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Corollaries of the Gauss-Bonnet theorem for surfaces in $\mathbb{R}^n$

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Abstract

The Gauss-Bonnet theorem for surfaces states that integrating Gaussian curvature over a surface without boundary yields a topological constant. Consequently, the variation of Gaussian curvature should be identically zero. In this report we conduct the extrinsic variation of Gaussian curvature, and see that it encodes the two fundamental equations of submanifold geometry: the Gauss equation and the Codazzi equation.

1 Introduction

The Gauss-Bonnet theorem for surfaces is a classical result stating that the total Gaussian curvature of a surface without boundary is a topological invariant. Consequently, any deformation of the surface will not change the total Gaussian curvature. In section 2, we establish essential definitions and notation. We vary the Gaussian curvature in section 3, yielding a quantity that is identically zero. From this point we derive the Gauss equation and contracted Codazzi-Mainardi equations, which is done in section 4. Finally, in section 5, we look at how the Gauss-Codazzi equations present in higher dimensions, and propose further applications of this variational method.

Statement of Authorship

The equations derived in this report are classical; see [2], for example. The variation of Gaussian curvature and subsequent derivation of these classical equations was carried out by Annalisa Calvi, under the guidance of her supervisor Dr. Yann Bernard.

2 Preliminary definitions

We consider a smooth, orientable, compact surface Σ with no boundary. Let $\vec{\Phi}$ be an immersion of Σ in $\mathbb{R}^n$, with $m \geq 3$.

The *metric tensor*, denoted g, is a 2×2 matrix with entries g_{ij}, where $g_{ij} := \partial_i \vec{\Phi} \cdot \partial_j \vec{\Phi}$. The entries of its inverse are denoted g^{ij}, and the *area element* is given by $|g|^{1/2} := \sqrt{\det g}$. Let ∇_k denote the covariant derivative compatible with the metric, so that $\nabla_k g_{ij} = \nabla_k g^{ij} = 0$ for all $i, j, k \in \{1, 2\}$. Let Γ^k_{ij} be the corresponding Christoffel symbols.

At any point on the surface Σ, there are tangent vectors $\nabla_i \vec{\Phi} = \partial_i \vec{\Phi}$, for $i = 1, 2$. Let π_T denote the projection map onto the tangent space, so that $\pi_T \vec{F} = (\vec{F} \cdot \nabla^i \vec{\Phi}) \nabla_i \vec{\Phi}$ for any vector field $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$.

Let π_N denote the projection onto the normal space of Σ. Note also that $\pi_T + \pi_N = \text{id}$, the identity map. We can then define a normal Laplacian $\Delta_\perp := \pi_N \nabla^i \pi_N \nabla_i$, and similarly a normal double derivative $(\nabla^2_\perp)_{ij} := \pi_N \nabla_i \pi_N \nabla_j$, to use later on.

Let the *second fundamental form*, denoted $\vec{h}$, be a 2×2 matrix with vector entries $\vec{h}_{ij} := \nabla_i \nabla_j \vec{\Phi}$. Note that $\vec{h}_{ij} = \vec{h}_{ji}$ for all $i, j \in \{1, 2\}$. We let

$$|\vec{h}|^2 := \text{tr}_g(\vec{h} \cdot \vec{h}) = \vec{h}_{ij} \cdot \vec{h}^{ij}.$$

The *mean curvature*, denoted $\vec{H}$, is given by

$$\vec{H} := \frac{1}{2}\vec{h}^i_i = \frac{1}{2}g^{ij}\vec{h}_{ij},$$

so that $2\vec{H} = \mathrm{tr}_g(\vec{h})$. Then $|\vec{H}|^2 := \vec{H}\cdot\vec{H}$ is equal to $\frac{1}{4}\,\mathrm{tr}_g(\vec{h})^2$.

Gaussian curvature in arbitrary codimension, denoted K, will be defined as $K := 2|\vec{H}|^2 - \frac{1}{2}|\vec{h}|^2$. The Gauss-Bonnet theorem states that

$$\int_\Sigma K dA = 2\pi\chi(\Sigma),$$

where the Euler characteristic $\chi(\Sigma)$ is defined as $2-2g$, where g is the genus of Σ. Supposing Σ has coordinates x^1, x^2, we have $dA = |g|^{1/2}dx^1 \wedge dx^2$. Then

$$\int_\Sigma K dA = \int_\Sigma K|g|^{1/2}dx^1 \wedge dx^2 = 2\pi\chi(\Sigma).$$

We can consider K and $|g|^{1/2}$ to be functions of the immersion $\vec{\Phi}$, so that $K = K(\vec{\Phi})$ and $|g|^{1/2} = |g|^{1/2}(\vec{\Phi})$. The topological constant $2\pi\chi(\Sigma)$, on the other hand, does not depend on $\vec{\Phi}$.

Consider a variation of the form

$$\vec{\Phi}_t := \vec{\Phi} + t(A^j\nabla_j\vec{\Phi} + \vec{B}),$$

for some tensor A^j, and normal vector $\vec{B}$. For any function f of $\vec{\Phi}$, let

$$\delta f := \frac{d}{dt}f(\Phi_t),$$

so that δ denotes our variation. Then by the Gauss-Bonnet theorem, we have

$$\int_\Sigma \delta(K|g|^{1/2})\; dx^1 \wedge dx^2 = \delta(2\pi\chi(\Sigma)) = 0. \tag{2.1}$$

3 Variation of $\int K$

In this section we evaluate $\delta(K|g|^{1/2})$. Using our expression $K = 2|\vec{H}|^2 - \frac{1}{2}|\vec{h}|^2$, we can rewrite this as

$$\delta(K|g|^{1/2}) = 2\delta(|\vec{H}|^2|g|^{1/2}) - \frac{1}{2}\delta(|\vec{h}|^2|g|^{1/2}), \tag{3.1}$$

and proceed to expand each of these terms.

3.1 Evaluating $\delta(|\vec{H}|^2|g|^{1/2})$

The variation $\delta(|\vec{H}|^2|g|^{1/2})$ is carried out in [1], arriving at the result

$$\delta\int|\vec{H}|^2 = \int \vec{B}\cdot\vec{\mathcal{W}} + \nabla_i\left(\vec{H}\cdot\nabla^i\vec{B} - \vec{B}\cdot\nabla^i\vec{H} + A^i|\vec{H}|^2\right),$$

with

$$\vec{\mathcal{W}} := \Delta_\perp\vec{H} + (\vec{H}\cdot\vec{h}^i_j)\vec{h}^j_i - 2|\vec{H}|^2\vec{H}.$$

However, in this variation, the contracted Codazzi-Mainardi equation is used. In particular, $\pi_N\nabla^j\vec{h}_{js}$ is replaced by $2\pi_N\nabla_s\vec{H}$ early in the variation.

We will modify the variation of $|\vec{H}|^2$ from the paper as follows. As our aim is to derive the contracted Codazzi-Mainardi equation, we will not use it; we will introduce an error term instead. Let $\vec{C}_s := \pi_N \nabla^j \vec{h}_{js} - 2\pi_N \nabla_s \vec{H}$. Then

$$\pi_N \nabla^j \vec{h}_{js} = 2\pi_N \nabla_s \vec{H} + \vec{C}_s.$$

The variation is done exactly as in [1], except that the extra term $\vec{C}_s$ carries through. Ultimately we obtain the variation

$$\delta \int |\vec{H}|^2 = \int \vec{B} \cdot \vec{\mathcal{W}} + \nabla_i \left(\vec{H} \cdot \nabla^i \vec{B} - \vec{B} \cdot \nabla^i \vec{H} + A^i |\vec{H}|^2 \right) + A^s \vec{H} \cdot \vec{C}_s \tag{3.2}$$

3.2 Evaluating $\delta(|\vec{h}|^2 |g|^{1/2})$

We begin by expanding $|\vec{h}|^2$, obtaining

$$|\vec{h}|^2 = \vec{h}_{ij} \cdot \vec{h}^{ij} = \vec{h}_{ij} \cdot \left(g^{ik} g^{jl} \vec{h}_{kl} \right) = g^{ik} g^{jl} \left(\vec{h}_{ij} \cdot \vec{h}_{kl} \right),$$

where we are summing over i, j, k, l. Now

$$\begin{aligned}
\delta |\vec{h}|^2 &= \delta(g^{ik} g^{jl}) \vec{h}_{ij} \cdot \vec{h}_{kl} + g^{ik} g^{jl} \delta(\vec{h}_{ij} \cdot \vec{h}_{kl}) \\
&= (\delta g^{ik}) \vec{h}_{ij} \cdot \vec{h}^j_k + (\delta g^{jl}) \vec{h}_{ij} \cdot \vec{h}^i_l + g^{ik} g^{jl} \vec{h}_{kl} \cdot \delta \vec{h}_{ij} + g^{ik} g^{jl} \vec{h}_{ij} \cdot \delta \vec{h}_{kl} \\
&= 2\delta(g^{ik}) \vec{h}_{ij} \cdot \vec{h}^j_k + 2\vec{h}^{ij} \cdot \delta \vec{h}_{ij}
\end{aligned} \tag{3.3}$$

by repeated application of the product rule.

3.2.1 Finding $\vec{h}^{ij} \cdot \delta \vec{h}_{ij}$

We focus first on $\vec{h}^{ij} \cdot \delta \vec{h}_{ij}$. As $\vec{h}^{ij}$ exists in the normal space we have $\vec{h}^{ij} \cdot \delta \vec{h}_{ij} = \vec{h}^{ij} \cdot \pi_N \delta \vec{h}_{ij}$. Using the definition of the covariant derivative, we can expand and contract to commute δ and ∇_i:

$$\begin{aligned}
\pi_N \delta \vec{h}_{ij} &= \pi_N \delta \nabla_i \nabla_j \vec{\Phi} \\
&= \pi_N \delta \left(\partial_i \nabla_j \vec{\Phi} - \Gamma^p_{ji} \nabla_p \vec{\Phi} \right) \\
&= \pi_N \left(\partial_i \delta \nabla_j \vec{\Phi} - (\delta \Gamma^p_{ji}) \nabla_p \vec{\Phi} - \Gamma^p_{ji} (\delta \nabla_p \vec{\Phi}) \right) \\
&= \pi_N \left(\partial_i \delta \nabla_j \vec{\Phi} - \Gamma^p_{ji} (\delta \nabla_p \vec{\Phi}) \right) \\
&= \pi_N \nabla_i \delta \nabla_j \vec{\Phi}.
\end{aligned} \tag{3.4}$$

The normal projection is important here; δ does not commute with covariant derivatives in general. Now we can evaluate

$$\delta \nabla_j \vec{\Phi} = \nabla_j \delta \vec{\Phi} = \nabla_j A^s \nabla_s \vec{\Phi} + A^s \vec{h}_{js} + \nabla_j \vec{B}.$$

since $\nabla_j \vec{\Phi} = \partial_j \vec{\Phi}$, and the flat derivatives δ and ∂_j commute. We substitute this in to find

$$\begin{aligned} \pi_N \nabla_i \delta \nabla_j \vec{\Phi} &= \pi_N \nabla_i \left(\nabla_j A^s \nabla_s \vec{\Phi} + A^s \vec{h}_{js} + \nabla_j \vec{B} \right) \\ &= \pi_N \left[(\nabla_i \nabla_j A^s) \nabla_s \vec{\Phi} + (\nabla_j A^s) \nabla_i \nabla_s \vec{\Phi} + (\nabla_i A^s) \vec{h}_{js} + A^s \nabla_i \vec{h}_{js} + \nabla_i \nabla_j \vec{B} \right] \\ &= \pi_N \left[(\nabla_j A^s) \vec{h}_{is} + (\nabla_i A^s) \vec{h}_{js} + A^s \nabla_i \vec{h}_{js} + \nabla_i \nabla_j \vec{B} \right] \\ &= (\nabla_j A^s) \vec{h}_{is} + (\nabla_i A^s) \vec{h}_{js} + A^s \pi_N \nabla_i \vec{h}_{js} + \pi_N \nabla_i \nabla_j \vec{B}. \end{aligned} \tag{3.5}$$

We will now derive a helpful identity; for any $\vec{u}$ normal to the surface, we have

$$0 = \nabla_j (\vec{u} \cdot \nabla^s \vec{\Phi}) \nabla_s \vec{\Phi} = \pi_T \nabla_j \vec{u} + (\vec{u} \cdot \vec{h}^s_j) \nabla_s \vec{\Phi},$$

so that

$$\pi_T \nabla_j \vec{u} = -(\vec{u} \cdot \vec{h}^s_j) \nabla_s \vec{\Phi}. \tag{3.6}$$

We use this equation with $\vec{u} := \vec{B}$ to obtain

$$\begin{aligned} \pi_N \nabla_i \nabla_j \vec{B} &= \pi_N \nabla_i \pi_N \nabla_j \vec{B} + \pi_N \nabla_i \pi_T \nabla_j \vec{B} \\ &= (\nabla^2_\perp)_{ij} \vec{B} + \pi_N \nabla_i (-(\vec{B} \cdot \vec{h}^s_j) \nabla_s \vec{\Phi}) \\ &= (\nabla^2_\perp)_{ij} \vec{B} - \pi_N \left(\nabla_i (\vec{B} \cdot \vec{h}^s_j) \nabla_s \vec{\Phi} + (\vec{B} \cdot \vec{h}^s_j) \nabla_i \nabla_s \vec{\Phi} \right) \\ &= (\nabla^2_\perp)_{ij} \vec{B} - (\vec{B} \cdot \vec{h}^s_j) \vec{h}_{is}. \end{aligned} \tag{3.7}$$

Recall that $(\nabla^2_\perp)_{ij} = \pi_N \nabla_i \pi_N \nabla_j$, as defined in section 2. We now combine (3.4), (3.5) and (3.7) to compute $\vec{h}^{ij} \cdot \delta \vec{h}_{ij}$ as

$$\begin{aligned} \vec{h}^{ij} \cdot \delta \vec{h}_{ij} =& \vec{h}^{ij} \cdot \left[(\nabla_j A^s) \vec{h}_{is} + (\nabla_i A^s) \vec{h}_{js} + A^s \nabla_i \vec{h}_{js} + (\nabla^2_\perp)_{ij} \vec{B} - (\vec{B} \cdot \vec{h}^s_j) \vec{h}_{is} \right] \\ =& (\nabla_j A^s)(\vec{h}_{is} \cdot \vec{h}^{ij}) + (\nabla_i A^s)(\vec{h}_{js} \cdot \vec{h}^{ij}) + A^s (\nabla_i \vec{h}_{js}) \cdot \vec{h}^{ij} + (\nabla^2_\perp)_{ij} \vec{B} \cdot \vec{h}^{ij} - (\vec{B} \cdot \vec{h}^s_j)(\vec{h}_{is} \cdot \vec{h}^{ij}) \\ =& 2(\nabla_i A^s)(\vec{h}_{js} \cdot \vec{h}^{ij}) + A^s (\nabla_i \vec{h}_{js}) \cdot \vec{h}^{ij} + (\nabla^2_\perp)_{ij} \vec{B} \cdot \vec{h}^{ij} - (\vec{B} \cdot \vec{h}^s_j)(\vec{h}_{is} \cdot \vec{h}^{ij}), \end{aligned}$$

by switching indices i and j in the first term. We conclude that

$$2\vec{h}^{ij} \cdot \delta \vec{h}_{ij} = 4(\nabla_i A^s)(\vec{h}_{js} \cdot \vec{h}^{ij}) + 2A^s \vec{h}^{ij} \cdot (\nabla_i \vec{h}_{js}) + 2((\nabla^2_\perp)_{ij} \vec{B}) \cdot \vec{h}^{ij} - 2(\vec{B} \cdot \vec{h}^s_j)(\vec{h}_{is} \cdot \vec{h}^{ij}). \tag{3.8}$$

3.2.2 Adding $\delta(g^{ik}) \vec{h}_{ij} \cdot \vec{h}^j_k$

Now we compute $\delta(g^{ik}) \vec{h}_{ij} \cdot \vec{h}^j_k$. The paper [1] provides the variation

$$\delta g^{ik} = -\nabla^i A^k - \nabla^k A^i + 2\vec{B} \cdot \vec{h}^{ik}.$$

Making this substitution gives

$$\begin{aligned} 2(\vec{h}_{ij} \cdot \vec{h}^j_k)(\delta g^{ik}) &= 2(\vec{h}_{ij} \cdot \vec{h}^j_k)(-\nabla^i A^k - \nabla^k A^i + 2\vec{B} \cdot \vec{h}^{ik}) \\ &= -2(\nabla^i A^k)(\vec{h}_{ij} \cdot \vec{h}^j_k) - 2(\nabla^k A^i)(\vec{h}_{ij} \cdot \vec{h}^j_k) + 2(\vec{B} \cdot \vec{h}^{ik})(\vec{h}_{ij} \cdot \vec{h}^j_k) \\ &= -2(\nabla_i A^s)(\vec{h}^{ij} \cdot \vec{h}_{js}) - 2(\nabla_i A^s)(\vec{h}_{sj} \cdot \vec{h}^{ij}) + 2(\vec{B} \cdot \vec{h}^s_j)(\vec{h}^{ij} \cdot \vec{h}_{is}) \\ &= -4(\nabla_i A^s)(\vec{h}_{js} \cdot \vec{h}^{ij}) + 2(\vec{B} \cdot \vec{h}^s_j)(\vec{h}_{is} \cdot \vec{h}^{ij}). \end{aligned} \tag{3.9}$$

From the second line to the third: in the first term we switch k with s; in the second term we switch i with s, then k with i; and in the third term we switch j with i and k with s.

By equation (3.3), we can obtain $\delta|\vec{h}|^2$ by adding equations (3.8) and (3.9). Then the terms $4(\nabla_i A^s)(\vec{h}_{js}\cdot\vec{h}^{ij})$ and $-2(\vec{B}\cdot\vec{h}^s_j)(\vec{h}_{is}\cdot\vec{h}^{ij})$ in (3.8) will be cancelled by the corresponding terms in (3.9), leaving:

$$\delta|\vec{h}|^2 = 2A^s\vec{h}^{ij}\cdot(\nabla_i\vec{h}_{js}) + 2((\nabla^2_\perp)_{ij}\vec{B})\cdot\vec{h}^{ij}. \tag{3.10}$$

Sections 3.2.3 and 3.2.4 derive results which are useful for arriving at the final form of this variation.

3.2.3 Preliminary: Reforming $((\nabla^2_\perp)_{ij}\vec{B})\cdot\vec{h}^{ij}$

We will first show that

$$((\nabla^2_\perp)_{ij}\vec{B})\cdot\vec{h}^{ij} = ((\nabla^2_\perp)_{ij}\vec{h}^{ij})\cdot\vec{B} - (\nabla_i\nabla_j\vec{h}^{ij})\cdot\vec{B} + (\nabla_i\nabla_j\vec{B})\cdot\vec{h}^{ij}. \tag{3.11}$$

Using the definition of $(\nabla^2_\perp)_{ij}$ and equation (3.6), we see that

$$\begin{aligned}
& ((\nabla^2_\perp)_{ij}\vec{B})\cdot\vec{h}^{ij} = ((\nabla^2_\perp)_{ij}\vec{h}^{ij})\cdot\vec{B} - (\nabla_i\nabla_j\vec{h}^{ij})\cdot\vec{B} + (\nabla_i\nabla_j\vec{B})\cdot\vec{h}^{ij} \\
\iff\quad & \vec{B}\cdot\left(\nabla_i\nabla_j\vec{h}^{ij} - \pi_N\nabla_i\pi_N\nabla_j\vec{h}^{ij}\right) = \vec{h}^{ij}\cdot\left(\nabla_i\nabla_j\vec{B} - \pi_N\nabla_i\pi_N\nabla_j\vec{B}\right) \\
\iff\quad & \vec{B}\cdot\left(\nabla_i\pi_T\nabla_j\vec{h}^{ij}\right) = \vec{h}^{ij}\cdot\left(\nabla_i\pi_T\nabla_j\vec{B}\right) \\
\iff\quad & -\vec{B}\cdot\left(\nabla_i(\vec{h}^{ij}\cdot\vec{h}^s_j\nabla_s\vec{\Phi})\right) = -\vec{h}^{ij}\cdot\left(\nabla_i(\vec{B}\cdot\vec{h}^s_j\nabla_s\vec{\Phi})\right) \\
\iff\quad & (\vec{B}\cdot\vec{h}_{is})(\vec{h}^{ij}\cdot\vec{h}^s_j) = (\vec{h}^{ij}\cdot\vec{h}_{is})(\vec{B}\cdot\vec{h}^s_j).
\end{aligned}$$

This final equation is true, as we can switch indices i and j on the LHS, and raise/lower s. Incorporating with (3.11) the fact that

$$-(\nabla_i\nabla_j\vec{h}^{ij})\cdot\vec{B} + (\nabla_i\nabla_j\vec{B})\cdot\vec{h}^{ij} = \nabla_i\left[\vec{h}^{ij}\cdot\nabla_j\vec{B} - \vec{B}\cdot\nabla_j\vec{h}^{ij}\right],$$

which can be shown by applying the chain rule, we have

$$((\nabla^2_\perp)_{ij}\vec{B})\cdot\vec{h}^{ij} = ((\nabla^2_\perp)_{ij}\vec{h}^{ij})\cdot\vec{B} + \nabla_i\left[\vec{h}^{ij}\cdot\nabla_j\vec{B} - \vec{B}\cdot\nabla_j\vec{h}^{ij}\right]. \tag{3.12}$$

3.2.4 Preliminary: Collapsing $2A^s\vec{h}^{ij}\cdot(\nabla_i\vec{h}_{js}) + |\vec{h}|^2\nabla_i A^i$

Let $\vec{C_{ijs}} := \nabla_i\vec{h}_{js} - \nabla_s\vec{h}_{ij}$. Then

$$2A^s\vec{h}^{ij}\cdot(\nabla_i\vec{h}_{js}) = 2A^s\vec{h}^{ij}\cdot(\nabla_s\vec{h}_{ij}) + 2A^s\vec{h}^{ij}\cdot\vec{C_{ijs}}. \tag{3.13}$$

We then see that

$$\begin{aligned}
2A^s\vec{h}^{ij}\cdot(\nabla_s\vec{h}_{ij}) + |\vec{h}|^2\nabla_i A^i &= A^s\nabla_s(\vec{h}^{ij}\cdot\vec{h}_{ij}) + |\vec{h}|^2\nabla_i A^i \\
&= A^s\nabla_s|\vec{h}|^2 + |\vec{h}|^2\nabla_i A^i \\
&= \nabla_i(A^i|\vec{h}|^2).
\end{aligned} \tag{3.14}$$

using the product rule. By combining (3.13) and (3.14), we obtain

$$2A^s\vec{h}^{ij}\cdot(\nabla_i\vec{h}_{js}) + |\vec{h}|^2\nabla_s A^s = \nabla_i(A^i|\vec{h}|^2) + 2A^s\vec{h}^{ij}\cdot\vec{C_{ijs}}. \tag{3.15}$$

3.2.5 Factoring in $|g|^{1/2}$

First, we have the variation

$$\delta|g|^{1/2} = |g|^{1/2}[\nabla_i A^i - 2\vec{B}\cdot\vec{H}]. \tag{3.16}$$

as provided in [1]. Now, we can combine the variations (3.10) and (3.16) with the identities (3.12) and (3.15), yielding

$$\begin{aligned}
\delta(|\vec{h}|^2|g|^{1/2}) &= |g|^{1/2}\delta|\vec{h}|^2 + |\vec{h}|^2\delta|g|^{1/2} \\
&= |g|^{1/2}\left[2((\nabla_\perp^2)_{ij}\vec{B})\cdot\vec{h}^{ij} + 2A^s\vec{h}^{ij}\cdot(\nabla_i\vec{h}_{js}) + |\vec{h}|^2\nabla_i A^i - 2|\vec{h}|^2(\vec{B}\cdot\vec{H})\right] \\
&= |g|^{1/2}\left[2((\nabla_\perp^2)_{ij}\vec{h}^{ij})\cdot\vec{B} + \nabla_i\left[2\vec{h}^{ij}\cdot\nabla_j\vec{B} - 2\vec{B}\cdot\nabla_j\vec{h}^{ij}\right] + \nabla_i(A^i|\vec{h}|^2) - 2|\vec{h}|^2(\vec{B}\cdot\vec{H}) + 2A^s\vec{h}^{ij}\cdot\vec{C_{ijs}}\right] \\
&= |g|^{1/2}\left[\vec{B}\cdot\left(2(\nabla_\perp^2)_{ij}\vec{h}^{ij} - 2|\vec{h}|^2\vec{H}\right) + \nabla_i\left(2\vec{h}^{ij}\cdot\nabla_j\vec{B} - 2\vec{B}\cdot\nabla_j\vec{h}^{ij} + A^i|\vec{h}|^2\right) + 2A^s\vec{h}^{ij}\cdot\vec{C_{ijs}}\right].
\end{aligned}$$

Let $\vec{\mathcal{Q}} := 2(\nabla_\perp^2)_{ij}\vec{h}^{ij} - 2|\vec{h}|^2\vec{H}$. Then we have

$$\delta\int|\vec{h}|^2 = \int\vec{B}\cdot\vec{\mathcal{Q}} + \nabla_i\left(2\vec{h}^{ij}\cdot\nabla_j\vec{B} - 2\vec{B}\cdot\nabla_j\vec{h}^{ij} + A^i|\vec{h}|^2\right) + 2A^s\vec{h}^{ij}\cdot\vec{C_{ijs}}. \tag{3.17}$$

3.3 Combining subsections 3.1 and 3.2 to vary $\int K$

We have now obtained the variations $\delta(|\vec{H}|^2|g|^{1/2})$ and $\delta(|\vec{h}|^2|g|^{1/2})$ in a suitable form. Recall equation 3.1, which stated

$$\delta(K|g|^{1/2}) = 2\delta(|\vec{H}|^2|g|^{1/2}) - \frac{1}{2}\delta(|\vec{h}|^2|g|^{1/2}).$$

We will use (3.2) and (3.17) to vary $\int K$. Firstly, let

$$\begin{aligned}
\vec{\mathcal{U}} &:= 2\vec{\mathcal{W}} - \frac{1}{2}\vec{\mathcal{Q}} \\
&= 2\Delta_\perp\vec{H} + 2(\vec{H}\cdot\vec{h}^i_j)\vec{h}^j_i - 4|\vec{H}|^2\vec{H} - (\nabla_\perp^2)_{ij}\vec{h}^{ij} + |\vec{h}|^2\vec{H} \\
&= 2(\vec{H}\cdot\vec{h}^i_j)\vec{h}^j_i + 2\Delta_\perp\vec{H} - (\nabla_\perp^2)_{ij}\vec{h}^{ij} - 2K\vec{H}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{V}^i &:= 2\left[\vec{H}\cdot\nabla^i\vec{B} - \vec{B}\cdot\nabla^i\vec{H} + A^i|\vec{H}|^2\right] - \frac{1}{2}\left[2\vec{h}^{ij}\cdot\nabla_j\vec{B} - 2\vec{B}\cdot\nabla_j\vec{h}^{ij} + A^i|\vec{h}|^2\right] \\
&= 2\vec{H}\cdot\nabla^i\vec{B} - \vec{h}^{ij}\cdot\nabla_j\vec{B} - \vec{B}\cdot\nabla^i 2\vec{H} + \vec{B}\cdot\nabla_j\vec{h}^{ij} + A^i K.
\end{aligned} \tag{3.18}$$

Then, using these definitions and our previous variations, we obtain

$$\begin{aligned}
\delta\int K &= \delta\int 2|\vec{H}|^2 - \frac{1}{2}|\vec{h}|^2 \\
&= \int\vec{B}\cdot\vec{\mathcal{U}} + \nabla_i\mathcal{V}^i + A^s\left[2\vec{H}\cdot\vec{C_s} - \vec{h}^{ij}\cdot\vec{C_{ijs}}\right]
\end{aligned} \tag{3.19}$$

as our variation of $\int K$.

4 Obtaining the Gauss and contracted Codazzi equations

4.1 Simplifying the variation using the Gauss-Bonnet theorem

Suppose our transformation satisfies $A^s = 0$ for $s = 1, 2$ everywhere on the surface, so that we are only varying the shape in the normal direction. Then the variation in (3.19) becomes

$$\delta \int_\Sigma K = \int_\Sigma \vec{B} \cdot \vec{\mathcal{U}} + \nabla_i \mathcal{V}^i,$$

where $\vec{\mathcal{U}}$ is unchanged, and

$$\mathcal{V}^i = 2\vec{H}\nabla^i \vec{B} - \vec{h}^{ij} \cdot \nabla_j \vec{B} - \vec{B} \cdot \nabla^i 2\vec{H} + \vec{B} \cdot \nabla_j \vec{h}^{ij}.$$

Recall from (2.1) that, by the Gauss-Bonnet theorem, we have $\delta \int_\Sigma K = 0$. As $\int_\Sigma \nabla_i \mathcal{V}^i$ is the integral of a divergence over the closed surface Σ, it is equal to 0. Therefore

$$\int_\Sigma \vec{B} \cdot \vec{\mathcal{U}} = 0$$

for all $\vec{B}$. Indeed, this is true for $\vec{B} = \vec{\mathcal{U}}$; and as $\int_\Sigma \vec{\mathcal{U}} \cdot \vec{\mathcal{U}} = 0$, we conclude that $\vec{\mathcal{U}} = \vec{0}$ everywhere.

Now, consider again an arbitrary transformation. As $\vec{\mathcal{U}}$ is identically zero, equation 3.19 becomes

$$0 = \delta \int_\Sigma K = \int_\Sigma \nabla_i \vec{\mathcal{V}}^i + A^s \left[2\vec{H} \cdot \vec{C_s} - \vec{h}^{ij} \cdot \vec{C_{ijs}}\right],$$

and as $\nabla_i \vec{\mathcal{V}}^i$ is a divergence integral over a closed surface, and hence 0, we have

$$\int_\Sigma A^s \left[2\vec{H} \cdot \vec{C_s} - \vec{h}^{ij} \cdot \vec{C_{ijs}}\right] = 0$$

for all A^s. For fixed $k \in \{1, 2\}$, we can set $A^s = \delta^s_k [2\vec{H} \cdot C_k - \vec{h}^{ij} \cdot C_{ijk}]$, where δ^s_k is the Kronecker delta. Then,

$$0 = \int_\Sigma A^s \left[2\vec{H} \cdot \vec{C_s} - \vec{h}^{ij} \cdot \vec{C_{ijs}}\right] = \int_\Sigma \left[2\vec{H} \cdot C_k - \vec{h}^{ij} \cdot C_{ijk}\right]^2$$

so that $2\vec{H} \cdot C_k - \vec{h}^{ij} \cdot C_{ijk} = 0$ at every point on Σ. We conclude that

$$0 = \int_\Sigma \delta K = \int_\Sigma \nabla_i \mathcal{V}^i \tag{4.1}$$

4.2 Considering a smooth deformation $\vec{F}$

Let $\vec{F} : \Sigma \to \mathbb{R}^n$ be a smooth vector field over Σ. Consider a deformation of the form $\vec{\Phi}_t := \vec{\Phi} + t\vec{F}$. Then, as A^i and $\vec{B}$ are the tangential and normal parts of the deformation respectively, we have $A^i := \vec{F} \cdot \nabla^i \vec{\Phi}$ and $\vec{B} = \pi_N \vec{F}$. Recall the definition of $\mathcal{V}^i$ in (3.18). We then calculate

$$\begin{aligned}\mathcal{V}^i &= 2\vec{H} \cdot \nabla^i \vec{B} - \vec{h}^{ij} \cdot \nabla_j \vec{B} - \vec{B} \cdot \nabla^i 2\vec{H} + \vec{B} \cdot \nabla_j \vec{h}^{ij} + A^i K \\ &= 2\vec{H} \cdot \nabla^i (\pi_N \vec{F}) - \vec{h}^{ij} \cdot \nabla_j (\pi_N \vec{F}) - (\pi_N \vec{F}) \cdot \nabla^i 2\vec{H} + (\pi_N \vec{F}) \cdot \nabla_j \vec{h}^{ij} + (\vec{F} \cdot \nabla^i \vec{\Phi}) K \end{aligned} \tag{4.2}$$

We can expand and contract these first two terms, obtaining

$$
\begin{aligned}
2\vec{H}\cdot\nabla^i(\pi_N\vec{F}) - \vec{h}^{ij}\cdot\nabla_j(\pi_N\vec{F}) &= \nabla^i(2\vec{H}\cdot\pi_N\vec{F}) - 2(\pi_N\vec{F})\cdot\nabla^i\vec{H} - \nabla_j(\vec{h}^{ij}\cdot\pi_N\vec{F}) + (\pi_N\vec{F})\cdot\nabla_j\vec{h}^{ij} \\
&= \nabla^i(2\vec{H}\cdot\vec{F}) - 2\vec{F}\cdot\pi_N\nabla^i\vec{H} - \nabla_j(\vec{h}^{ij}\cdot\vec{F}) + \vec{F}\cdot\pi_N\nabla_j\vec{h}^{ij} \\
&= 2\vec{H}\cdot(\nabla^i\vec{F}) + 2\vec{F}\cdot\nabla^i\vec{H} - 2\vec{F}\cdot\pi_N\nabla^i\vec{H} - \vec{h}^{ij}\cdot\nabla_j\vec{F} - \vec{F}\cdot\nabla_j\vec{h}^{ij} + \vec{F}\cdot\pi_N\nabla_j\vec{h}^{ij} \\
&= 2\vec{H}\cdot(\nabla^i\vec{F}) + 2\vec{F}\cdot\pi_T\nabla^i\vec{H} - \vec{h}^{ij}\cdot\nabla_j\vec{F} - \vec{F}\cdot\pi_T\nabla_j\vec{h}^{ij}. \qquad (4.3)
\end{aligned}
$$

Then, substituting (4.3) into (4.2) yields

$$
\begin{aligned}
\mathcal{V}^i &= \vec{F}\cdot\left[2\pi_T\nabla^i\vec{H} - \pi_T\nabla_j\vec{h}^{ij} - 2\pi_N\nabla^i\vec{H} + \pi_N\nabla_j\vec{h}^{ij} + K\nabla^i\vec{\Phi}\right] + 2(\nabla^i\vec{F})\cdot\vec{H} - (\nabla_j\vec{F})\cdot\vec{h}^{ij} \\
&= \vec{F}\cdot(\vec{T}^i + \vec{N}^i) + 2(\nabla^i\vec{F})\cdot\vec{H} - (\nabla_j\vec{F})\cdot\vec{h}^{ij}, \qquad (4.4)
\end{aligned}
$$

where

$$\vec{T}^i := 2\pi_T\nabla^i\vec{H} - \pi_T\nabla_j\vec{h}^{ij} + K\nabla^i\vec{\Phi}$$

is tangent to Σ, and

$$\vec{N}^i := \pi_N\nabla_j\vec{h}^{ij} - 2\pi_N\nabla^i\vec{H}$$

is normal to Σ. Using (4.4) to substitute for $\mathcal{V}^i$, we find that

$$\nabla_i\mathcal{V}^i = \vec{F}\cdot\nabla_i(\vec{T}^i + \vec{N}^i) + (\nabla_i\vec{F})\cdot(\vec{T}^i + \vec{N}^i) + \nabla_i\left[2(\nabla^i\vec{F})\cdot\vec{H} - (\nabla_j\vec{F})\cdot\vec{h}^{ij}\right]$$

for all $\vec{F}$. Substituting this into 4.1 yields

$$0 = \int_\Sigma \nabla_i\mathcal{V}^i = \int_\Sigma \vec{F}\cdot\nabla_i(\vec{T}^i + \vec{N}^i) + (\nabla_i\vec{F})\cdot(\vec{T}^i + \vec{N}^i) + \nabla_i\left[2(\nabla^i\vec{F})\cdot\vec{H} - (\nabla_j\vec{F})\cdot\vec{h}^{ij}\right]. \qquad (4.5)$$

Suppose $\vec{F}$ is a constant, so that $\vec{F} = \vec{a}$ with $\vec{a} \in \vec{\mathbb{R}}$. This amounts to a translation of the surface. Indeed, we can apply this to any patch $\Sigma_0 \subset \Sigma$ of the surface also, as the translation should not change the total Gaussian curvature over that patch. That is, for all constant $\vec{a} \in \mathbb{R}^n$ and patches $\Sigma_0 \subset \Sigma$, we have

$$0 = \int_{\Sigma_0} \delta K = \int_{\Sigma_0} \vec{a}\cdot\nabla_i(\vec{T}^i + \vec{N}^i) \qquad (4.6)$$

This shows that $\nabla_i(\vec{T}^i + \vec{N}^i)$ is identically zero by the following argument: suppose for a contradiction that $\nabla_i(\vec{T}^i + \vec{N}^i)$ is not identically zero. As $\nabla_i(\vec{T}^i + \vec{N}^i)$ is continuous, there must be at least a small patch $\Sigma_0 \subset \Sigma$ and a constant $\vec{a} \in \mathbb{R}^n$ for which $\vec{a}\cdot\nabla_i(\vec{T}^i + \vec{N}^i)$ is positive-valued everywhere on Σ_0. Either way, $\int_{\Sigma_0} \vec{a}\cdot\nabla_i(\vec{T}^i + \vec{N}^i) \neq 0$, contradicting (4.6).

Returning our attention to (4.5), and again letting $\vec{F}$ be any smooth function, we see that the term $\vec{F}\cdot\nabla_i(\vec{T}^i + \vec{N}^i)$ now vanishes. Additionally, as $\nabla_i\left[2(\nabla^i\vec{F})\cdot\vec{H} - (\nabla_j\vec{F})\cdot\vec{h}^{ij}\right]$ is a divergence being integrated over a surface with no boundary, (4.5) becomes

$$0 = \int_\Sigma (\nabla_i\vec{F})\cdot(\vec{T}^i + \vec{N}^i).$$

As this holds for all $\vec{F}$, it must be the case that $\vec{T}^i + \vec{N}^i = \vec{0}$.

4.3 Final steps to extract the Gauss and Codazzi equations

Recall that $\vec{T}^i$ is tangential to the surface and $\vec{N}^i$ is normal. Then, as $\vec{T}^i + \vec{N}^i = \vec{0}$, it must be the case that $\vec{T}^i = \vec{0}$ and $\vec{N}^i = \vec{0}$. From $\vec{T}^i = \vec{0}$ we obtain

$$\begin{aligned}\vec{T}^i &= 2\pi_T \nabla^i \vec{H} - \pi_T \nabla_j \vec{h}^{ij} + K \nabla^i \vec{\Phi} \\ &= -2(\vec{H} \cdot \vec{h}^{is}) \nabla_s \vec{\Phi} + (\vec{h}^{ij} \cdot \vec{h}^s_j) \nabla_s \vec{\Phi} + K g^{is} \nabla_s \vec{\Phi};\end{aligned}$$

therefore, for each $i, s \in \{1, 2\}$,

$$-2\vec{H} \cdot \vec{h}^{is} + \vec{h}^{ij} \cdot \vec{h}^s_j + K g^{is} = 0. \tag{4.7}$$

This is the Gauss equation. Then $\vec{N}^i = \vec{0}$ immediately yields

$$\pi_N \nabla_j \vec{h}^{ij} - 2\pi_N \nabla^i \vec{H} = \vec{0},$$

which is the contracted Codazzi-Mainardi equation.

5 Generalisation to higher dimensions

We have shown that, by varying Gaussian curvature, we can obtain the Gauss equation and the contracted Codazzi-Mainardi equation in dimension 2 and any codimension. Indeed, the Gauss equation is only true in dimension 2; as it incorporates Gaussian curvature, it does not make sense in any other dimension. However, the contracted Codazzi-Mainardi equation in dimension 2 is a special case of the far more general Codazzi equation.

5.1 General contracted Codazzi-Mainardi equation

Let $\vec{\Phi} : \Sigma^k \to \mathbb{R}^n$ be an immersion of any smooth k-dimensional manifold Σ^k in $\mathbb{R}^n$. As before, for any $1 \leq i, j \leq k$, define the second fundamental form $\vec{h}_{ij} := \nabla_i \nabla_j \vec{\Phi}$ and mean curvature $\vec{H} := \frac{1}{k} g^{ij} \vec{h}_{ij}$. We will also define the antisymmetric tensor $\vec{\eta}_{ij} := \nabla_i \vec{\Phi} \wedge \nabla_j \vec{\Phi}$, where $\wedge$ denotes the wedge product. Finally, we introduce the Riemann curvature tensor $R^a{}_{ecd}$ satisfying

$$(\nabla_c \nabla_d - \nabla_d \nabla_c) \tau^{ab} = -R^a{}_{ecd} \tau^{eb} - R^b{}_{ecd} \tau^{ae} \tag{5.1}$$

for rank 2 tensors τ^{ab}. We see that

$$\begin{aligned}2\nabla_i \nabla_j \vec{\eta}^{ij} &= (\nabla_i \nabla_j - \nabla_j \nabla_i) \vec{\eta}^{ij} \\ &= -R^i{}_{sij} \vec{\eta}^{sj} - R^j{}_{sij} \vec{\eta}^{is} \\ &= R^i{}_{sji} \vec{\eta}^{sj} - R^i{}_{jsi} \vec{\eta}^{sj} \\ &= \vec{0},\end{aligned}$$

so that $\nabla_i \nabla_j \vec{\eta}^{ij} = \vec{0}$. We then see that

$$\nabla_j \vec{\eta}^{ij} = \nabla_j (\nabla^i \vec{\Phi} \wedge \nabla^j \vec{\Phi}) = \vec{h}^i_j \wedge \nabla^j \vec{\Phi} - k\vec{H} \wedge \nabla^i \vec{\Phi} = (\vec{h}^i_j - k\vec{H} g^i_j) \wedge \nabla^j \vec{\Phi},$$

and furthermore,

$$\begin{aligned}\vec{0} &= \nabla_i \nabla_j \vec{\eta}^{ij} \\ &= \nabla_i((\vec{h}^i_j - k\vec{H}g^i_j) \wedge \nabla^j \vec{\Phi}) \\ &= \pi_T \nabla_i(\vec{h}^i_j - k\vec{H}g^i_j) \wedge \nabla^j \vec{\Phi} + \pi_N \nabla_i(\vec{h}^i_j - k\vec{H}g^i_j) \wedge \nabla^j \vec{\Phi} + (\vec{h}^i_j - k\vec{H}g^i_j) \wedge \vec{h}^j_i. \end{aligned} \tag{5.2}$$

To simplify, note that

$$(\vec{h}^i_j - k\vec{H}g^i_j) \wedge \vec{h}^j_i = \vec{h}^i_j \wedge \vec{h}^j_i - k\vec{H} \wedge k\vec{H} = \vec{0}$$

and that

$$\begin{aligned}\pi_T \nabla_i(\vec{h}^i_j - k\vec{H}g^i_j) \wedge \nabla^j \vec{\Phi} &= -((\vec{h}^i_j - k\vec{H}g^i_j) \cdot \vec{h}_{is})\nabla^s \vec{\Phi} \wedge \nabla^j \vec{\Phi} \\ &= (\vec{h}^i_j \cdot \vec{h}_{is} - k\vec{H} \cdot \vec{h}_{js})(\nabla^j \vec{\Phi} \wedge \nabla^s \vec{\Phi}) \\ &= \vec{0},\end{aligned}$$

as $\vec{h}^i_j \cdot \vec{h}_{is} - k\vec{H} \cdot \vec{h}_{js}$ is symmetric in j and s, whereas $\nabla^j \vec{\Phi} \wedge \nabla^s \vec{\Phi}$ is antisymmetric in j and s. Therefore, from (5.2) we deduce

$$\pi_N \nabla_i(\vec{h}^i_j - k\vec{H}g^i_j) \wedge \nabla^j \vec{\Phi} = \vec{0},$$

and therefore, as $\nabla^1 \vec{\Phi}, \dots, \nabla^k \vec{\Phi}$ are all linearly independent, we have

$$\pi_N \nabla_i(\vec{h}^i_j - k\vec{H}g^i_j) = \vec{0}$$

for all $1 \leq j \leq k$. This gives the contracted Codazzi-Mainardi equation in dimension k.

5.2 Generalisation of Gauss equation

As stated previously, the Gauss equation does not make sense in higher dimensions, as Gaussian curvature is defined for 2-dimensional surfaces only.

One way we might try to generalise the Gauss equation is by replacing K with the k-dimensional scalar curvature R, given by $R = g_{ij}R^{ij} = k^2|\vec{H}|^2 - |\vec{h}|^2$, where $R^{ij} = R^{isj}{}_s$ is the Ricci tensor. When $k = 2$, we have $R = 2K$, and can write

$$-k\vec{H} \cdot \vec{h}^{ij} + \vec{h}^{is} \cdot \vec{h}^j_s + \frac{1}{2}Rg^{ij} = 0 \tag{5.3}$$

for all $1 \leq i, j \leq k$, by reformulating (4.7).

It is natural to wonder if this holds for $k > 2$. However, this turns out not to be the case. For a general k-dimensional manifold Σ^k immersed in $\mathbb{R}^n$, the left-hand side of (5.3) is equal to the Einstein tensor E^{ij}, which happens to be identically zero in dimension 2 only.

It is, however, true that the Einstein tensor is divergence free. One way to see this is to carry out the variation of scalar curvature $\delta \int R$, as done in section 3. Note that the Gauss-Bonnet theorem does not generalise to say $\delta \int R = 0$ in dimensions higher than 2, so we cannot use an arbitrary deformation $\vec{F}$ and expect to get zero. Instead, we can perform a translation, which does preserve scalar curvature. By a line of reasoning similar to that in section 4, and by using the contracted Codazzi-Mainardi equation when needed, we can obtain the identity $\nabla_i E^{ij} = 0$.

6 Conclusion

In this report we have performed an extrinsic variation of Gaussian curvature, which is a topological invariant. This allowed us to deduce corollaries of the Gauss-Bonnet theorem, which include the two fundamental equations of submanifold geometry: the Gauss equation and the Codazzi-Mainardi equation. Although these equations are well-known, this establishes a new link between the Gauss-Bonnet theorem and the Gauss-Codazzi equations, and demonstrates a useful technique for deriving identities in differential geometry. We then investigated the Gauss-Codazzi equations in higher dimension, and looked briefly at the Einstein tensor. It is proposed that following the methodology in this report can show a similar link between the invariance of scalar curvature under translation, and the fact that the Einstein tensor is divergence-free.

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