mathcal{S}_n$ is (an isomorphic copy of) the symmetric group on $\mathbf{n}$. Two approaches are taken, both of which rely on using pre-existing presentations for $\mathcal{S}_n$, $\mathcal{T}_n\backslash\mathcal{S}_n$, and $\mathcal{F}_n$ in the aforementioned product decomposition.

The first approach proves its presentations directly and is self-contained, providing all necessary definitions, results and proofs. Our main results, including presentations for $\mathcal{P}_n^{\mathrm{fd}}\backslash\mathcal{S}_n$ (Theorem 2.3) and $\mathcal{P}_n^{\mathrm{fd}}$ (Theorem 2.4), can be found in Section 2 along with all necessary definitions. The proofs of Theorems 2.3 and 2.4 be found in Sections 3 and 4, respectively.

The second approach, found in Section 5, obtains presentations using the results in [2], which provides a generalised methodology for constructing presentations for restriction monoids by decomposing them into products arising from *action pairs*. The main result of this approach is an alternative presentation for $\mathcal{P}_n^{\mathrm{fd}}$ (Theorem 5.5), as well as an alternative proof for Theorem 2.3. Whilst we do provide some context, this section is intended to be read in conjunction with [2].

Other than that which is appropriately referenced, all work contained in this paper is my own. However, I would like to acknowledge the many contributions of my supervisors James East and Matthias Fresacher, who provided valuable guidance and feedback throughout the project and inspired many of the ideas and arguments used in this paper.

2 Preliminaries

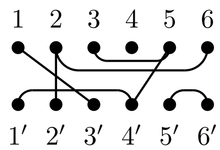
A *binary operation* on a set X is a map $X \times X \to X$ that is typically denoted by juxtaposition. We say this operation is associative if $(xy)z = x(yz)$ for all $x, y, x \in X$. A *semigroup* S is a set with an associative binary operation. A *monoid* M is a semigroup that contains an *identity element* e, meaning that $ae = ea = a$ for all $a \in M$. A *group* G is a monoid where, for each $a \in G$, there exists an inverse a^{-1}, meaning that $aa^{-1} = a^{-1}a = e$, where e is the identity element in G. If S is a semigroup (resp., monoid or group) with the binary operation $\circ$ and $T \subseteq S$ is a semigroup (resp., monoid or group) with the binary operation $\circ|_T$ (i.e., the operation of S restricted to T) we say that T is a *subsemigroup* (resp., *submonoid* or *subgroup*) of S. If X is a subset of a semigroup S, we will write $\langle X \rangle$ to denote the subsemigroup generated by X (i.e., the smallest subsemigroup of S that contains X). In contrast to traditional functional notation, we will write maps on the right. Given semigroups (resp., monoids or groups) S and T, a *semigroup* (resp., *monoid* or *group*) *homomorphism* is a map $\phi : S \to T$ such that $(xy)\phi = (x\phi)(y\phi)$ for all $x, y \in S$ (and, for a monoid homomorphism, $e\phi = f$, where e and f are the identity elements of S and T respectively). The *image* and *kernel* of ϕ are defined by

$$\mathrm{im}(\phi) = \{y \in T : y = x\phi \text{ for some } x \in S\} \quad \text{and} \quad \ker(\phi) = \{(x, y) \in S \times S : x\phi = y\phi\}.$$

If ϕ is surjective, we say it is a *surmorphism.* If ϕ is bijective, we say it is an *isomorphism.* If there exists an isomorphism $\phi : S \to T$ between two semigroups (resp., monoids or groups) S and T, we say they are *isomorphic.*

An *equivalence relation* ε on a set X is a subset of $X \times X$ that is *reflexive* (i.e., $(x, x) \in \varepsilon$ for all $x \in X$), *symmetric* (i.e., $(x, y) \in \varepsilon$ implies that $(y, x) \in \varepsilon$ for all $x, y \in X$) and *transitive* (i.e., $(x, y), (y, z) \in \varepsilon$ implies that $(x, z) \in \varepsilon$ for all $x, y, z \in X$). The set of all equivalence relations on X, denoted $\mathfrak{Eq}_X$, forms a monoid under the *join* operation. The join $\varepsilon \vee \eta$ of $\varepsilon, \eta \in \mathfrak{Eq}_X$ is defined as the smallest equivalence relation on X containing $\varepsilon \cup \eta$. Using the notation $x \xrightarrow{\varepsilon} y$ to indicate $(x, y) \in \varepsilon$ for a given $\varepsilon \in \mathfrak{Eq}_X$, a pair (x, y) is in $\varepsilon \vee \eta$ if and only if there exists a chain $x \xrightarrow{\varepsilon} z_1 \xrightarrow{\eta} z_2 \xrightarrow{\varepsilon} \cdots \xrightarrow{\varepsilon} z_{p-1} \xrightarrow{\eta} y$ for some even $p \in \mathbb{N}$. The identity element in $\mathfrak{Eq}_X$ is $\Delta_X = \{(x, x) : x \in X\}$, the *trivial relation* on X, and $\mathfrak{Eq}_X$ is both *commutative* (i.e., $\varepsilon \vee \eta = \eta \vee \varepsilon$ for all $\varepsilon, \eta \in \mathfrak{Eq}_X$), and consists entirely of *idempotents* (i.e., $\varepsilon \vee \varepsilon = \varepsilon$ for all $\varepsilon \in \mathfrak{Eq}_X$), making it a *semilattice.* A *partition* of X is a set $\rho = \{A_1, A_2, \ldots, A_p\}$ such that each *block* A_i is a non-empty subset of X, the blocks are pairwise disjoint, and the union of the blocks is X. An equivalence relation $\varepsilon \in \mathfrak{Eq}_X$ induces a partition of X, where each block of the partition is an *equivalence class* consisting only of elements related to one another in ε. Conversely, any partition of X induces an equivalence relation on X. Given a partition ρ of a set X, we write $[x]_\rho$ to indicate the block of ρ that contains the element $x \in X$. Analagously, we write $[x]_\varepsilon$ to indicate the equivalence class of x in $\varepsilon \in \mathfrak{Eq}_X$.

For an integer $n \geq 2$ ($n < 2$ being trivial), define $\mathbf{n} = \{1, 2, \ldots, n\}$ and, for $A \subseteq \mathbf{n}$, define $A' = \{a' : a \in A\}$. The *partition monoid of degree n*, denoted $\mathcal{P}_n$, is the monoid whose elements are partitions of the set $\mathbf{n} \cup \mathbf{n}'$, with *concatenation* as the binary operation, which will be described shortly. Elements of $\mathcal{P}_n$ may be represented as a graph consisting of two rows of n vertices corresponding to the sets $\mathbf{n}$ and $\mathbf{n}'$, and whose connected components correspond to the blocks of the partition. For example, the graph

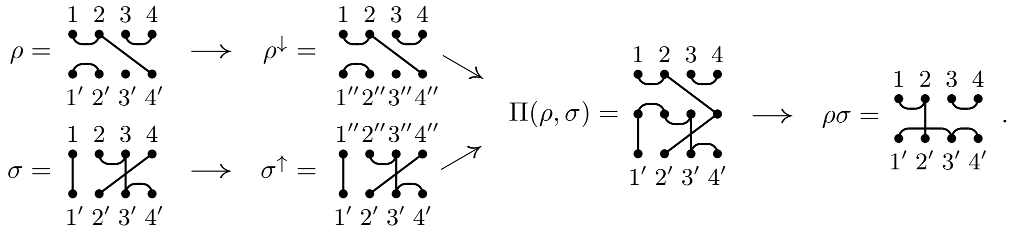


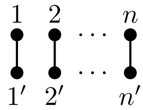
, (2.1)

represents the partition

$$\rho = \big\{\{1, 3'\}, \{2, 2', 6\}, \{1', 3, 4', 5\}, \{4\}, \{5', 6'\}\big\} \in \mathcal{P}_6\,.$$

These graphical representations are not necessarily unique, as different sets of edges can form the same connected components. To concatenate ρ followed by σ where $\rho, \sigma \in \mathcal{P}_n$, let $\rho^{\downarrow}$ be the graph ρ modified such that each lower vertex i' is relabeled i'', and $\sigma^{\uparrow}$ be σ modified such that each upper vertex i is relabeled i''. Then, by identifying each lower vertex i'' of $\rho^{\downarrow}$ with the corresponding upper vertex i'' of $\sigma^{\uparrow}$, these two graphs can be combined to form the product graph $\Pi(\rho, \sigma)$ on the vertex set $\mathbf{n} \cup \mathbf{n}' \cup \mathbf{n}''$. The concatenation $\rho\sigma$ is then defined as the partition that, for all $x, y \in \mathbf{n} \cup \mathbf{n}'$, $[x]_{\rho\sigma} = [y]_{\rho\sigma}$ if and only if there is a path between x and y in $\Pi(\rho, \sigma)$. For example,



The identity element of $\mathcal{P}_n$ is the partition $\mathrm{id}_{\mathbf{n}} =$  .

If a block of a partition contains elements from both $\mathbf{n}$ and $\mathbf{n}'$, it is called a *transversal*, while an *upper* or *lower non-transversal* contains only elements from $\mathbf{n}$ or $\mathbf{n}'$, respectively. For example, the partition ρ given in (2.1) consists of three transversals, namely $\{1, 3'\}, \{2, 2', 6\}$ and $\{1', 3, 4', 5\}$, one upper non-transversal $\{4\}$ (a *singleton*), and one lower non-transversal $\{5', 6'\}$. For $\rho \in \mathcal{P}_n$, the *domain* is the subset of $\mathbf{n}$ whose elements in the upper row of ρ are in a transversal of ρ, whilst the *codomain* is similarly defined for the lower row of ρ. That is,

$$\mathrm{dom}(\rho) = \{x \in \mathbf{n} : [x]_\rho \cap \mathbf{n}' \neq \emptyset\} \quad \text{and} \quad \mathrm{codom}(\rho) = \{x \in \mathbf{n} : [x']_\rho \cap \mathbf{n} \neq \emptyset\}.$$

The *kernel* and *cokernel* of ρ are defined as the equivalence relations

$$\ker(\rho) = \{(x, y) \in \mathbf{n} \times \mathbf{n} : [x]_\rho = [y]_\rho\} \quad \text{and} \quad \mathrm{coker}(\rho) = \{(x, y) \in \mathbf{n} \times \mathbf{n} : [x']_\rho = [y']_\rho\}.$$

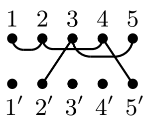
Note, however, that this definition of $\ker(\rho)$ is distinct from that used for homomorphisms. The equivalence classes of $\ker(\rho)$ are the subsets of $\mathbf{n}$ whose elements in the upper row of ρ share a block in ρ, whilst the classes in $\mathrm{coker}(\rho)$ are similarly defined for the lower row of ρ. Continuing with our example in (2.1), we have $\mathrm{dom}(\rho) = \{1, 2, 3, 5, 6\}$ and $\mathrm{codom}(\rho) = \{1, 2, 3, 4\}$, whilst $\ker(\rho)$ and $\mathrm{coker}(\rho)$ have equivalence classes $\{1\}, \{2, 6\}, \{4\}, \{3, 5\}$, and $\{1, 4\}, \{2\}, \{3\}, \{5, 6\}$, respectively.

At times, we will use the block notation $\rho = \left(\begin{array}{c|c|c|c|c|c} A_1 & \cdots & A_p & C_1 & \cdots & C_q \\ B_1 & \cdots & B_p & D_1 & \cdots & D_r \end{array}\right)$ to represent a partition $\rho \in \mathcal{P}_n$ with transversals $A_1 \cup B_1', A_2 \cup B_2' \ldots, A_p \cup B_p'$, upper non-transversals $C_1, C_2, \ldots, C_q$, and lower non-transversals $D_1', D_2', \ldots, D_r'$, where each $A_i, B_i, C_i, D_i \subseteq \mathbf{n}$.

The submonoid $\mathcal{T}_n$ of $\mathcal{P}_n$ is defined by

$$\mathcal{T}_n = \{\alpha \in \mathcal{P}_n : \mathrm{dom}(\alpha) = \mathbf{n},\, \mathrm{coker}(\alpha) = \Delta_{\mathbf{n}}\}.$$

This monoid is isomoporphic to the *full-transformation monoid* on $\mathbf{n}$, the set of all mappings $\mathbf{n} \to \mathbf{n}$ under composition. As such, it will be convenient to identify a partition $\alpha \in \mathcal{T}_n$ with its corresponding map $\mathbf{n} \to \mathbf{n}$ and use the notation $x\alpha = y$ to indicate the unique element $y \in \mathbf{n}$ for a given $x \in \mathbf{n}$ such that $[x]_\alpha = [y']_\alpha$. For example, for $\alpha \in \mathcal{T}_5$ with graphical representation

$\alpha =$  ,

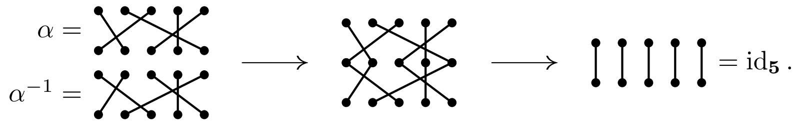
we have $1\alpha = 2\alpha = 4\alpha = 5$ and $3\alpha = 5\alpha = 2$. For $\alpha \in \mathcal{T}_n$, we will use the simplified block notation $\alpha = \left[\begin{array}{c|c|c|c} A_1 & A_2 & \cdots & A_p \\ b_1 & b_2 & \cdots & b_p \end{array}\right]$, indicating that α has transversals $A_1 \cup \{b_1'\}, A_2 \cup \{b_2'\}, \ldots, A_p \cup \{b_p'\}$, where

$A_1, A_2, \ldots, A_p \subseteq \mathbf{n}$ and $b_1, b_2, \ldots, b_p \in \mathbf{n}$ are singletons. Any omitted vertices are assumed to be elements of a lower non-transversal singleton, as each $\alpha \in \mathcal{T}_n$ has *full domain* (i.e., each $x \in \mathbf{n}$ is part of a transversal in α).

We will also be considering two complementary subsemigroups of $\mathcal{T}_n$ with definitions:

$$\mathcal{S}_n = \{\alpha \in \mathcal{T}_n : \ker(\alpha) = \Delta_{\mathbf{n}}\} \quad \text{and} \quad \mathcal{T}_n \backslash \mathcal{S}_n = \{\alpha \in \mathcal{T}_n : \ker(\alpha) \neq \Delta_{\mathbf{n}}\}.$$

The first, $\mathcal{S}_n$, is the group of *units* (i.e., invertible elements) in $\mathcal{P}_n$ and is isomorphic to the *symmetric group*, the set of all bijective maps $\mathbf{n} \to \mathbf{n}$ under composition. The graphical representation of the inverse α^{-1} of any $\alpha \in \mathcal{S}_n$ can be obtained from that of α via vertical reflection. For example,

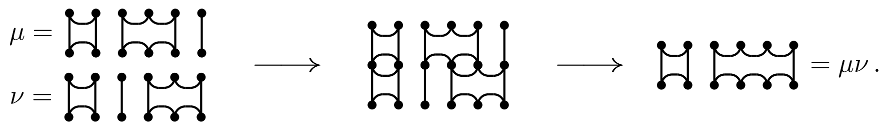


The second is $\mathcal{T}_n \backslash \mathcal{S}_n$, is the semigroup of all *singular* (i.e., non-invertible) partitions in $\mathcal{T}_n$. This semigroup is an *ideal* of $\mathcal{T}_n$ (i.e., $\alpha\beta, \beta\alpha \in \mathcal{T}_n \backslash \mathcal{S}_n$ for all $\alpha \in \mathcal{T}_n$ and $\beta \in \mathcal{T}_n \backslash \mathcal{S}_n$).

For an equivalence relation $\varepsilon \in \mathfrak{Eq}_{\mathbf{n}}$ with equivalence classes $A_1, A_2, \ldots, A_p$, we will use id_ε to represent the partition $\mathrm{id}_\varepsilon = \begin{pmatrix} A_1 & A_2 & \cdots & A_p \\ A_1 & A_2 & \cdots & A_p \end{pmatrix} \in \mathcal{P}_n$. The submonoid $\mathcal{F}_n$ of $\mathcal{P}_n$ is defined by

$$\mathcal{F}_n = \{\mathrm{id}_\varepsilon : \varepsilon \in \mathfrak{Eq}_{\mathbf{n}}\}.$$

It is clear from these definitions that $\mu = \mathrm{id}_{\ker(\mu)} = \mathrm{id}_{\mathrm{coker}(\mu)}$ for each $\mu \in \mathcal{F}_n$. Furthermore, $\mathcal{F}_n$ is isomporphic to the semilattice $\mathfrak{Eq}_{\mathbf{n}}$ as $\mathrm{id}_\varepsilon \mathrm{id}_\eta = \mathrm{id}_{\varepsilon \vee \eta}$ for all $\varepsilon, \eta \in \mathfrak{Eq}_{\mathbf{n}}$. For example, if $\mu, \nu \in \mathcal{F}_6$ are such that $\mu = \mathrm{id}_\varepsilon$ and $\nu = \mathrm{id}_\eta$, where $\varepsilon, \eta \in \mathfrak{Eq}_{\mathbf{6}}$ have equivalence classes $\{1,2\}, \{3,4,5\}, \{6\}$ and $\{1,2\}, \{3\}, \{4,5,6\}$ respectively, then $\mu\nu = \mathrm{id}_{\varepsilon \vee \eta}$, where $\varepsilon \vee \eta$ has equivalence classes $\{1,2\}, \{3,4,5,6\}$. Pictorially,



The focus of this paper is the *full-domain partition monoid* $\mathcal{P}_n^{\mathrm{fd}}$, a submonoid of $\mathcal{P}_n$, and its singular ideal $\mathcal{P}_n^{\mathrm{fd}} \backslash \mathcal{S}_n$, a subsemigroup of $\mathcal{P}_n^{\mathrm{fd}}$, with definitions:

$$\mathcal{P}_n^{\mathrm{fd}} = \{\rho \in \mathcal{P}_n : \mathrm{dom}(\rho) = \mathbf{n}\} \quad \text{and} \quad \mathcal{P}_n^{\mathrm{fd}} \backslash \mathcal{S}_n = \{\rho \in \mathcal{P}_n^{\mathrm{fd}} : \ker(\rho) \neq \Delta_{\mathbf{n}}\}.$$

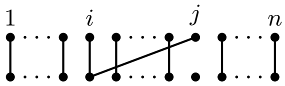
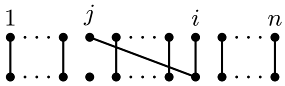
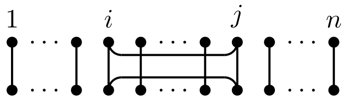
A partition $\rho \in \mathcal{P}_n^{\mathrm{fd}}$ has simplified block notation $\rho = \begin{pmatrix} A_1 & \cdots & A_p & & & \\ B_1 & \cdots & B_p & C_1 & \cdots & C_q \end{pmatrix}$, where $p < n$ in the case that $\rho \in \mathcal{P}_n^{\mathrm{fd}} \backslash \mathcal{S}_n$. As all $\alpha \in \mathcal{T}_n$ and $\mu \in \mathcal{F}_n$ have full domain it follows that $\mathcal{T}_n, \mathcal{F}_n \subseteq \mathcal{P}_n^{\mathrm{fd}}$. Indeed, an important property of $\mathcal{P}_n^{\mathrm{fd}}$ and $\mathcal{P}_n^{\mathrm{fd}} \backslash \mathcal{S}_n$ we will make use of are the following product decompositions, first alluded to in [5, page 344]:

$$\mathcal{P}_n^{\mathrm{fd}} = \mathcal{T}_n \mathcal{F}_n \quad \text{and} \quad \mathcal{P}_n^{\mathrm{fd}} \backslash \mathcal{S}_n = (\mathcal{T}_n \backslash \mathcal{S}_n)\mathcal{F}_n \,. \tag{2.2}$$

The forward inclusion for these decompositions is demonstrated in the proof of Proposition 3.1, whilst the reverse inclusion becomes apparent by observing that the product of two partitions with full domain must itself have full domain.

Fix some semigroup S. Roughly speaking, a *semigroup presentation* for S reduces S to a set of *generators* which can be used to construct any element of S, and a set of *relations* which can be used to describe any equivalence amongst elements of S. Formally, a *congruence* $\sim$ is an equivalence relation on S such that, for all $a, b, c, d \in S$, if $a \sim c$ and $b \sim d$, then $ab \sim cd$. The set of all $\sim$–classes then form the quotient semigroup $S/{\sim}$ under the induced operation (i.e., $[a]_\sim \cdot [b]_\sim = [ab]_\sim$ for all $a, b \in S$). The *Fundamental Homomorphism Theorem* states that if $\phi : S \to T$ is a semigroup homomorphism, then $\ker(\phi)$ is a congruence on S, and $S/\ker(\phi)$ is isomorphic to $\mathrm{im}(\phi)$. The *free semigroup* on an *alphabet* (i.e., set of symbols) X, denoted X^+, is the set of all possible non-empty strings consisting of *letters* (i.e., symbols) in X under the juxtaposition operation. Elements of X^+ are called *words*, and the *length* of a word w, denoted by $|w|$, is the number of letters it contains. Similarly, we denote the *free monoid* as $X^* = X^+ \cup \{\iota\}$, where ι, the *empty word*, is the identity element in X^*, having length 0. Let X be an alphabet, and $R \subseteq X^+ \times X^+$ (resp., $R \subseteq X^* \times X^*$) be a set of relations on the free semigroup (resp., monoid) on X. We denote by $R^\sharp$ the congruence on X^+ (resp., X^*) *generated* by R; i.e., the smallest congruence on X^+ (resp., X^*) containing R. We then say a semigroup (resp., monoid) S has *semigroup* (resp., *monoid*) *presentation* $\langle X : R\rangle$ if $S \cong X^+/R^\sharp$ (resp., $S \cong X^*/R^\sharp$), or, equivalently, if there is a surmorphism $X^+ \to S$ (resp., $X^* \to S$) with kernel $R^\sharp$. If ϕ is such a surmorphism, we say S has *presentation* $\langle X : R\rangle$ via ϕ. Whether a presentation is a semigroup or monoid presentation will be explicitly stated. For convenience, a relation $(w_1, w_2) \in R$ is depicted as the equation $w_1 = w_2$.

For $i, j \in \mathbf{n}$ where $i \neq j$, define the following partitions in $\mathcal{P}_n^{\mathrm{fd}} \backslash \mathcal{S}_n$:

$\widehat{e}_{ij} =$  if $i < j$;  if $i > j$ and $\widehat{t}_{ij} =$  .

Note that $\widehat{t}_{ij} = \widehat{t}_{ji}$ for all $i, j \in \mathbf{n}$. Consider the corresponding alphabets $E = \{e_{ij} : i, j \in \mathbf{n},\ i \neq j\}$ and $T = \{t_{ij} = t_{ji} : i, j \in \mathbf{n},\ i \neq j\}$. Define a semigroup homomorphism

$$\phi : (E \cup T)^+ \longrightarrow \mathcal{P}_n^{\mathrm{fd}} \backslash \mathcal{S}_n$$

by $e_{ij} \mapsto \widehat{e}_{ij}$ and $t_{ij} \mapsto \widehat{t}_{ij}$. Consider the relations

$$e_{ij}^2 = e_{ij} = e_{ji}e_{ij}\,, \qquad \text{(T1)}$$
$$e_{ij}e_{kl} = e_{kl}e_{ij}\,, \qquad \text{(T2)}$$
$$e_{ik}e_{jk} = e_{ik}\,, \qquad \text{(T3)}$$
$$e_{ij}e_{ik} = e_{ik}e_{ij} = e_{jk}e_{ij}\,, \qquad \text{(T4)}$$
$$e_{ki}e_{ij}e_{jk} = e_{ik}e_{kj}e_{ji}e_{ik}\,, \qquad \text{(T5)}$$
$$e_{ki}e_{ij}e_{jk}e_{kl} = e_{ik}e_{kl}e_{li}e_{ij}e_{jl}\,, \qquad \text{(T6)}$$

$$t_{ij}^2 = t_{ij}\,, \qquad \text{(F1)}$$
$$t_{ij}t_{kl} = t_{kl}t_{ij}\,, \qquad \text{(F2)}$$
$$t_{ij}t_{jk} = t_{jk}t_{ki}\,, \qquad \text{(F3)}$$

$$t_{ij}e_{ij} = e_{ij}\,, \qquad \text{(PS1)}$$
$$t_{jk}e_{ij} = e_{ij}t_{ik}\,, \qquad \text{(PS2)}$$
$$t_{kl}e_{ij} = e_{ij}t_{kl}\,, \qquad \text{(PS3)}$$
$$e_{ij}t_{ij} = t_{ij}\,, \qquad \text{(PS4)}$$

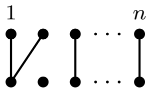
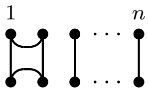
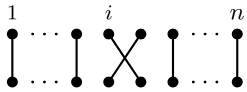
where $i, j, k, l \in \mathbf{n}$ are all distinct, except for (F2) which only requires $i \neq j$ and $k \neq l$. Define

$$R_{PS} = \text{(T1–T6)} \cup \text{(F1–F3)} \cup \text{(PS1–PS4)}.$$

With these definitions, we can now state our first main result, the proof of which is found in Section 3 (with an alternative proof using action pairs located in Section 5.1).

Theorem 2.3. *The semigroup $\mathcal{P}_n^{\mathrm{fd}} \backslash \mathcal{S}_n$ has semigroup presentation $\langle E \cup T : R_{PS}\rangle$ via ϕ.*

For $1 \le i < n$, define the following partitions in $\mathcal{P}_n^{\mathrm{fd}}$:

$\overline{e} = \widehat{e}_{1,2} =$  , $\overline{t} = \widehat{t}_{1,2} =$  , and $\overline{s}_i =$  .

Consider the corresponding alphabet $S \cup \{e, t\}$ where $S = \{s_i : 1 \le i < n\}$. Define a monoid homomorphism

$$\Phi : (S \cup \{e, t\})^* \longrightarrow \mathcal{P}_n^{\mathrm{fd}}$$

by $e \mapsto \overline{e}$, $t \mapsto \overline{t}$, and $s_i \mapsto \overline{s}_i$. Consider the relations

$$s_i^2 = \iota\,, \tag{S1}$$
$$s_i s_j = s_j s_i \quad \text{if } |i-j| > 1\,, \tag{S2}$$
$$s_i s_j s_i = s_j s_i s_j \quad \text{if } |i-j| = 1\,, \tag{S3}$$
$$e = te\,, \tag{P1}$$
$$t = et = s_1 t = ts_1\,, \tag{P2}$$
$$s_i e = es_i \quad \text{if } i > 2\,, \tag{P3}$$
$$es_1 s_2 e = es_1 s_2 s_1\,, \tag{P4}$$
$$es_2 es_2 = s_2 es_2 e\,, \tag{P5}$$
$$e\kappa e\kappa = \kappa e\kappa e\,, \tag{P6}$$
$$s_i t = ts_i \quad \text{if } i > 2\,, \tag{P7}$$
$$ts_2 ts_2 = s_2 ts_2 t\,, \tag{P8}$$
$$t\kappa t\kappa = \kappa t\kappa t\,, \tag{P9}$$
$$es_2 ts_2 = s_2 ts_2 e\,, \tag{P10}$$
$$e\kappa t\kappa = \kappa t\kappa e\,, \tag{P11}$$

where $1 \leq i, j < n$ and $\kappa = s_2 s_3 s_1 s_2$. Define $R_P =$ (S1–S3) $\cup$ (P11–P11). These allow us to state our second main result, the proof of which is found in Section 4.

Theorem 2.4. *The monoid $\mathcal{P}_n^{\text{fd}}$ has monoid presentation $\langle S \cup \{e, t\} : R_P \rangle$ via Φ.*

Given the product decompositions of $\mathcal{P}_n^{\text{fd}}$ and $\mathcal{P}_n^{\text{fd}} \backslash \mathcal{S}_n$ stated in (2.2), we will also make use of existing presentations for isomorphic copies of $\mathcal{T}_n \backslash \mathcal{S}_n$, $\mathcal{F}_n$ and $\mathcal{S}_n$.

Theorem 2.5 ([3, Theorem 3])**.** *The semigroup $\mathcal{T}_n \backslash \mathcal{S}_n$ has semigroup presentation $\langle E : $ (T1–T6)$\rangle$ via $\phi|_{E^+}$.*

Theorem 2.6 ([6, Theorem 2])**.** *The monoid $\mathcal{F}_n$ has monoid presentation $\langle T : $ (F1–F3)$\rangle$ via $\phi|_{T^*}$.*

Theorem 2.7 ([8, Theorem A])**.** *The group $\mathcal{S}_n$ has monoid presentation $\langle S : $ (S1–S3)$\rangle$ via $\Phi|_{S^*}$.*

3 Presentation for $\mathcal{P}_n^{\text{fd}} \backslash \mathcal{S}_n$

In this section, we provide the proof of Theorem 2.3. Let $\sim$ be the congruence on $(E \cup T)^+$ generated by R_{PS}. To prove Theorem 2.3, we require ϕ to be a surmorphism such that $\ker(\phi) = {\sim}$. It is clear from the definition of ϕ that it is a homomorphism, so it remains to be shown that ϕ is surjective (i.e., that $(E \cup T)\phi$ generates $\mathcal{P}_n^{\text{fd}} \backslash \mathcal{S}_n$), which is proven in Proposition 3.1, and that $\ker(\phi) = {\sim}$, which the remainder of this section is dedicated to. Throughout this section, it will be convenient to extend $\sim$ and ϕ to $(E \cup T)^*$ in such a way that $\iota \sim \iota$ and $\iota\phi = \text{id}_{\mathbf{n}}$, despite the fact that $\iota \notin (E \cup T)^+$ and $\text{id}_{\mathbf{n}} \notin \mathcal{T}_n \backslash \mathcal{S}_n$. However, contradictions will be avoided as ι and $\text{id}_{\mathbf{n}}$ will only appear in products that are not equal to ι or $\text{id}_{\mathbf{n}}$. For $w \in (E \cup T)^*$, we will write $\widehat{w} = w\phi$.

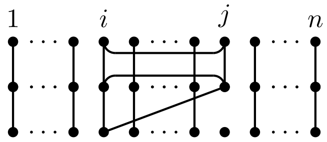
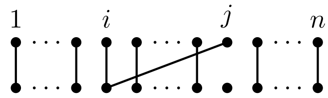
Proposition 3.1. *The map ϕ is surjective.*

Proof. Any element $\rho = \begin{pmatrix} A_1 | \cdots | A_p | \\ B_1 | \cdots | B_p | C_1 | \cdots | C_q \end{pmatrix} \in \mathcal{P}_n^{\text{fd}} \backslash \mathcal{S}_n$ can be expressed as a product $\rho = \alpha\mu$, where $\alpha = \begin{bmatrix} A_1 | A_2 | \cdots | A_p \\ b_1 | b_2 | \cdots | b_p \end{bmatrix} \in \mathcal{T}_n \backslash \mathcal{S}_n$, for some $b_i \in B_i$ for each $1 \leq i \leq p$, and $\mu = \begin{pmatrix} B_1 | \cdots | B_p | C_1 | \cdots | C_p \\ B_1 | \cdots | B_p | C_1 | \cdots | C_q \end{pmatrix} \in \mathcal{F}_n$. Given $\mathcal{T}_n \backslash \mathcal{S}_n$ is generated by $E\phi$ and $\mathcal{F}_n$ is generated by $T\phi$, according to Theorems 2.5 and 2.6, we have $\alpha = u\phi$ for some $u \in E^+$ and $\mu = v\phi$ for some $v \in T^*$, and thus $uv \in (E \cup T)^+$ such that $(uv)\phi = \rho$. □

Note that the decomposition for $\rho \in \mathcal{P}_n^{\text{fd}} \backslash \mathcal{S}_n$ in the previous proof is not necessarily unique. We will explore this in further detail in Corollary 3.10, Lemma 3.12 and throughout Section 5.

Proposition 3.2. *We have the inclusion $\sim \subseteq \ker(\phi)$.*

Proof. It must be shown that $w_1 \sim w_2$ implies that $\widehat{w}_1 = \widehat{w}_2$ for all $w_1, w_2 \in (E \cup T)^+$. That is, that each relation in R_{PS} holds for its image under ϕ. This is easily checked diagramatically. For the $i < j$ case of (PS1),

$\widehat{t}_{ij}\widehat{e}_{ij}$ =  =  = $\widehat{e}_{ij}$.

Further examples can be found in Appendix 8.1. We leave the rest to the reader. ∎

Our goal now is to use the existing presentations for $\mathcal{T}_n \backslash \mathcal{S}_n$ and $\mathcal{F}_n$ in Theorems 2.5 and 2.6 to show that $\ker(\phi) \subseteq \sim$. First, we will show that any word $w \in (E \cup T)^+$ can be sorted into a product of two subwords $u \in E^+$ and $v \in T^*$ using the relations in R_{PS}. This is done by moving each letter $e_{ij} \in E$ to the left of each letter $t_{kl} \in T$. Relations (PS1–PS3) describe this for three of the four distinct arrangements of $i, j, k, l \in \mathbf{n}$. For the fourth case, we require the following relation which is a consequence of relations in R_{PS}.

Lemma 3.3. *For distinct $i, j, k \in \mathbf{n}$, we have*

$$t_{ik}e_{ij} \sim e_{ij}t_{ik} \,. \qquad \text{(PS2}'\text{)}$$

Proof. Using (T1) and (PS2), we have $t_{ik}e_{ij} \sim t_{ik}e_{ji}e_{ij} \sim e_{ji}t_{jk}e_{ij} \sim e_{ji}e_{ij}t_{ik} \sim e_{ij}t_{ik}$. ∎

Lemma 3.4. *If $v \in T^+$ and $u \in E$, then $vu \sim uv'$ for some $v' \in T^*$.*

Proof. Use strong induction on $|v|$. Relations (PS1–PS3) and (PS2′) show this is true for all cases where $|v| = 1$. If $|v| > 1$, we have $v = v_1v_2$ for some $v_1, v_2 \in T^+$ where $|v_1|, |v_2| < |v|$. Then, $vu = v_1v_2u \sim v_1uv_2' \sim uv_1'v_2'$, for some $v_1', v_2' \in T^*$, using the inductive hypothesis. ∎

Lemma 3.5. *If $w \in (E \cup T)^+$, then $w \sim uv$ for some $u \in E^*$ and $v \in T^*$.*

Proof. Use strong induction on $|w|$. If $|w| = 1$, then either $w = v \in T$ and $u = \iota \in E^*$, or $w = u \in E$ and $v = \iota \in T^*$. Suppose instead that $|w| > 1$. Then, $w = w_1w_2$ for some $w_1, w_2 \in (E \cup T)^+$ such that $|w_2| = 1$. By the inductive hypothesis, $w_1w_2 \sim uvw_2$ for some $u \in E^*$ and $v \in T^*$. If $w_2 \in T$ then $vw_2 \in T^*$ and we are done. If, however, $w_2 \in E$, then $vw_2 \sim w_2v'$ for some $v' \in T^*$, by Lemma 3.4, and thus $w \sim uw_2v$ with $uw_2 \in E^*$ and $v \in T^*$. ∎

Corollary 3.6. *If $w \in (E \cup T)^+$, then $w \sim uv$ for some $u \in E^+$ and $v \in T^*$.*

Proof. By Lemma 3.5, $w \sim uv$, for some $u \in E^*$ and $v \in T^*$. The result is trivial if $u \neq \iota$, so suppose $u = \iota$, implying that $v \neq \iota$. It follows that $v = t_{ij}v'$ for some $v' \in T^*$ and $i, j \in \mathbf{n}$ where $i \neq j$. Using (PS4), we have $v = t_{ij}v' \sim e_{ij}t_{ij}v'$ as required. ∎

Once we have sorted two words $w_1, w_2 \in (E \cup T)^+$, we require equality between the $\mathcal{T}_n \backslash \mathcal{S}_n$ and $\mathcal{F}_n$ elements of the resulting product decompositions of $\widehat{w}_1$ and $\widehat{w}_2$ before we can use the presentations for $\mathcal{T}_n \backslash \mathcal{S}_n$ and $\mathcal{F}_n$. As will be demonstrated by Corollary 3.8, equality between the $\mathcal{F}_n$ elements is guaranteed, whilst the $\mathcal{T}_n \backslash \mathcal{S}_n$ elements can be replaced by equal elements using our relations.

Lemma 3.7. *If $\rho, \sigma \in \mathcal{P}_n$ are such that $\mathrm{coker}(\rho) = \Delta_{\mathbf{n}}$, then $\mathrm{coker}(\rho\sigma) = \mathrm{coker}(\sigma)$.*

Proof. It suffices to show the forward inclusion, as $\mathrm{coker}(\sigma) \subseteq \mathrm{coker}(\rho\sigma)$ for all $\rho, \sigma \in \mathcal{P}_n$. Let $(x, y) \in \mathrm{coker}(\rho\sigma)$, and let E_ρ and E_σ be the sets of edges in the product graph $\Pi(\rho, \sigma)$ originating in the graphs for ρ and σ, respectively. By definition, $(x, y) \in \mathrm{coker}(\rho\sigma)$ implies there exists a path $\mathfrak{p}$ from x' to y' in $\Pi(\rho, \sigma)$. Observing that no edges in $\Pi(\rho, \sigma)$ can be incident with vertices from both $\mathbf{n}$ and $\mathbf{n}'$, any subpath $\mathfrak{q}$ of $\mathfrak{p}$ whose internal vertices are a subset of $\mathbf{n}$ and whose endpoints are not in $\mathbf{n}$ must have the form $r'' \to t_1 \to t_2 \to \cdots \to t_m \to s''$ where each $r, s, t_i \in \mathbf{n}$. As a consequence of $\mathrm{coker}(\rho) = \Delta_{\mathbf{n}}$, an edge is in E_ρ if and only if it is incident with a vertex in $\mathbf{n}$. It follows that each edge in $\mathfrak{q}$ must also be in in E_ρ. We then have $r = s$, as $r \neq s$ would contradict $\operatorname{coker}(\rho) = \Delta_{\mathbf{n}}$. This means $\mathfrak{p}$ can be modified by removing each edge from $\mathfrak{q}$ in $\mathfrak{p}$ to produce a new path $\mathfrak{p}'$. Repeating this process for any remaining subpaths with the form of $\mathfrak{q}$ in $\mathfrak{p}'$, a path $\widetilde{\mathfrak{p}}$ from x' to y' that does not intersect $\mathbf{n}$ can be obtained. The edges in $\widetilde{\mathfrak{p}}$ must then be a subset of E_σ and thus $(x, y) \in \operatorname{coker}(\sigma)$. □

Corollary 3.8. *If $\alpha, \beta \in \mathcal{T}_n$ and $\mu, \nu \in \mathcal{F}_n$ are such that $\alpha\mu = \beta\nu$, then $\mu = \nu$.*

Proof. Here, $\alpha\mu = \beta\nu$ implies $\operatorname{coker}(\alpha\mu) = \operatorname{coker}(\beta\nu)$. Then, recalling that all elements of $\mathcal{T}_n$ have trivial cokernel, we have $\operatorname{coker}(\mu) = \operatorname{coker}(\nu)$ by Lemma 3.7, and thus $\mu = \nu$. □

To assist with the remainder of this section, we will extend the notation used for the letters in our alphabets E and T. For $A = \{a_1, a_2, \ldots, a_p\} \subseteq \mathbf{n}$ such that $|A| \geq 2$ and $a_1 < a_2 < \cdots < a_p$, define $e_A = e_{a_1a_2}e_{a_1a_3}\ldots e_{a_1a_p}$. If $|A| \leq 1$, instead define $e_A = \iota$. In the case that $A = \{i, j\}$, e_A is just the letter e_{ij} or e_{ji}, depending on whether $i < j$ or $i > j$. For $\varepsilon \in \mathfrak{Eq}_n$ with equivalence classes $A_1, A_2, \ldots, A_q$ such that $\min(A_1) < \min(A_2) < \cdots < \min(A_q)$, define $e_\varepsilon = e_{A_1}e_{A_2}\ldots e_{A_q}$. Define t_A and t_ε in an analagous way. Observe that $\widehat{e}_\varepsilon = \left[\begin{array}{c|c|c|c} A_1 & A_2 & \cdots & A_p \\ \min(A_1) & \min(A_2) & \cdots & \min(A_p) \end{array}\right]$ and $\widehat{t}_\varepsilon = \mathrm{id}_\varepsilon$.

The following lemma and its corollary will allow us to conjure words in E^+ from words in T^*, via relation (PS4). These conjured words can then be used to induce equality between the $\mathcal{T}_n \backslash \mathcal{S}_n$ elements of the images under ϕ of two equivalent words in $(E \cup T)^+$, as will be shown in Lemma 3.12.

Lemma 3.9. *For all $A \subseteq \mathbf{n}$, we have $t_A \sim e_A t_A$.*

Proof. Use induction on $|A|$. In the trivial case that $|A| \leq 1$, we have $t_A = \iota = e_A t_A$. Suppose $A = \{a_1, a_2, \ldots, a_p\}$, where $a_1 < a_2 < \cdots < a_p$ and $p > 1$. Then, by definition

$$\begin{aligned}
t_A &= t_{A\backslash\{a_p\}} t_{a_1a_p} \\
&\sim (e_{A\backslash\{a_p\}} t_{A\backslash\{a_p\}})\, t_{a_1a_p} && \text{by inductive hypothesis} \\
&\sim e_{A\backslash\{a_p\}}\, (t_{a_1a_p} t_{A\backslash\{a_p\}}) && \text{by (F2)} \\
&\sim e_{A\backslash\{a_p\}}\, (e_{a_1a_p} t_{a_1a_p})\, t_{A\backslash\{a_p\}} && \text{by (PS4)} \\
&\sim e_{A\backslash\{a_p\}} e_{a_1a_p}\, (t_{A\backslash\{a_p\}} t_{a_1a_p}) && \text{by (F2)} \\
&= e_A t_A\,.
\end{aligned}$$

□

Corollary 3.10. *For all $\varepsilon \in \mathfrak{Eq}_{\mathbf{n}}$, we have $t_\varepsilon \sim e_\varepsilon t_\varepsilon$.*

Proof. Let $A_1, A_2, \ldots, A_p \subseteq \mathbf{n}$ for some $p \geq 1$ be such that $\min(A_1) < \min(A_2) < \cdots < \min(A_p)$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$. Note that the equivalence classes of any $\varepsilon \in \mathfrak{Eq}_{\mathbf{n}}$ will satisfy the above criteria, so it suffices to show that, for all $p \geq 1$,

$$t_{A_1}t_{A_2}\ldots t_{A_p} \sim e_{A_1}e_{A_2}\ldots e_{A_p}t_{A_1}t_{A_2}\ldots t_{A_p}\,.$$

Use induction on p. When $p = 1$, we have $t_{A_1} \sim e_{A_1}t_{A_1}$ by Lemma 3.9. Suppose $p > 2$. Then

$$\begin{aligned}
t_{A_1}t_{A_2}\ldots t_{A_p} &\sim (e_{A_1}e_{A_2}\ldots e_{A_{p-1}}t_{A_1}t_{A_2}\ldots t_{A_{p-1}})t_{A_p} && \text{by inductive hypothesis} \\
&\sim e_{A_1}e_{A_2}\ldots e_{A_{p-1}}(t_{A_p}t_{A_1}t_{A_2}\ldots t_{A_{p-1}}) && \text{by (F2)} \\
&\sim e_{A_1}e_{A_2}\ldots e_{A_{p-1}}(e_{A_p}t_{A_p})t_{A_1}t_{A_2}\ldots t_{A_{p-1}} && \text{by Lemma 3.9} \\
&\sim e_{A_1}e_{A_2}\ldots e_{A_p}(t_{A_1}t_{A_2}\ldots t_{A_p}) && \text{by (F2).}
\end{aligned}$$

□

Lemma 3.11. *For all $\varepsilon \in \mathfrak{Eq}_{\mathbf{n}}$, we have $e_\varepsilon \sim t_\varepsilon e_\varepsilon$.*

Proof. The proof of this is analagous to those used for Lemma 3.9 and Corollary 3.10, however the roles of the words in E^* and T^* are reversed, and relations (T4) and (PS1) are used in place of (F2) and (PS4), respectively. □

Lemma 3.12. *If $\alpha, \beta \in \mathcal{T}_n$ and $\mu \in \mathcal{F}_n$ are such that $\alpha\mu = \beta\mu$, then $\alpha\gamma = \beta\gamma$, where $\gamma = \widehat{e}_{\ker(\mu)}$.*

Proof. Let $\varepsilon = \ker(\mu)$ hence $\mu = \mathrm{id}_\varepsilon = \widehat{t}_\varepsilon$ and $\gamma = \widehat{e}_\varepsilon$. Then $\mu\gamma = \widehat{t}_\varepsilon \widehat{e}_\varepsilon = \widehat{e}_\varepsilon = \gamma$ by Lemma 3.11 and Proposition 3.2, which gives us $\alpha\gamma = \alpha\mu\gamma = \beta\mu\gamma = \beta\gamma$ as required. □

The final two pieces required are a direct consequence of Theorems 2.5 and 2.6.

Lemma 3.13. *If $u_1, u_2 \in E^+$, then $\widehat{u}_1 = \widehat{u}_2 \implies u_1 \sim u_2$.*

Lemma 3.14. *If $v_1, v_2 \in T^*$, then $\widehat{v}_1 = \widehat{v}_2 \implies v_1 \sim v_2$.*

All that remains to prove Theorem 2.3 is to show $\ker(\phi) \subseteq \sim$ and thus $\ker(\phi) = \sim$.

Proposition 3.15. *We have the inclusion $\ker(\phi) \subseteq \sim$.*

Proof. It must be shown that $\widehat{w}_1 = \widehat{w}_2$ implies that $w_1 \sim w_1$ for all $w_1, w_2 \in (E \cup T)^+$. First, recall that $w_1 \sim w_2$ implies that $\widehat{w}_1 = \widehat{w}_2$ for all $w_1, w_2 \in (E \cup T)^+$ by Proposition 3.2, as this will be used throughout the proof. Let $w_1, w_2 \in (E \cup T)^+$ be such that $\widehat{w}_1 = \widehat{w}_2$. By Corollary 3.6, $w_1 \sim u_1 v_1$ and $w_2 \sim u_2 v_2$ for some $u_1, u_2 \in E^+$ and $v_1, v_2 \in T^*$, hence $\widehat{u}_1 \widehat{v}_1 = \widehat{w}_1 = \widehat{w}_2 = \widehat{u}_2 \widehat{v}_2$. It follows from Corollary 3.8 that $\widehat{v}_1 = \widehat{v}_2$ and thus $v_1 \sim v_2$ by Lemma 3.14. Let $\varepsilon = \ker(\widehat{v}_1)$ so that $\widehat{v}_1 = \widehat{t}_\varepsilon$ and $v_1 \sim v_2 \sim t_\varepsilon \sim e_\varepsilon t_\varepsilon$ by Lemma 3.14 and Corollary 3.10. We now have $w_1 \sim u_1 v_1 \sim u_1 e_\varepsilon t_\varepsilon$ and similarly $w_2 \sim u_2 e_\varepsilon t_\varepsilon$. Since $\widehat{u}_1 \widehat{t}_\varepsilon = \widehat{u}_1 \widehat{v}_1 = \widehat{u}_2 \widehat{v}_2 = \widehat{u}_2 \widehat{t}_\varepsilon$, it must also be the case that $\widehat{u}_1 \widehat{e}_\varepsilon = \widehat{u}_2 \widehat{e}_\varepsilon$ according to Lemma 3.12. This gives $u_1 e_\varepsilon \sim u_2 e_\varepsilon$ by Lemma 3.13 because $u_1 e_\varepsilon, u_2 e_\varepsilon \in E^+$. Putting it all together we have $w_1 \sim u_1 e_\varepsilon t_\varepsilon \sim u_2 e_\varepsilon t_\varepsilon \sim w_2$. □

4 Presentation for $\mathcal{P}_n^{\mathrm{fd}}$

In this section we prove Theorem 2.4. To do this, we adapt the method used to prove [4, Theorem 2.2]. The central idea is to use the *transpositions* in $\mathcal{S}_n$ (i.e., the $\overline{s}_i$ partitions) to construct the generators required for the presentation of $\mathcal{P}_n^{\mathrm{fd}} \backslash \mathcal{S}_n$ from just e and t, with $\mathcal{S}_n$ generated from the transpositions alone.

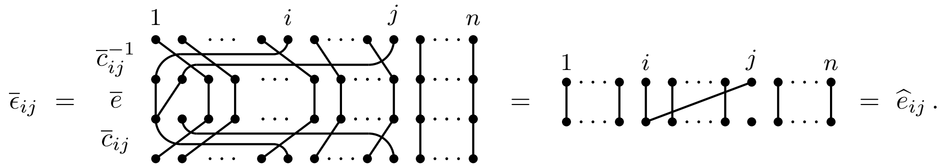
Let $\approx$ be the congruence on $(S \cup \{e, t\})^*$ generated by R_P. We require Φ to be a surmorphism such that $\ker(\Phi) = \approx$. It is clear from the definition of Φ that it is a homomorphism, so it remains to be shown that Φ is surjective and $\ker(\Phi) = \approx$. Surjectivity and the inclusion $\approx \subseteq \ker(\Phi)$ are proven in Propositions 4.2 and 4.3, while the remainder of this section is dedicated to proving that $\ker(\Phi) \subseteq \approx$.

We will write $\overline{w} = w\Phi$ for $w \in (S \cup \{e, t\})^*$. For $w = s_{i_1} s_{i_2} \ldots s_{i_p} \in S^*$, we will write $w^{-1} = s_{i_p} s_{i_{p-1}} \ldots s_{i_1}$, giving the relation $ww^{-1} \approx w^{-1}w \approx \iota$, by (S1). For $i, j \in \mathbf{n}$ where $i \neq j$, define the words

$$c_{ij} = (s_2 s_3 \ldots s_{j-1})(s_1 s_2 \ldots s_{i-1}) \text{ for } i < j,$$

$$\epsilon_{ij} = \begin{cases} c_{ij}^{-1} e c_{ij} & \text{if } i < j \\ c_{ji}^{-1} e s_1 c_{ji} & \text{if } i > j, \end{cases} \quad \text{and} \quad \tau_{ij} = \tau_{ji} = c_{ij}^{-1} t c_{ij} \text{ for } i < j\,.$$

Observe that $c_{1,2} = \iota$, $\epsilon_{1,2} = e$ and $\tau_{1,2} = t$. *Conjugating* e by the word c_{ij} (i.e., $c_{ij}^{-1} e c_{ij}$) transforms its image under Φ into $\widehat{e}_{ij}$ where $i < j$. Pictorially,



(4.1)

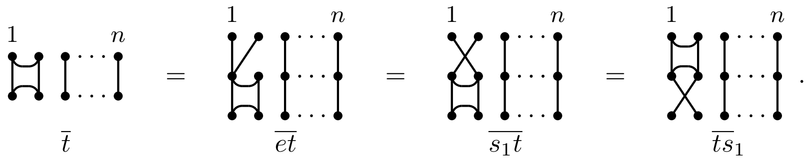
Conjugating t by c_{ij} has an analagous effect. However, to transform the image of $\overline{e}$ into $\widehat{e}_{ij}$ where $i > j$ we must combine e with s_1 to transpose $\overline{e}$ before conjugation.

Proposition 4.2. *The map Φ is surjective.*

Proof. As in (4.1), one may check diagramatically that $\widehat{e}_{ij} = \overline{\epsilon}_{ij}$ and $\widehat{t}_{ij} = \overline{\tau}_{ij}$. By Proposition 3.1, these elements generate $\mathcal{P}_n^{\mathrm{fd}} \backslash \mathcal{S}_n$, hence $\mathcal{P}_n^{\mathrm{fd}} \backslash \mathcal{S}_n \subseteq \operatorname{im}(\Phi)$. We also know from Theorem 2.7 that the set of transpositions $S\Phi = \{\overline{s}_i : 1 \leq i < n\}$ generates $\mathcal{S}_n$. Together, these show that $\mathcal{P}_n^{\mathrm{fd}}$ is generated by $(S \cup \{e, t\})\Phi$. □

Proposition 4.3. *We have the inclusion $\approx \subseteq \ker(\Phi)$.*

Proof. It must be shown that $w_1 \approx w_2$ implies that $\overline{w}_1 = \overline{w}_2$ for all $w_1, w_2 \in (S \cup \{e, t\})^*$. That is, that the relations in R_P hold for their image under Φ. This is checked diagramatically. For (P2)



Further examples can be found in Appendix 8.2, and we leave the rest to the reader. □

The following lemma is a direct consequence of Theorem 2.7.

Lemma 4.4. *If $w_1, w_2 \in S^*$, then $\overline{w}_1 = \overline{w}_2 \implies w_1 \approx w_2$.*

Define a semigroup homomorphism $\psi : (E \cup T)^+ \to (S \cup \{e, t\})^*$, by $e_{ij} \mapsto \epsilon_{ij}$ and $t_{ij} \mapsto \tau_{ij}$. Then, $\widehat{w} = \overline{w\psi}$ (i.e., $w\phi = w\psi\Phi$) for all $w \in (E \cup T)^+$, so that $(E \cup T)^+\psi\Phi = \mathcal{P}_n^{\mathrm{fd}} \backslash \mathcal{S}_n$. To prove $\ker(\Phi) \subseteq \approx$, we aim to show that the image of the relations in R_{PS} under ψ hold for words in $(E \cup T)^+\psi$ using the relations in R_P. This will allow us to incorporate the presentation in Theorem 2.3 here. We will often need to consider how a letter $s_k \in S$ interacts with another word. To understand the possible cases, it is best to think in terms of the partitions to which they correspond. Recall that a transposition $\overline{s}_k \in S\Phi$ corresponds to the map $k \mapsto k+1$, $k+1 \mapsto k$, and $x \mapsto x$ for all $x \in \mathbf{n} \backslash \{k, k+1\}$. As such, when combining s_k with some word c_{ij}, ϵ_{ij} or τ_{ij}, we need to consider all ways that the sets $\{k, k+1\}$ and $\{i, j\}$ can intersect. In each instance, Cases 1–4 will describe partial intersections, Case 5 complete intersections, and Cases ≥ 6 empty intersections. We will need the following relations that describe how each letter s_k combines with each word c_{ij}. For $1 \leq i < j \leq n$ and $1 \leq k < n$, we have

$$c_{ij}s_k \approx \begin{cases} c_{i+1,j} & \text{if } k = i \neq j-1 \quad \text{(C1)} \\ c_{i-1,j} & \text{if } k = i-1 \quad \text{(C2)} \\ c_{i,j+1} & \text{if } k = j \quad \text{(C3)} \\ c_{i,j-1} & \text{if } k = j-1 \neq i, \quad \text{(C4)} \end{cases} \qquad c_{ij}s_k \approx \begin{cases} s_1 c_{ij} & \text{if } \{k, k+1\} = \{i, j\} \quad \text{(C5)} \\ s_{k+2}c_{ij} & \text{if } k < i-1 \quad \text{(C6)} \\ s_{k+1}c_{ij} & \text{if } i < k < j-1 \quad \text{(C7)} \\ s_k c_{ij} & \text{if } j < k. \quad \text{(C8)} \end{cases}$$

Each relation (C1–C8) consists only of words in S^*. Hence, due to Lemma 4.4, they can be proven diagramatically. Furthermore, because $w_1 \approx w_2$ implies that $w_1^{-1} \approx w_2^{-1}$ for all $w_1, w_2 \in S^*$ the inverse of each relation (C1–C8) also holds and will be referred to in the same manner. We will need two more relations:

$$e \approx e^2 \approx s_1 e, \tag{P1$'$}$$

$$t \approx t^2. \tag{P2$'$}$$

These are both consequences of the relations (P1) and (P2). For (P1′), we have $e \approx te \approx ete \approx e^2$ and $e \approx te \approx s_1te \approx s_1e$. For (P2′), we have $t \approx et \approx tet \approx t^2$.

The next step is to show that the subscripts of ϵ_{ij} or τ_{ij} conjugated by a word $w \in S^*$ are transformed in accordance with the permutation $\overline{w}$, thus allowing us to use a specific set of subscripts to prove a relation in general. For the remainder of this section, recall that, if $\overline{w} \in \mathcal{T}_n$ and $i \in \mathbf{n}$, we write $i\overline{w}$ to indicate the element of $\mathbf{n}$ that i is mapped to via $\overline{w}$.

Lemma 4.5. *If $i, j \in \mathbf{n}$ where $i \neq j$ and $1 \leq k < n$, then $s_k \epsilon_{ij} s_k \approx \epsilon_{i\overline{s}_k, j\overline{s}_k}$.*

Proof. It must be shown that

$$s_k \epsilon_{ij} s_k \approx \begin{cases} \epsilon_{i+1,j} & \text{if } k = i \neq j-1 & \text{(Case 1)} \\ \epsilon_{i-1,j} & \text{if } k = i-1 \neq j & \text{(Case 2)} \\ \epsilon_{i,j+1} & \text{if } k = j \neq i-1 & \text{(Case 3)} \\ \epsilon_{i,j-1} & \text{if } k = j-1 \neq i & \text{(Case 4)} \\ \epsilon_{ji} & \text{if } \{k, k+1\} = \{i, j\} & \text{(Case 5)} \\ \epsilon_{ij} & \text{otherwise.} & \text{(Case 6)} \end{cases}$$

Cases 1-4 follow quickly from (C1–C4). Case 5 can be shown using (C7), (P1′) and, in the subcase where $i > j$, (S1). Case 6 has six subcases (one for each ordering of i, j and k). The subcase where $j < k < i - 1$ is shown here; the others are treated similarly;

$$\begin{aligned} s_k \epsilon_{ij} s_k &= s_k (c_{ji}^{-1} e s_1 c_{ji}) s_k \\ &\approx (c_{ji}^{-1} s_{k+1}) e s_1 (s_{k+1} c_{ji}) && \text{by (C7)} \\ &\approx c_{ji}^{-1} (e s_1 s_{k+1}) s_{k+1} c_{ji} && \text{by (P3) and (S2), as } k > j \geq 1 \\ &\approx c_{ji}^{-1} e s_1 c_{ji} && \text{by (S1)} \\ &= \epsilon_{ij}\,. \end{aligned}$$

□

Lemma 4.6. *If $i, j \in \mathbf{n}$ where $i \neq j$ and $1 \leq k < n$, then $s_k \tau_{ij} s_k \approx \tau_{i\overline{s}_k, j\overline{s}_k}$.*

Proof. It is clear that the proof for Lemma 4.5 can be adapted for this result, where ϵ_{ij} is replaced with τ_{ij}, e with t, (P1′) with (P2), and (P3) with (P7). □

Corollary 4.7. *If $i, j \in \mathbf{n}$ where $i \neq j$ and $w \in S^*$, then $w^{-1} \epsilon_{ij} w \approx \epsilon_{i\overline{w}, j\overline{w}}$ and $w^{-1} \tau_{ij} w \approx \tau_{i\overline{w}, j\overline{w}}$.*

Proof. Using induction on $|w|$, this follows quickly from Lemmas 4.5 and 4.6. □

We can now incorporate the presentation for $\mathcal{P}_n^{\text{fd}} \backslash \mathcal{S}_n$ from Theorem 2.3.

Lemma 4.8. *If $u, v \in (E \cup T)^+$, then $u \sim v \implies u\psi \approx v\psi$.*

Proof. Recall that ψ maps each word $(E \cup T)^+$ to their counterparts in $(S \cup \{e, t\})^*$. It must be shown that each relation in R_{PS} holds for its image under ψ in $(S \cup \{e, t\})^*$ via the relations in R_P. For (T3), choose $w \in S^*$ such that $1\overline{w} = i$, $2\overline{w} = k$, and $3\overline{w} = j$. Then

$$\begin{aligned} \epsilon_{ik} \epsilon_{jk} &\approx (w^{-1} \epsilon_{1,2} w)(w^{-1} \epsilon_{3,2} w) && \text{by Corollary 4.7} \\ &\approx w^{-1} (e)(c_{2,3}^{-1} e s_1 c_{2,3}) w && \text{by (S1)} \\ &\approx w^{-1} e (s_1 s_2) e s_1 (s_2 s_1) w \\ &\approx w^{-1} (e s_1 s_2 s_1) s_1 s_2 s_1 w && \text{by (P4)} \\ &\approx w^{-1} e w && \text{by (S1)} \\ &\approx w^{-1} \epsilon_{1,2} w \\ &\approx \epsilon_{ik} && \text{by Corollary 4.7.} \end{aligned}$$

For each of the remaining relations in R_{PS}, Corollary 4.7 can be applied in the same way to transform the subscripts to a fixed set of values so that the relations from R_P can be used to prove the congruence in general. See Appendix 8.3 for further examples. □

The next two lemmas show that the combination of any word $w \in \operatorname{im}(\psi)$ and a letter $s_k \in S$ must also be congruent to some $w' \in \operatorname{im}(\psi)$. Lemma 4.11 extends this by showing that any $w \in (S \cup \{e, t\})^*$ containing a letter $u \notin S$ must also be congruent to some $w' \in \operatorname{im}(\psi)$. As a consequence, every word $w \in (S \cup \{e,t\})^*$ is either strictly in S^*, so that $\overline{w} \in \mathcal{S}_n$ and can be explained using the presentation in Theorem 2.7, or it is congruent to some $w' \in \operatorname{im}(\psi)$ so that $\overline{w'} \in \mathcal{P}_n^{\mathrm{fd}} \backslash \mathcal{S}_n$ and can be explained using the presentation in Theorem 2.3.

Lemma 4.9. *If $i,j \in \mathbf{n}$ where $i \neq j$ and $1 \leq k < n$, then $\epsilon_{ij}s_k \approx w_1$ and $s_k\epsilon_{ij} \approx w_2$ for some $w_1, w_2 \in \operatorname{im}(\psi)$.*

Proof. For $\epsilon_{ij}s_k$, it can be shown using the relations (S1–S3), (P1′), (P3) and (P4) that

$$\epsilon_{ij}s_k \approx \begin{cases} \epsilon_{ij}\epsilon_{j,i+1}\epsilon_{i+1,i}\epsilon_{ij} & \text{if } k = i \neq j-1 & \text{(Case 1)} \\ \epsilon_{ij}\epsilon_{j,i-1}\epsilon_{i-1,i}\epsilon_{ij} & \text{if } k = i-1 \neq j & \text{(Case 2)} \\ \epsilon_{ij}\epsilon_{j,j+1} & \text{if } k = j \neq i-1 & \text{(Case 3)} \\ \epsilon_{ij}\epsilon_{j,j-1} & \text{if } k = j-1 \neq i & \text{(Case 4)} \\ \epsilon_{ji} & \text{if } \{k,k+1\} = \{i,j\} & \text{(Case 5)} \\ \epsilon_{ij}\epsilon_{j,k+1}\epsilon_{k+1,k}\epsilon_{jk} & \text{otherwise.} & \text{(Case 6)} \end{cases}$$

As in Lemma 4.8, Corollary 4.7 is used to transform the subscripts for each case. Case 3 is shown here using the transformation $(i,j) = (1,2)$:

$$\begin{aligned} \epsilon_{1,2}s_2 &\approx es_2 \\ &\approx e(s_1s_1)s_2(s_1s_1) && \text{by (S1)} \\ &\approx es_1(s_2s_1s_2)s_1 && \text{by (S3)} \\ &\approx (es_1s_2e)s_2s_1 && \text{by (P3)} \\ &\approx e(c_{2,3}^{-1})e(c_{2,3}) \\ &\approx \epsilon_{1,2}\epsilon_{2,3}\,. \end{aligned}$$

Further examples can be found in Appendix 8.4. By Lemma 4.5 and (S1),

$$s_k\epsilon_{ij} \approx s_k\epsilon_{ij}s_ks_k \approx \epsilon_{i\overline{s}_k,j\overline{s}_k}s_k\,,$$

hence $s_k\epsilon_{ij}$ is also considered above. □

Lemma 4.10. *If $i,j \in \mathbf{n}$ where $i \neq j$ and $1 \leq k < n$, then $\tau_{ij}s_k \approx w_1$ and $s_k\tau_{ij} \approx w_2$ for some $w_1, w_2 \in \operatorname{im}(\psi)$.*

Proof. According to (PS4), $t_{ij} \sim e_{ij}t_{ij}$ which implies $s_k\tau_{ij} \approx s_k\epsilon_{ij}\tau_{ij}$ by Lemma 4.8. Using Lemma 4.9, $s_k\epsilon_{ij}\tau_{ij} \approx w\tau_{ij}$ for some $w \in \operatorname{im}(\psi)$, hence $w\tau_{ij} \in \operatorname{im}(\psi)$. Furthermore, we have $\tau_{ij}s_k \approx s_ks_k\tau_{ij}s_k \approx s_k\tau_{i\overline{s}_k,j\overline{s}_k}$, so each $\tau_{ij}s_k$ case has also been considered. □

Lemmas 4.9 and 4.10 allow us to adapt the key result [4, Lemma 4.7] to this setting.

Lemma 4.11. *If $w \in (S \cup \{e,t\})^* \backslash S^*$, then w is $\approx$-equivalent to an element of $\operatorname{im}(\psi)$.*

Proof. Let $\Sigma = (E \cup T)\psi = \{\epsilon_{ij} : i,j \in \mathbf{n}, i \neq j\} \cup \{\tau_{ij} : i,j \in \mathbf{n}, i \neq j\}$, noting that $\operatorname{im}(\psi) = \langle \Sigma \rangle$. Since $e = \epsilon_{1,2} \in \Sigma$ and $t = t_{1,2} \in \Sigma$, it suffices to show that every element of $\langle \Sigma \cup S \rangle \backslash S^*$ is $\approx$–equivalent to an element of $\langle \Sigma \rangle$. With this is in mind, let $w \in \langle \Sigma \cup S \rangle \backslash S^*$, and write $w = x_1 \ldots x_k$, where $x_1, \ldots, x_k \in \Sigma \cup S$. Denote by l the number of factors x_i that belong to S. We proceed by induction on l. If $l = 0$, then we already have $w \in \langle \Sigma \rangle$, so suppose $l \geq 1$. Since $w \notin S^*$, there exists $1 \leq i \leq k-1$ such that either (i) $x_i \in S$ and $x_{i+1} \in \Sigma$, or (ii) $x_i \in \Sigma$ and $x_{i+1} \in S$. In either case, Lemmas 4.9 and 4.10 tell us that $x_ix_{i+1} \approx u$ for some $u \in \operatorname{im}\psi = \langle \Sigma \rangle$. But then $w \approx (x_1 \ldots x_{i-1})u(x_{i+2} \ldots x_k)$, and we are done, after applying the inductive hypothesis (noting that $(x_1 \ldots x_{i-1})u(x_{i+2} \ldots x_k)$ has $l-1$ factors from S). □

We can now show $\ker(\Phi) \subseteq \approx$, and thus $\ker(\Phi) = \approx$, completing the proof of Theorem 2.4.

Proposition 4.12. *We have the inclusion* $\ker(\Phi) \subseteq \approx$.

Proof. Let $w_1, w_2 \in (S \cup \{e,t\})^*$ be such that $\overline{w}_1 = \overline{w}_2$. If $\overline{w}_1, \overline{w}_2 \in \mathcal{S}_n$ then $w_1, w_2 \in S^*$ and $w_1 \approx w_2$, by Lemma 4.4. If, however, $\overline{w}_1, \overline{w}_2 \notin \mathcal{S}_n$, then $w_1, w_2 \in (S \cup \{e,t\})^* \backslash S^*$, and thus $w_1 \approx u_1\psi$ and $w_2 \approx u_2\psi$ for some $u_1, u_2 \in (E \cup T)^+$ using Lemma 4.11. In this case, $\widehat{u}_1, \widehat{u}_2 \in \mathcal{P}_n^{\text{fd}} \backslash \mathcal{S}_n$, hence $\widehat{u}_1 = \overline{u_1\psi} = \overline{w}_1 = \overline{w}_2 = \overline{u_2\psi} = \widehat{u}_2$ implies that $u_1 \sim u_2$ by Proposition 3.15. Finally, we have $u_1\psi \approx u_2\psi$ by Lemma 4.8, showing $w_1 \approx w_2$. □

5 Alternative presentations using action pairs

In Section 3, our presentation for $\mathcal{P}_n^{\text{fd}} \backslash \mathcal{S}_n$ was proven directly using existing presentations for its product decomposition. In [2], however, many of these arguments have been generalised to handle any semigroup that factorises into so-called *action pairs*, including *right restriction* semigroups such as $\mathcal{P}_n^{\text{fd}}$, both of which we will define shortly. This section applies the techniques of [2] to $\mathcal{P}_n^{\text{fd}} \backslash \mathcal{S}_n$ and $\mathcal{P}_n^{\text{fd}}$. Through this approach we provide an alternative proof for the presentation of $\mathcal{P}_n^{\text{fd}} \backslash \mathcal{S}_n$ in Theorem 2.3 (see Section 5.1) and obtain an alternative presentation for $\mathcal{P}_n^{\text{fd}}$ in Theorem 5.5 (see Section 5.2). Whilst we include some context for these results, this section is intended to be read in conjunction with [2] which provides a more comprehensive description of concepts involved. We will be using a left-right dual of the notation, definitions, and concepts found in [2], as they are specific to left rather than right restricion semigroups such as $\mathcal{P}_n^{\text{fd}}$.

Given a semigroup S, we write S^1 to denote the *monoid completion* of S. That is, $S^1 = S$ if S is already a monoid, otherwise $S^1 = S \cup \{e\}$, where $e \notin S$ and e acts as an identity element in S^1. If S is a subsemigroup of a monoid M, we will assume the identity element in S^1 is the same as that in M. A *right action* of a semigroup S on a set M is a map $M \times S \to M : (x, s) \mapsto x^s$ such that $(x^s)^t = x^{st}$ for all $x \in M$ and $s, t \in S$. If M is a monoid with identity element e and S has a right action on M such that $(xy)^s = x^s y^s$ and $e^s = e$ for all $s \in S$ and $x, y \in M$, we say the action is *by monoid morphisms*. A *strong right action pair* in a monoid M is a pair (U, S) of subsemigroups of M satisfying:

$$S \text{ has a right action on } U^1 \text{ by monoid morphisms such that } us = su^s \text{ for all } s \in S \text{ and } u \in U^1, \tag{SA1}$$

$$su = tv \text{ implies that } u = v \text{ for all } u, v \in U^1 \text{ and } s, t \in S. \tag{SA2}$$

A *right restriction semigroup* S is a semigroup that, in addition to its binary operation, is equipped with a unary operation $x \mapsto R(x)$ such that, for all $x, y \in S$,

$$x = xR(x), \quad \text{(R1)} \qquad R(x)R(y) = R(xR(y)), \quad \text{(R3)}$$

$$R(x)R(y) = R(y)R(x), \quad \text{(R2)} \qquad R(x)y = yR(xy). \quad \text{(R4)}$$

We refer to $R(x)$ as the *range* of x. A right restriction monoid M also contains the submonoids

$$P(M) = \{R(x) : x \in M\} \quad \text{and} \quad T(M) = \{x \in M : R(x) = e\},$$

where e is the identity element in M. The monoid (indeed, semilattice) $P(M)$ is called the set of *projections* in M. The monoid M acts on $P(M)$ by the right action $p^x = R(px)$ for all $x \in M$ and $p \in P(M)$. Given a right restriction monoid M, the pair (Q, S) is a strong action pair for any $Q \leq P(M)$ and $S \leq T(M)$ such that Q^1 is closed under the action of S [2, Proposition 4.44 (ii)].

Proposition 5.1. *The pairs* $(\mathcal{F}_n, \mathcal{T}_n)$ *and* $(\mathcal{F}_n, \mathcal{T}_n \backslash \mathcal{S}_n)$ *are strong action pairs with the right action* $\mu^\alpha = \mathrm{id}_{\mathrm{coker}(\mu\alpha)}$.

Proof. According to [5, Proposition 4.23], $\mathcal{P}_n^{\mathrm{fd}}$ is a right restriction monoid with range operation $R(\rho) = \mathrm{id}_{\mathrm{coker}(\rho)}$, hence $P(\mathcal{P}_n^{\mathrm{fd}}) = \mathcal{F}_n$ and $T(\mathcal{P}_n^{\mathrm{fd}}) = \mathcal{T}_n$. We also have that $\mathcal{T}_n \backslash \mathcal{S}_n \leq \mathcal{T}_n$ and $\mathcal{F}_n$ is closed under the action of $\mathcal{T}_n$. By [2, Proposition 4.44 (ii)], $(\mathcal{F}_n, \mathcal{T}_n \backslash \mathcal{S}_n)$ and $(\mathcal{F}_n, \mathcal{T}_n)$ are strong action pairs with the right action $\mu^\alpha = R(\mu\alpha) = \mathrm{id}_{\mathrm{coker}(\mu\alpha)}$. $\square$

For each $\mu \in \mathcal{F}_n$, consider the left congruences

$$\theta_\mu = \{(\alpha, \beta) \in \mathcal{T}_n \times \mathcal{T}_n : \alpha\mu = \beta\mu\},$$
$$\Theta_\mu = \{(\alpha, \beta) \in (\mathcal{T}_n \backslash \mathcal{S}_n)^1 \times (\mathcal{T}_n \backslash \mathcal{S}_n)^1 : \alpha\mu = \beta\mu\}.$$

Note that $(\alpha, \beta) \in \theta_\mu$ or $(\alpha, \beta) \in \Theta_\mu$ if and only if for each $x \in \mathbf{n}$ we have

$$[x]_{\alpha\mu} = [x]_{\beta\mu} \iff [x\alpha]_\mu = [x\beta]_\mu \iff (x\alpha, x\beta) \in \ker(\mu)\,.$$

When applying the techniques in [2] to our setting, we need to find subsets of θ_μ and Θ_μ for each $\mu \in \mathcal{F}_n$ that generate θ_μ and Θ_μ as left congruences. These generating sets can be simplified as a result of the following lemma.

Lemma 5.2. *For all* $\mu, \nu \in \mathcal{F}_n$*, we have*

$$\text{(i) } \theta_{\mu\nu} = \theta_\mu \vee \theta_\nu \quad \textit{and} \quad \text{(ii) } \Theta_{\mu\nu} = \Theta_\mu \vee \Theta_\nu\,.$$

Proof. Let $\mu, \nu \in \mathcal{F}_n$ and $\varepsilon, \eta \in \mathfrak{Eq}_{\mathbf{n}}$ be such that $\mu = \mathrm{id}_\varepsilon$ and $\nu = \mathrm{id}_\eta$. As discussed in the proof for [2, Lemma 4.71 (ii)], it suffices to show the forward inclusions due to the commutativity of $\mathcal{F}_n$.

(i) Let $(\alpha, \beta) \in \theta_{\mu\nu}$. This implies $(x\alpha, x\beta) \in \varepsilon \vee \eta$ for all $x \in \mathbf{n}$ and thus, for each $x \in \mathbf{n}$, there exists a chain $x\alpha \xrightarrow{\varepsilon} y_{x,1} \xrightarrow{\eta} y_{x,2} \xrightarrow{\varepsilon} \cdots \xrightarrow{\varepsilon} y_{x,m_x-1} \xrightarrow{\eta} x\beta$ of some even length $m_x \in \mathbb{N}$. By the reflexive property, any such chain can be extended to an arbitrary even length by appending some number of $x\beta \xrightarrow{\varepsilon} x\beta \xrightarrow{\eta} x\beta$ to the chain. Let $m = \max\{m_x : x \in \mathbf{n}\}$ and choose a chain $x\alpha \to x\beta$ in $\varepsilon \cup \eta$ of length m for each $x \in \mathbf{n}$. In addition, if $x_1\alpha = x_2\alpha$ and $x_1\beta = x_2\beta$ for some $x_1, x_2 \in \mathbf{n}$, choose the same chain $x_1\alpha \to x_1\beta$ in $\varepsilon \cup \eta$ for both x_1 and x_2 so that $y_{x_1,k} = y_{x_2,k}$ for all $1 \leq k < m$. Define $\gamma_k \in \mathcal{T}_n$ by $x\gamma_k = y_{x,k}$ for $1 \leq k < m$. We then have $(\alpha, \beta) \in \theta_\mu \vee \theta_\nu$ by the chain $\alpha \xrightarrow{\theta_\mu} \gamma_1 \xrightarrow{\theta_\nu} \gamma_2 \xrightarrow{\theta_\mu} \cdots \xrightarrow{\theta_\mu} \gamma_{m-1} \xrightarrow{\theta_\nu} \beta$.

(ii) This proof is similar to (i), except extra steps may be necessary to ensure each $\gamma_k \notin \mathcal{S}_n \backslash \{\mathrm{id}_{\mathbf{n}}\}$. Let $(\alpha, \beta) \in \Theta_{\mu\nu}$. If $\alpha = \beta = \mathrm{id}_{\mathbf{n}}$ then trivially $(\alpha, \beta) \in \Theta_\mu \vee \Theta_\nu$, so we will assume $\beta \in \mathcal{T}_n \backslash \mathcal{S}_n$ and thus there exist distinct elements $x_1, x_2 \in \mathbf{n}$ such that $x_1\beta = x_2\beta$. There are two cases to consider.

Case 1: Suppose $x_1\alpha = x_2\alpha$, and define γ_k as in (i). Then, $x_1\gamma_k = x_2\gamma_k$ and thus $\gamma_k \notin \mathcal{S}_n$ for $1 \leq k < m$, showing $(\alpha, \beta) \in \Theta_\mu \vee \Theta_\nu$ as per the proof for (i).

Case 2: Suppose instead that $x_1\alpha \neq x_2\alpha$. If $\alpha \neq \mathrm{id}_{\mathbf{n}}$, choose distinct $z_1, z_2 \in \mathbf{n}$ such that $z_1\alpha = z_2\alpha$. Then, it could be that either x_1 or x_2 is in $\{z_1, z_2\}$, but $x_1\alpha \neq x_2\alpha$ implies it cannot be both, so we will assume without loss of generality that $x_1 \notin \{z_1, z_2\}$. By the symmetric and transitive properties, $x_1\beta = x_2\beta$ and $(x_1\alpha, x_1\beta), (x_2\alpha, x_2\beta) \in \varepsilon \vee \eta$ imply $(x_1\alpha, x_2\alpha) \in \varepsilon \vee \eta$, so there is a chain $x_1\alpha \xrightarrow{\varepsilon} y_1 \xrightarrow{\eta} y_2 \xrightarrow{\varepsilon} \cdots \xrightarrow{\varepsilon} y_{p-1} \xrightarrow{\eta} x_2\alpha$ of some even length $p \in \mathbb{N}$. For all $x \in \mathbf{n}$ and $1 \leq k < p$, define the maps λ_k and α' by

$$x\lambda_k = \begin{cases} y_k & \text{if } x = x_1 \\ x\alpha & \text{otherwise} \end{cases} \quad \text{and} \quad x\alpha' = \begin{cases} x_2\alpha & \text{if } x \in \{x_1, x_2\} \\ x\alpha & \text{otherwise.} \end{cases}$$

If $\alpha \neq \mathrm{id}_{\mathbf{n}}$, we have $z_1\lambda_k = z_2\lambda_k$ for all $1 \leq k < p$. If $\alpha = \mathrm{id}_{\mathbf{n}}$, either $x_1\lambda_k = x_1$ so that $\lambda_k = \mathrm{id}_{\mathbf{n}}$, or $x_1\lambda_k = x' = x'\lambda_k$ for some $x' \in \mathbf{n}\backslash\{x_1\}$. In either case, each $\lambda_k \notin \mathcal{S}_n \backslash \{\mathrm{id}_{\mathbf{n}}\}$. Hence,

$(\alpha, \alpha') \in \Theta_\mu \vee \Theta_\nu$ by the chain $\alpha \xrightarrow{\Theta_\mu} \lambda_1 \xrightarrow{\Theta_\nu} \lambda_2 \xrightarrow{\Theta_\mu} \cdots \xrightarrow{\Theta_\mu} \lambda_{p-1} \xrightarrow{\Theta_\nu} \alpha'$. Furthermore, we have $(\alpha', \beta) \in \Theta_{\mu\nu}$ by transitivity, $x_1\alpha' = x_2\alpha'$ and $x_1\beta = x_2\beta$, therefore $(\alpha', \beta) \in \Theta_\mu \vee \Theta_\nu$ by Case 1. It follows that $(\alpha, \beta) \in \Theta_\mu \vee \Theta_\nu$ by another appeal to transitivity. □

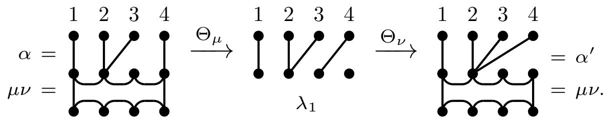
The following example is intended to help illustrate part (ii) of Lemma 5.2.

Example 5.3. Let $\mu, \nu \in \mathcal{F}_4$ be such that $\mu = \left(\begin{array}{c|c} \{1,2\} & \{3,4\} \\ \{1,2\} & \{3,4\} \end{array}\right)$ and $\nu = \left(\begin{array}{c|c|c} \{1\} & \{2,3\} & \{4\} \\ \{1\} & \{2,3\} & \{4\} \end{array}\right)$, let $\alpha, \beta \in \mathcal{T}_4 \backslash \mathcal{S}_4$ be such that $\alpha = \left[\begin{array}{c|c|c} \{1\} & \{2,3\} & \{4\} \\ 1 & 2 & 4 \end{array}\right]$ and $\beta = \left[\begin{array}{c|c|c} \{1\} & \{2,4\} & \{3\} \\ 2 & 4 & 1 \end{array}\right]$, and let $\varepsilon = \ker(\mu)$ and $\eta = \ker(\nu)$. Suppose, knowing that $(\alpha, \beta) \in \Theta_{\mu\nu}$, we want to show that $(\alpha, \beta) \in \Theta_\mu \vee \Theta_\nu$. Here $m = 4$ and the most direct chains of relations $x\alpha \to x\beta$ in $\varepsilon \cup \eta$ of length four for each $x \in \mathbf{4}$ are

(i) $1\alpha = 1 \xrightarrow{\varepsilon} 2 \xrightarrow{\eta} \boxed{2} \xrightarrow{\varepsilon} 2 \xrightarrow{\eta} 2 = 1\beta$, (iii) $3\alpha = 2 \xrightarrow{\varepsilon} 1 \xrightarrow{\eta} \boxed{1} \xrightarrow{\varepsilon} 1 \xrightarrow{\eta} 1 = 3\beta$,

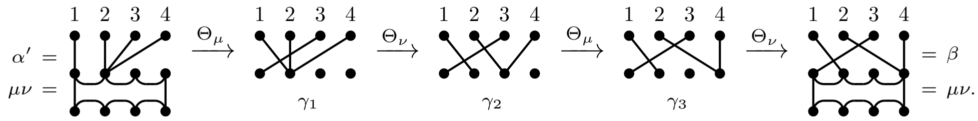
(ii) $2\alpha = 2 \xrightarrow{\varepsilon} 2 \xrightarrow{\eta} \boxed{3} \xrightarrow{\varepsilon} 4 \xrightarrow{\eta} 4 = 2\beta$, (iv) $4\alpha = 4 \xrightarrow{\varepsilon} 4 \xrightarrow{\eta} \boxed{4} \xrightarrow{\varepsilon} 4 \xrightarrow{\eta} 4 = 4\beta$.

However, defining γ_k using these chains would not show that $(\alpha, \beta) \in \Theta_\mu \vee \Theta_\nu$, as $\gamma_3 \in \mathcal{S}_4 \backslash \{\mathrm{id}_{\mathbf{4}}\}$ (see each $x\gamma_3$ in boxes above). Rather, this example fits Case 2, where $x_1 = 4, x_2 = z_1 = 2$ and $z_2 = 3$. First, let $\alpha' = \left[\begin{array}{c|c} \{1\} & \{2,3,4\} \\ 1 & 2 \end{array}\right]$. Then, use the chain $4\alpha = 4 \xrightarrow{\varepsilon} 3 \xrightarrow{\eta} 2 = 4\alpha'$ to define λ_1 in order to use the chain $\alpha \xrightarrow{\Theta_\mu} \lambda_1 \xrightarrow{\Theta_\nu} \alpha'$ to show that $(\alpha, \alpha') \in \Theta_\mu \vee \Theta_\nu$. Pictorially,



In addition, notice how it was necessary to choose $x_1 \notin \{z_1, z_2\}$ to ensure that $\lambda_1 \notin \mathcal{S}_4 \backslash \{\mathrm{id}_{\mathbf{4}}\}$.

Hence α' and β satisfy the conditions for Case 1, as $4\alpha' = 2\alpha'$ and $4\beta = 2\beta$, so both $4\alpha'$ and $2\alpha'$ can use chain (ii). When γ_k is defined using chains (i), (ii) and (iii), each $\gamma_k \notin \mathcal{S}_4 \backslash \{\mathrm{id}_{\mathbf{4}}\}$ as $2\gamma_k = 4\gamma_k$ for all $1 \le k < 4$. As a result, we can use the chain $\alpha' \xrightarrow{\Theta_\mu} \gamma_1 \xrightarrow{\Theta_\nu} \gamma_2 \xrightarrow{\Theta_\mu} \gamma_3 \xrightarrow{\Theta_\nu} \beta$ to show that $(\alpha', \beta) \in \Theta_\mu \vee \Theta_\nu$ and thus $(\alpha, \beta) \in \Theta_\mu \vee \Theta_\nu$. Pictorially,



5.1 Presentation for $\mathcal{P}_n^{\mathrm{fd}} \backslash \mathcal{S}_n$ using the action pair $(\mathcal{F}_n, \mathcal{T}_n \backslash \mathcal{S}_n)$

What follows is an alternative proof for Theorem 2.3 via the action pair approach. For letters $u \in E$ and $v \in T$, we will write $v^u = t_\varepsilon$, where $\varepsilon = \ker(\widehat{v}^{\,\widehat{u}})$.

Proof. We will use [2, Theorem 6.44 (ii)] to show that $\mathcal{P}_n^{\mathrm{fd}} \backslash \mathcal{S}_n$ has the presentation in Theorem 2.3 using the action pair $(\mathcal{F}_n, \mathcal{T}_n \backslash \mathcal{S}_n)$ and their presentations from Theorems 2.6 and 2.5. First, Proposition 5.1 and Lemma 5.2 show that $(\mathcal{F}_n, \mathcal{T}_n \backslash \mathcal{S}_n)$ satisfies the conditions of [2, Theorem 6.44 (ii)]. For $\mu \in \mathcal{F}_n$, let Ω_μ be a subset of $(\mathcal{T}_n \backslash \mathcal{S}_n)^1 \times (\mathcal{T}_n \backslash \mathcal{S}_n)^1$ that generates Θ_μ as a left congruence and define

$$R_1 = \{(vu, uv^u) : u \in E,\ v \in T\} \quad \text{and} \quad R_2 = \{(u_1 v, u_2 v) : v \in T,\ (\widehat{u}_1, \widehat{u}_2) \in \Omega_{\widehat{v}}\}.$$

By [2, Theorem 6.44 (ii)], $\mathcal{P}_n^{\mathrm{fd}} \backslash \mathcal{S}_n$ has semigroup presentation

$$\langle E \cup T : (\text{T1–T6}) \cup (\text{F1–F3}) \cup R_1 \cup R_2 \rangle \text{ via } \phi\,.$$

For R_1, recall that $t_{ij} = t_{ji}$ and observe that, for $i, j, k, l \in \mathbf{n}$ where $i \neq j$ and $k \neq l$,

$$t_{kl}^{e_{ij}} = \begin{cases} \iota & \text{if } \{i,j\} = \{k,l\} & \text{(Case 1)} \\ t_{il} & \text{if } k = j \text{ and } l \neq i & \text{(Case 2)} \\ t_{kl} & \text{if } k,l \notin \{i,j\} & \text{(Case 3)} \\ t_{kl} & \text{if } k = i \text{ and } l \neq j. & \text{(Case 4)} \end{cases}$$

Cases 1-3 correspond to relations (PS1–PS3) and Case 4 corresponds to (PS2′). However, as demonstrated in Lemma 3.3, (PS2′) is a consequence of (T2) and (PS1), so it can be omitted. Therefore $R_1 =$ (PS1–PS3).

For R_2, let $\mu = \widehat{t}_{ij} \in T\phi$ for some $i,j \in \mathbf{n}$ where $i \neq j$ and let $\vdash$ be the left congruence generated by $(\widehat{e}_{ij}, \mathrm{id}_{\mathbf{n}})$. One may check diagrammatically that $\widehat{e}_{ij}\widehat{t}_{ij} = \widehat{t}_{ij}$, showing that $(\widehat{e}_{ij}, \mathrm{id}_{\mathbf{n}}) \in \Theta_\mu$ and hence $\vdash \subseteq \Theta_\mu$. For the reverse inclusion, suppose $(\alpha, \beta) \in \Theta_\mu$ so that $\alpha\widehat{t}_{ij} = \beta\widehat{t}_{ij}$. It can also be shown diagrammatically that $\widehat{e}_{ij} = \widehat{t}_{ij}\widehat{e}_{ij}$, giving us $\alpha\widehat{e}_{ij} = \alpha\widehat{t}_{ij}\widehat{e}_{ij} = \beta\widehat{t}_{ij}\widehat{e} = \beta\widehat{e}_{ij}$. But then $\alpha \vdash \alpha\widehat{e}_{ij} = \beta\widehat{e}_{ij} \vdash \beta$, showing that $\Theta_\mu \subseteq \vdash$. Hence, for each $\mu = \widehat{t}_{ij} \in T\phi$, we have that $\Omega_\mu = \{(\widehat{e}_{ij}, \mathrm{id}_{\mathbf{n}})\}$ generates Θ_μ as left congruence and thus $R_2 =$ (PS4). □

5.2 Presentation for $\mathcal{P}_n^{\mathrm{fd}}$ using the action pair $(\mathcal{F}_n, \mathcal{T}_n)$

In this section, we obtain an alternative presentation for $\mathcal{P}_n^{\mathrm{fd}}$ using the action pair $(\mathcal{F}_n, \mathcal{T}_n)$. Given $w \in (S \cup \{e\})^*$ and $v \in T$, we will write $v^w = t_\varepsilon$, where $\varepsilon = \ker(\widehat{v}^{\,\overline{w}})$. In addition, for $i,j \in \mathbf{n}$ where $i \neq j$ we have $N(\widehat{e}_{ij}) = \epsilon_{ij}$. That is, each partition $\widehat{e}_{ij}$ has the *normal form* ϵ_{ij} in $(S \cup \{e\})^*$. Consider the relations

$$t_{1,2}e = e\,, \tag{A1}$$
$$t_{1,j}e = t_{2,j}e = et_{1,j} \quad \text{for } j > 2, \tag{A2}$$
$$t_{ij}e = et_{ij} \quad \text{for } i,j > 2, \tag{A3}$$
$$t_{i,i+1}s_i = s_it_{i,i+1} \quad \text{for } i < n, \tag{A4}$$
$$t_{ij}s_i = s_it_{i+1,j} \quad \text{for } j \neq i+1 \leq n, \tag{A5}$$
$$t_{ij}s_{i-1} = s_{i-1}t_{i-1,j} \quad \text{for } j \neq i-1 \geq 1, \tag{A6}$$
$$t_{ij}s_k = s_kt_{ij} \quad \text{for } k \notin \{i\text{-}1, i, j\text{-}1, j\}, \tag{A7}$$
$$\epsilon_{ij}t_{ij} = t_{ij} \quad \text{for distinct } i,j, \tag{B1}$$
$$e^2 = e = s_1e\,, \tag{P1″}$$

where $i,j \in \mathbf{n}$ and $1 \leq k < n$. Define

$$R_{\mathcal{T}} = \text{(S1–S3)} \cup \text{(P1″)} \cup \text{(P3–P6)},$$
$$R_{\mathcal{F}} = \text{(F1–F3)}, \quad R_A = \text{(A1–A7)}, \quad R_B = \text{(B1)},$$
$$\Psi_{\mathcal{F}} = \phi|_{T^*}\,, \quad \text{and} \quad \Psi_{\mathcal{T}} = \Phi|_{(S\cup\{e\})^*}\,.$$

The following presentation for $\mathcal{T}_n$ is shown by restricting the arguments in Section 4 to the presentation for $\mathcal{T}_n \backslash \mathcal{S}_n$ in Theorem 2.5 alone. It is analagous to that which is found in [7, Theorem 9].

Theorem 5.4. *The monoid $\mathcal{T}_n$ has monoid presentation $\langle S \cup \{e\} : R_{\mathcal{T}} \rangle$ via $\Psi_{\mathcal{T}}$.*

Define a monoid surmorphism

$$\Psi : (S \cup \{e\} \cup T)^* \to \mathcal{P}_n^{\mathrm{fd}} : x \mapsto \begin{cases} x\Psi_{\mathcal{F}} & \text{if } x \in T \\ x\Psi_{\mathcal{T}} & \text{if } x \in S \cup \{e\}. \end{cases}$$

We can now prove this alternative presentation for $\mathcal{P}_n^{\mathrm{fd}}$.

Theorem 5.5. *The monoid $\mathcal{P}_n^{\mathrm{fd}}$ has monoid presentation $\langle S \cup \{e\} \cup T : R_{\mathcal{T}} \cup R_{\mathcal{F}} \cup R_A \cup R_B \rangle$ via Ψ.*

Proof. We will use [2, Theorem 6.5] to show that $\mathcal{P}_n^{\mathrm{fd}}$ has the above presentation using the action pair $(\mathcal{F}_n, \mathcal{T}_n)$ and their presentations in Theorems 2.6 and 5.4. By Proposition 5.1, $(\mathcal{F}_n, \mathcal{T}_n)$ is a strong action pair and hence it satisfies the conditions of [2, Theorem 6.5]. For $\mu \in \mathcal{F}_n$, let Ω_μ be a subset of $\mathcal{T}_n \times \mathcal{T}_n$ that generates θ_μ as a left congruence and define

$$R_1 = \{(vw, wv^w) : w \in S \cup \{e\},\ v \in T\} \quad \text{and} \quad R_\Omega = \{(vN(w_1), vN(w_2)) : v \in T,\ (w_1, w_2) \in \Omega_{\widehat{v}}\}.$$

By [2, Theorem 6.5] and Lemma 5.2, $\mathcal{P}_n^{\text{fd}}$ has monoid presentation

$$\langle S \cup \{e\} \cup T : R_\mathcal{T} \cup R_\mathcal{F} \cup R_1 \cup R_\Omega \rangle,$$

via Ψ.

For R_1, recall that $t_{ij} = t_{ji}$ and observe that, for $i, j \in \mathbf{n}$ where $i \neq j$ and $1 \leq k < n$,

$$t_{ij}^e = \begin{cases} \iota & \text{if } \{i,j\} = \{1,2\} & \text{(Case 1)} \\ t_{1,j} & \text{if } i = 2 < j & \text{(Case 2.1)} \\ t_{1,j} & \text{if } i = 1 \text{ and } j > 2 & \text{(Case 2.2)} \\ t_{ij} & \text{if } i, j > 2, & \text{(Case 3)} \end{cases}$$

$$t_{ij}^{s_k} = \begin{cases} t_{i+1,j} & \text{if } k = i \neq j - 1 & \text{(Case 4)} \\ t_{i-1,j} & \text{if } k = i - 1 \neq j & \text{(Case 5)} \\ t_{ij} & \text{if } \{k, k+1\} = \{i,j\} & \text{(Case 6)} \\ t_{ij} & \text{for } k \notin \{i-1, i, j-1, j\}. & \text{(Case 7)} \end{cases}$$

Cases 1-7 correspond to (A1–A7), showing $R_1 = R_A$.

For R_Ω, let $\mu = \widehat{t}_{ij} \in T\Psi_\mathcal{F}$ for some $i, j \in \mathbf{n}$ where $i \neq j$ and recall that $\widehat{e}_{ij} = \overline{\epsilon}_{ij}$. It can be shown that $\Omega_\mu = \{(\overline{\epsilon}_{ij}, \text{id}_\mathbf{n})\}$ generates θ_μ using a proof analgous to that found Section 5.1, except each $\widehat{e}_{ij}$ is replaced by $\overline{\epsilon}_{ij}$ and each Θ_μ replaced by θ_μ. It follows that $R_\Omega = R_B$ as required. □

The presentation for $\mathcal{P}_n^{\text{fd}}$ in Theorem 5.5 can simplified to obtain one similar to that found in Theorem 2.4 by replacing the alphabet T with $\{t\}$ as in Section 4 and reducing the relations accordingly. We omit the details for reasons of space.

6 Discussion and Conclusion

The ease with which Theorem 2.3 was proven in Section 5.1 compared to Section 3 demonstrates the effectiveness of the techniques in [2] for semigroups satisfying the assumptions in [2, Theorem 6.44 (ii)]. On the other hand, the presentation for $\mathcal{P}_n^{\text{fd}}$ in Theorem 2.4 is significantly simpler than that found in Theorem 5.5 via [2, Theorem 6.5]. Using a different presentation for $\mathcal{T}_n$ (such as that which is found in [1]) in combination with [2, Theorem 6.5] could produce a more desirable presentation for $\mathcal{P}_n^{\text{fd}}$.

But there may be a better alternative. The method from [4] of using the group of units in a monoid to extend a presentation of its singular ideal was applied without difficulty, which suggests it could be generalised. This could then be used to extend the techniques in [2] to better handle monoids like $\mathcal{P}_n^{\text{fd}}$. As such, further research in this direction would be of significant interest.

In addition to the action pairs $(\mathcal{F}_n, \mathcal{T}_n \backslash \mathcal{S}_n)$ and $(\mathcal{F}_n, \mathcal{T}_n)$ discussed in this paper, [2] discusses the existence of other action pairs that can be found within $\mathcal{P}_n^{\text{fd}}$, including $(\mathcal{F}_n \backslash \{\text{id}_\mathbf{n}\}, \mathcal{T}_n)$. This produces the semigroup $\mathcal{P}_n^{\text{fd}} \backslash \mathcal{T}_n$, which is analogous to the semigroup of strictly partial transformations $\mathcal{PT}_n \backslash \mathcal{T}_n$. Finding a presentation for $\mathcal{P}_n^{\text{fd}} \backslash \mathcal{T}_n$ may provide important insights into semigroups such as these, which are typically more challenging to work with.

Finally, another intriguing monoid for further research is $\mathcal{P}_n^{\text{tc}} = \{\rho \in \mathcal{P}_n : \text{coker}(\rho) = \Delta_\mathbf{n}\}$. Similar to $\mathcal{P}_n^{\text{fd}}$ and $\mathcal{PT}_n$, this monoid has product decomposition $\mathcal{P}_n^{\text{tc}} = \mathcal{T}_n \mathcal{E}_n$, where $\mathcal{E}_n$ is the semilattice of partial identities. It is not a restriction monoid, however, so a new approach would be required to obtain a presentation.

7 References

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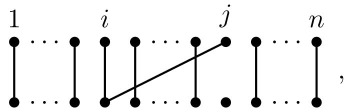
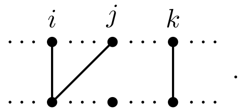
8 Appendices

These Appendices include further examples of the diagramatic and algebraic calculations involved in proving many of the results in this paper. Not all calculations omitted from the body of this paper are included here, however, due to their extent and similarity to one another.

To simplify the diagrams in our diagramatic proofs, we will typically omit vertices and edges that are inconsequential to the calculation. All vertices are assumed to be in **n** and in sequential order from left to right. All omitted vertices in an upper row are assumed to be adjacent only to their counterpart (also omitted) in the lower row and vice versa. For example, instead of representing $\widehat{e}_{ij}$ as

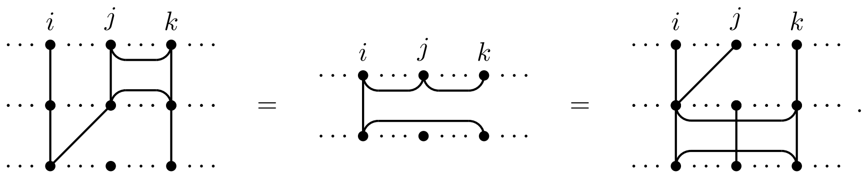


we will represent it as



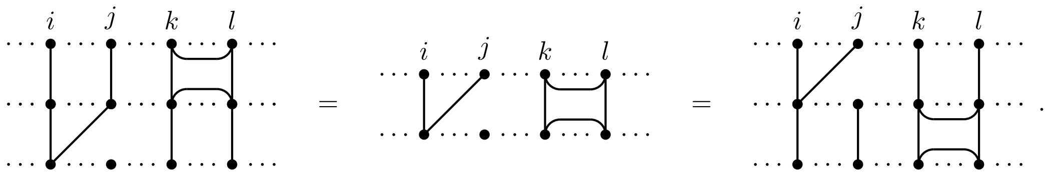
8.1 Further examples for Proposition 3.2

Example 8.1. Diagramatic proof of relation (PS2) in the case $i < j < k$:



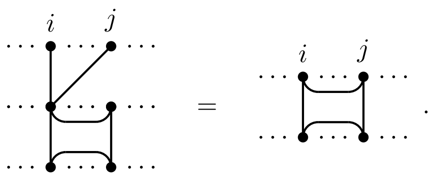
$\widehat{t}_{jk}\widehat{e}_{ij}$ $\widehat{e}_{ij}\widehat{t}_{ik}$

Example 8.2. Diagramatic proof of relation (PS3) in the case $i < j < k < l$:



$\widehat{t}_{kl}\widehat{e}_{ij}$ $\widehat{e}_{ij}\widehat{t}_{kl}$

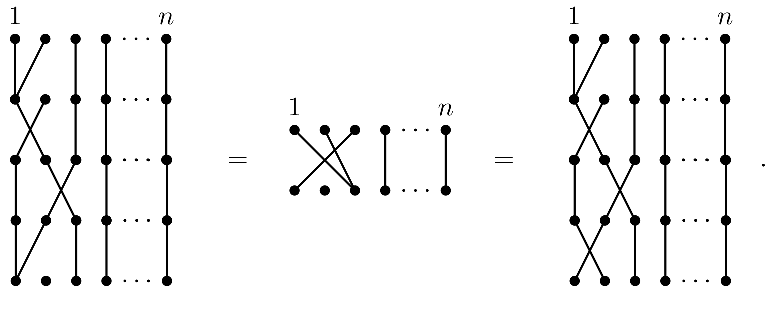
Example 8.3. Diagramatic proof of relation (PS4) in the case $i < j$:



$\widehat{e}_{ij}\widehat{t}_{ij}$ $\widehat{t}_{ij}$

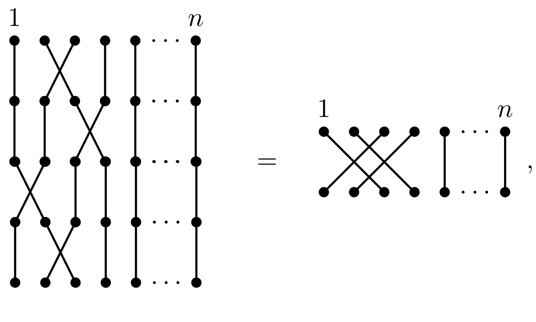
8.2 Further examples for Proposition 4.3

Example 8.4. Diagramatic proof of relation (P4):

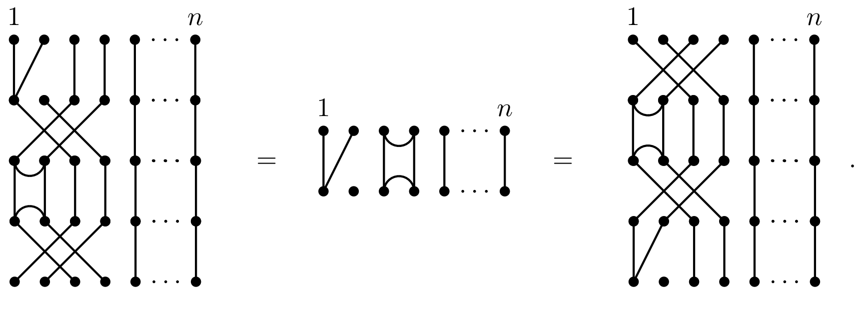


$\overline{e}\,\overline{s}_1\overline{s}_2\overline{e}$ $\qquad$ $\overline{e}\,\overline{s}_1\overline{s}_2\overline{s}_1$

Example 8.5. Diagramatic proof of relation (P11) (recall that $\kappa = s_1s_3s_1s_2$):



$\overline{s}_1\overline{s}_3\overline{s}_1\overline{s}_2$



$\overline{e}\,\overline{\kappa}\,\overline{t}\,\overline{\kappa}$ $\qquad$ $\overline{\kappa}\,\overline{t}\,\overline{\kappa}\,\overline{e}$

8.3 Further examples for Lemma 4.8

Recall that Corollary 4.7 allows us to transform the subscripts of the words ϵ_{ij} and τ_{ij}.

Example 8.6. To show relation (PS2) holds for its image under ψ, we will use the transformation $(i, j, k) = (1, 2, 3)$:

$$\begin{aligned}
\tau_{2,3}\epsilon_{1,2} &= (c_{2,3}^{-1}tc_{2,3})(e) \\
&= (s_1s_2)t(s_2s_1)e \\
&\approx s_1s_2ts_2(e) && \text{by (P1}'\text{)} \\
&\approx s_1(es_2ts_2) && \text{by (P10)} \\
&\approx (e)s_2ts_2 && \text{by (P1}'\text{)} \\
&= e(c_{1,3}^{-1})t(c_{1,3}) \\
&= \epsilon_{1,2}\tau_{1,3}\,.
\end{aligned}$$

Example 8.7. To show relation (F2) holds for its image under ψ, we will use the transformation $(i, j, k, l) = (1, 2, 3, 4)$:

$$\begin{aligned}
\tau_{1,2}\tau_{3,4} &= (t)(c_{3,4}^{-1}tc_{3,4}) \\
&= t(s_2s_1s_3s_2)t(s_2s_3s_1s_2) \\
&\approx ts_2(s_3s_1)s_2ts_2s_3s_1s_2 && \text{by (S2)} \\
&\approx s_2s_3s_1s_2ts_2s_3s_1s_2t && \text{by (P9)} \\
&\approx s_2(s_1s_3)s_2ts_2s_3s_1s_2t && \text{by (S2)} \\
&= (c_{3,4}^{-1})t(c_{3,4})t \\
&= \tau_{3,4}\tau_{1,2}\,.
\end{aligned}$$

Example 8.8. To show relation (T5) holds for its image under ψ, we will use the transformation $(i, j, k) = (1, 2, 3)$:

$$\begin{aligned}
\epsilon_{3,1}\epsilon_{1,2}\epsilon_{2,3} &= (c_{1,3}^{-1}es_1c_{1,3})(e)(c_{2,3}^{-1}ec_{2,3}) \\
&= (s_2)es_1(s_2)e(s_1s_2)e(s_2s_1) \\
&\approx s_2es_1s_2(s_1e)s_1s_2(s_1e)s_2s_1(s_2s_2) && \text{by (P1}'\text{) and (S1)} \\
&\approx s_2e(s_2s_1s_2)es_1s_2s_1e(s_1s_2s_1)s_2 && \text{by (S3)} \\
&\approx s_2es_2s_1s_2es_1s_2s_1(es_1s_2e)s_2 && \text{by (P4)} \\
&= (c_{1,3}^{-1})e(c_{1,3})(c_{2,3}^{-1})es_1(c_{2,3})es_1(c_{1,3}^{-1})e(c_{1,3}) \\
&= \epsilon_{1,3}\epsilon_{3,2}\epsilon_{2,1}\epsilon_{1,3}\,.
\end{aligned}$$

8.4 Further examples for Lemma 4.9

Recall that Corollary 4.7 allows us to transform the subscripts of the words ϵ_{ij} and τ_{ij}.

Example 8.9. To show $\epsilon_{i,i+1}s_i \approx \epsilon_{ji}$ (Case 1) we will use the transformation $i = 1$:

$$\begin{aligned} \epsilon_{1,2}s_1 &= es_1 \\ &= \epsilon_{2,1}\,. \end{aligned}$$

Example 8.10. To show $\epsilon_{ij}s_{i-1} \approx \epsilon_{ij}\epsilon_{j,i-1}\epsilon_{i-1,i}\epsilon_{ij}$ if $j \neq i-1$ (Case 2) we will use the transformation $(i, j) = (2, 3)$:

$$\begin{aligned} \epsilon_{2,3}\epsilon_{3,1}\epsilon_{1,2}\epsilon_{2,3} &= (c_{2,3}^{-1}ec_{2,3})(c_{1,3}^{-1}es_1c_{1,3})(e)(c_{2,3}^{-1}ec_{2,3}) \\ &= (s_1s_2)e(s_2s_1)(s_2)es_1(s_2)e(s_1s_2)e(s_2s_1) \\ &\approx s_1s_2e(s_1s_2s_1)(es_1s_2s_1)s_1s_2es_2s_1 && \text{by (S3) and (P4)} \\ &\approx s_1s_2es_1s_2(e)s_1es_2s_1 && \text{by (P1}') \text{ and (S1)} \\ &\approx s_1s_2(es_1s_2s_1)s_1es_2s_1 && \text{by (P4)} \\ &\approx s_1s_2es_1s_2es_2s_1 && \text{by (S1)} \\ &\approx s_1s_2(es_1s_2s_1)s_2s_1 && \text{by (P4)} \\ &\approx s_1s_2e(s_2s_1s_2)s_2s_1 && \text{by (S3)} \\ &\approx s_1s_2es_2s_1s_1 && \text{by (S1)} \\ &= (c_{2,3}^{-1})e(c_{2,3})s_1 \\ &= \epsilon_{2,3}s_1\,. \end{aligned}$$