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Evolution of Curves with Boundary

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Abstract

Longtime behaviour of embedded closed curves in $\mathbb{R}^2$ under curve shortening flow has been extensively studied, and the results encompassed by the Gage-Hamilton-Grayson theorem. In 1998, Huisken provided a chord-arc estimate which provides an alternative proof to the preservation of embeddedness under curve shortening flow. In 2011, Andrews and Bryan improved upon this estimate to arrive at sharp bounds on curvature which prove the Gage-Hamilton-Grayson theorem in its entirety. In 2020, Langford and Zhu adapted these ideas to understand longtime existence of curve shortening flow with Neumann-Neumann boundary conditions. We provide an overview of these techniques, and discuss their potential usage in a wider class of problem.

1 Introduction

Joseph Fourier first developed the heat equation to describe heat diffusion in a given region [3]. Since then, the heat equation has become the prototypical parabolic partial differential equation (PDE). It appears not only in physical evolution systems moving towards an equilibrium, but also more abstract settings such as options pricing via the Feynman-Kac connection. This paper explores the natural geometric extension of 1-dimensional heat flow, the curve shortening flow.

While the heat equation and various different boundary conditions and settings have been vastly studied, the same cannot be said for curve shortening flow. Foundational results have been known for the case of embedded closed curves since the 1980's due to Michael Gage, Richard Hamilton and Matthew Grayson [2]. Major improvements in the proof methodology were made by Gerhard Huisken in 1998 [4] and Ben Andrews and Paul Bryan in 2009 [1]. Since the advent of these new methods, new problems in curve shortening flow with boundary condition have been made accessible. This paper outlines these techniques, formulates new problems, and introduce potential methods to tackle them.

2 Statement of Authorship

All prose in this paper is written by myself, Lekh Bhatia. All theorems, lemmas, and ideas are credited to [2] and [5], with the exception of content related to Dirichlet-Neumann boundary conditions, which is a result of work done by myself and Mat Langford over the course of this program. I would also like to thank Mat Langford and Ben Andrews once again for guiding me throughout this project.

3 Curve Shortening Flow

In the 1-dimensional case, a function $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ is **caloric** or satisfies the heat equation if

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

which is exactly the gradient flow of the Dirichlet energy functional

$$E(u) := \frac{1}{2}\int_{\mathbb{R}} \left|\frac{\partial^2 u}{\partial x^2}\right|^2 \mathrm{d}\,x,$$

which is to say that it is the quickest way to vary u to reduce $E(u)$ (up to a scalar multiple). Caloric functions also exhibit a maximum principle, and there is instantaneous smoothing behaviour on all derivatives. These are intuitive concepts which arise from our physical understanding of heat, and they present themselves naturally in the mathematics.

Curve shortening flow is the natural geometric version of the heat flow. Let $X : M^1 \times [0, T) \to \mathbb{R}^2$ be a smooth map which is immersed for each t, that is that $|X'(\cdot, t)| \neq 0$ for each t, where $M^1 \cong S^1$. We can call this is a smooth 1-parameter family of immersions. X is a solution to curve shortening flow if

$$\frac{\partial X}{\partial t} = -\kappa\,\mathrm{N}$$

where N is the unit outward pointing normal vector and κ is the curvature with respect to this choice of normal vector (as per the Frenet-Serret equations for a curve in a plane). The neat observation is that this flow can be identically written as

$$\frac{\partial X}{\partial t} = \frac{\mathrm{d}^2 X}{\mathrm{d}s^2}$$

where $\frac{\mathrm{d}}{\mathrm{d}s}$ is the derivative with respect to the arc-length parameter. From this point, we use a prime symbol to denote a derivative with respect to an arbitrary parametrisation, and specify s for arc-length derivatives. The relationship between these derivatives is given by $\frac{\mathrm{d}}{\mathrm{d}s} = \frac{1}{|X'|}\frac{\mathrm{d}}{\mathrm{d}u}$. In this formulation, it becomes clear that this geometric PDE is a heat-type equation. Unfortunately, since the arc-length derivative changes with time, the PDE is no longer linear. However, we can still derive a number of useful results.

1. **Gradient flow of the length functional:** The curve shortening flow is the quickest way to minimise the length of the curve, given by

$$L := \int_{M^1} \mathrm{d}\,s$$

2. **Heat-type behaviour:** We can prove that several derived quantities from the flow demonstrate heat-type equations such as

$$\kappa_t = \kappa_{ss} + \kappa^3$$

3. **Smoothing:** Curve shortening flow instantly smooths all the derivatives with any C^2 curve as the initial condition. For all $m \geq 1$, there exist some $C_m < \infty$ such that if $|\kappa| \leq K$ on $M^1 \times [0, t_0)$, where $t_0 \in (0, K^{-2}]$ and $K < \infty$, then

$$\left|\kappa^{(m)}\right| \leq \frac{C_m K}{t^{m/2}}$$

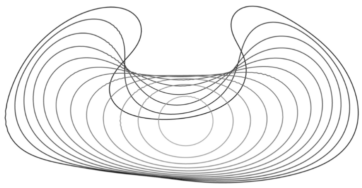

Figure 1: Example curve shortening flow, demonstrating convergence to a unit circle.

4. **Area Decrease:** It can be shown that

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -\int_{M^1} \kappa \,\mathrm{d}\, s = -2\pi$$

where A is the area bounded by each immersion, and the second equality arises from the Euler characteristic of shapes in $\mathbb{R}^2$.

Theorem 1 (Gage-Hamilton-Grayson Theorem): Let $X_0 : M^1 \to \mathbb{R}^2$ be a smooth embedding of a compact connected 1-manifold $M^1 \cong S^1$. Then the solution of the curve shortening flow with initial data X_0 exists on a maximal time interval $[0, T)$ and converges to a point $p \in \mathbb{R}^2$ as $t \to T$. The rescaled embeddings

$$\tilde{X}(\cdot, t) = \frac{X(\cdot, t) - p}{\sqrt{2(T - t)}}$$

converges in the C^∞ norm to a limit embedding $\tilde{X}_T$ with image equal to the unit circle.

In 1984, Gage first proved that if a curve is to converge to a point, then the scaled flow must converge to a unit circle. In 1986, Gage and Hamilton proved that all convex curves converge to a point. Finally, in 1987, Grayson proved that all non-convex curves eventually become convex, completing the theorem. This paper discusses an entirely different piece of technology which simplifies this proof, and leads to a easy applications to new problems.

4 The Chord-Arc Profile

Before we can introduce the primary focus of this paper, it is instructive to consider the proof of an intriguing result.

Theorem 2 (Avoidance Principle): Let $X_i : M_i^1 \times [0, T) \to \mathbb{R}^2$ be solution to curve shortening flow such that $X_1(M_1^1, 0) \cap X_2(M_2^1, 0) = \emptyset$, then $X_1(M_1^1, t) \cap X_2(M_2^1, t) = \emptyset$ for all $t \in [0, T)$.

Proof of Theorem 2:

We prove instead that the length of the shortest line segment between the two curves is non-decreasing in time, which establishes the theorem. Define $d : M_1^1 \times M_2^1 \times [0, T) \to \mathbb{R}$ to be

$$d(x, y, t) = |X_2(y, t) - X_1(x, t)|.$$

Since the curves we selected are initially compact, and non-intersecting, the initial minimal distance is positive.

$$d_0 := \inf\{d(x, y, 0) : (x, y) \in M_1^1 \times M_2^2\}$$

Effectively, we wish to prove that $d \geq d_0$ for all $t \in [0, T)$, which we will do via contradiction. The issue with directly attempting to prove this inequality is that the first time for which this might not be true may not be an interior point, and difficult to pin-point. Hence, we instead try and prove an equivalent statement, that $de^{\epsilon(1+t)} > d_0$ for every $\epsilon > 0$. We approach this problem using a second derivative test argument which is common in maximum principle proofs.

Assume that $de^{\epsilon(1+t)}$ does equal to d_0 for some first coordinates (x_0, y_0, t_0), and some fixed ϵ. We know

1. $\frac{\partial}{\partial t}(de^{\epsilon(1+t)}) \leq 0$ **2.** The spatial derivatives are 0 **3.** The spatial Hessian is non-negative

Hence, we can compute

$$\frac{\partial d}{\partial x} = -\langle \omega, \mathrm{T}_1 \rangle = 0 \qquad \frac{\partial d}{\partial y} = \langle \omega, \mathrm{T}_2 \rangle = 0$$

where ω is the unit vector pointing from $X_1(x_0, t_0)$ to $X_2(y_0, t_0)$, and $\mathrm{T}_1, \mathrm{T}_2$ are unit tangent vectors to the curves. Since these derivatives are equal, it implies that $T_1 = T_2$ (up to reversing parametrisation of one of these curves). Additionally, $\mathrm{N}_1 = \mathrm{N}_2 = \omega$ as well with this choice (assuming that N is chosen consistently to be 90° clockwise to T). Then, the second derivatives can be computed

$$\begin{aligned}
\frac{\partial^2 d}{\partial x^2} &= \frac{\langle \mathrm{T}_1 - \langle \omega, \mathrm{T}_1 \rangle \omega, \mathrm{T}_1 \rangle}{|X_1 - X_2|} - \langle \omega, \kappa_1 \mathrm{N}_1 \rangle \\
\frac{\partial^2 d}{\partial y^2} &= \frac{\langle \mathrm{T}_2 - \langle \omega, \mathrm{T}_2 \rangle \omega, \mathrm{T}_2 \rangle}{|X_1 - X_2|} + \langle \omega, \kappa_2 \mathrm{N}_2 \rangle \\
\frac{\partial^2 d}{\partial x\, \partial y} &= -\frac{\langle \mathrm{T}_1 - \langle \omega, \mathrm{T}_1 \rangle \omega, \mathrm{T}_2 \rangle}{|X_1 - X_2|}
\end{aligned}$$

where κ_1, κ_2 are the respective curvatures. Noting that we are are at (x_0, y_0, t_0), we can simplify and see that

$$\frac{\partial^2 d}{\partial x^2} = \frac{1}{d} - \kappa_1 \qquad \frac{\partial^2 d}{\partial x^2} = \frac{1}{d} + \kappa_2 \qquad \frac{\partial^2 d}{\partial x\, \partial y} = -\frac{1}{d}$$

We can also compute

$$\frac{\partial d}{\partial t} = \left\langle \frac{X_2 - X_1}{|X_2 - X_1|}, -\kappa_1 \mathrm{N}_1 + \kappa_2 \mathrm{N}_2 \right\rangle = \langle \omega, -\kappa_1 \mathrm{N}_1 + \kappa_2 \mathrm{N}_2 \rangle = \kappa_2 - \kappa_1$$

We see here that we have identified the following heat-type equation for d in this situation

$$\frac{\partial d}{\partial t} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)^2 d = \kappa_2 - \kappa_1$$

We would have hoped to immediately derive a contradiction by noting the non-positivity of the time derivative and the non-negativity of the Hessian. However, equality is in fact possible since these are not strict inequalities. Hence, as is standard in proofs of this type, we can analyse the modified quantity instead.

$$0 \geq e^{-\epsilon(1+t_0)} \frac{\partial}{\partial t} \left(de^{\epsilon(1+t)} \right) = \epsilon d + \kappa_2 - \kappa_1 > \kappa_2 - \kappa_1 = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)^2 d \geq 0$$

This adjustment allows us to coerce a strict inequality, and arrive at a contradiction. Hence, $de^{\epsilon(1+t)} > d_0$ for all $\epsilon > 0$, which implies that $d \geq d_0$, as required. □

Remark 3: We can derive another useful result from this avoidance behaviour. If $X_0 : M^1 \to \mathbb{R}^2$ is an immersion, then the curve shortening flow with X_0 as initial data has a finite maximal time of existence. We can conclude this result by noting that circles shrink and only exist for $\frac{r^2}{2}$ time due to the constant area decrease of -2π. Hence, any initial data can be bounded by a large enough circle, which shrinks, and due to the avoidance principle, also constricts the entire flow.

In addition to the avoidance principle, it is also true that embeddedness of a single curve shortening flow is preserved. We can use a similar 2-point maximum principle argument on the "chord-arc profile" to derive this result.

Theorem 4 (Huisken's Estimate): Let $X : M^1 \times [0,T) \to \mathbb{R}^2$ where $M^1 \cong S^1$ be a smooth solution to curve shortening flow, such that $X(\cdot,0)$ is an embedding. Define a function Z on $S \times [0,T) \to \mathbb{R}$ where $S = (M^1 \times M^1) \setminus \{(x,x) : x \in M^1\}$

$$Z(x,y,t) = \frac{L(t)}{d(x,y,t)} \sin\left(\frac{\pi l(x,y,t)}{L(t)}\right)$$

where $d(x,y,t) = |X(y,t) - X(x,t)|$, $l(x,y,t)$ is the standard arc length between x, y at time t, and $L(t)$ is the total length of the curve at time t. Then $\sup\limits_{x,y} Z$ is non-increasing in t, strictly unless the curve is a round circle.

Remark 5: This result proves that embeddedness is preserved. For any embedded initial data, the value of Z is some finite quantity which is non-increasing. Hence, it is impossible for $d(x,y,t) \to 0$ while $l(x,y,t) \not\to 0$.

We call the bound on Z here a **chord-arc estimate** since Z is roughly describing the ratio between chord length and arc length. The occurrence of the sine function here is not by happenstance. It is a convenient function to use since it remains invariant for any choice of x, y on a circle, and all curves are eventually approaching a circle in this type of flow. Looking for invariants on special self-similar cases like circles gives rise to ratios that might be nicer to analyse. We note without proof the following lemmata.

Lemma 6: If $X(\cdot,t)$ is embedded, then Z is smooth and extends to a continuous function on $M^1 \times M^1 \times [0,T)$ with $Z(x,x,t) = \pi$.

Lemma 7: If $X(M^1,t)$ is not a circle, then $\sup\limits_{x,y} Z > \pi$.

These lemmata can be proved by considering the Taylor expansion of the sine function and the geometric Taylor expansion of the immersion respectively. They show that Z only remains constant if the curve is circular.

Proof sketch of Theorem 4:

It only remains to prove that Z is strictly decreasing if the curve is not circular. We effectively argue via contradiction by assuming we have a new maximum and attempt to use a similar strategy as in the proof of the avoidance principle. Suppose there exist times $t_1 > t_0$ such that

$$\sup\{Z(x,y,t_1) : x,y \in M^1\} = \sup\{Z(x,y,t) : x,y \in M^1, t \in [t_0,t_1]\}$$

Then, there also exist x_0, y_0 such that $Z(x_0,y_0,t_1) = \sup\{Z(x,y,t) : x,y \in M^1, t \in [t_0,t_1]\}$. By Lemmata 6,7, we can note that $x_0 \neq y_0$. As before, we see that $\frac{\partial Z}{\partial t} \geq 0$ and the Hessian is non-positive, which lays the ground for a contradiction.

We first compute the time derivative, leaving most of the derivatives here with a factor of d on the left since they appear repeatedly.

$$d\frac{\partial Z}{\partial t} = Z\langle \omega, \kappa_y \mathrm{N}_y - \kappa_x \mathrm{N}_x \rangle - \left(\sin\left(\frac{\pi l}{L}\right) - \frac{\pi l}{L}\cos\left(\frac{\pi l}{L}\right)\right)\int_{M^1} \kappa^2 \, \mathrm{d}\, s - \pi\cos\left(\frac{\pi l}{L}\right)\int_x^y \kappa^2 \, \mathrm{d}\, s$$

where each term is non-positive except the first by noting that $0 \leq l \leq L/2$ without loss of generality. Then, we can compute the required derivatives at this extremum point (being at the extremum allows for simplifications).

$$d\frac{\partial Z}{\partial x} = Z\langle \omega, \mathrm{T}_x \rangle - \pi\cos\left(\frac{\pi l}{L}\right)$$
$$d\frac{\partial Z}{\partial y} = -Z\langle \omega, \mathrm{T}_y \rangle + \pi\cos\left(\frac{\pi l}{L}\right)$$

Then, can compute the second derivatives.

$$d\frac{\partial^2 Z}{\partial x^2} = -Z\langle w, \kappa_x \mathrm{N}_x \rangle - \frac{Z}{d}\left(1 - \langle w, \mathrm{T}_x\rangle^2\right) - \frac{\pi^2}{L}\sin\left(\frac{\pi l}{L}\right),$$
$$d\frac{\partial^2 Z}{\partial y^2} = Z\langle w, \kappa_y \mathrm{N}_y \rangle - \frac{Z}{d}\left(1 - \langle w, \mathrm{T}_y\rangle^2\right) - \frac{\pi^2}{L}\sin\left(\frac{\pi l}{L}\right),$$
$$d\frac{\partial x}{\partial y} = \frac{Z}{d}\left(\langle \mathrm{T}_y, \mathrm{T}_x\rangle - \langle w, \mathrm{T}_y\rangle\langle w, \mathrm{T}_x\rangle\right) + \frac{\pi^2}{L}\sin\left(\frac{\pi l}{L}\right).$$

Since both of the first derivatives vanish, we see that either $\mathrm{T}_x, \mathrm{T}_y$ are parallel or they are bisected by the direction ω. By a simple geometric argument we can conclude that $\mathrm{T}_x, \mathrm{T}_y$ cannot be parallel. If they were, then the curve would have to intersect the line between x and y at some other point, z. Then, picking x and z or y and z would lead to a larger value of Z, contradicting the fact that $Z(x,y,t_1)$ is a maximum. This is demonstrated in the below diagram.

Hence, we only need consider the case where $\mathrm{T}_x, \mathrm{T}_y$ are bisected by ω. We can then do some simple computations and substitutions to obtain the following heat-type inequality.

$$d\left(\frac{\partial Z}{\partial t} - \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 Z\right) < -\frac{4\pi}{l}\cos\left(\frac{\pi l}{L}\right)\left(\theta^2 - \left(\frac{\pi l}{L}\right)^2\right)$$

Note that the left hand side is strictly positive. However, a simple estimation shows that $\theta \geq \frac{\pi l}{L}$, which proves that the right hand side is non-positive. This provides the required contradiction. □

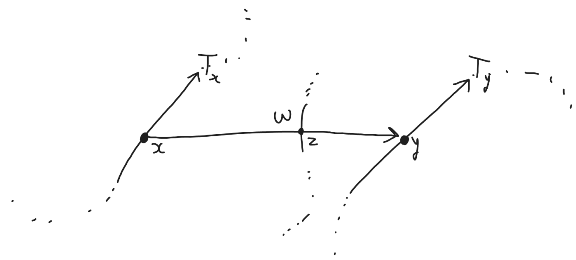

Figure 2: Demonstration of contradiction when $\mathrm{T}_x, \mathrm{T}_y$ are parallel.

Huisken's estimate lead to a shift in the types of proofs we see when studying curve shortening flow. By intelligently analysing invariant quantities, we can deduce useful bounds for curvature and embeddedness, rather than attempting to prove theorems in an ad-hoc fashion. However, Huisken's choice of Z is somewhat inefficient in that it does not provide the best possible bound.

With Huisken's estimate, we see that we have an upper bound on Z which gives us an inequality of the type $cL\sin\left(\frac{\pi l}{L}\right) \leq d$ where $c \in (0,1)$. This gives us that $cl \leq d$ as $l \to 0$. This is enough for to deduce the preservation of embeddedness, but not enough to deduce a bound on curvature. For large curvature κ, we see that $d \simeq \frac{2}{\kappa}\sin\left(\frac{\kappa l}{2}\right) \simeq l - \frac{\kappa^2 l^3}{24}$. Hence, we wish to prove a bound of the type $d \geq \phi(l)$ where $\phi(l) \geq l - Cl^3 + o(l^3)$ as $l \to 0$. This would prove a curvature bound of $\kappa^2 \leq 24C$, which would be enough to directly prove Grayson's theorem.

With this goal in mind we consider analysing the following chord-arc ratio instead.

$$Z(x,y,t) = d(x,y,t) - L(t)\phi\left(\frac{l(x,y,t)}{L(t)}, t\right)$$

for some smooth function ϕ which remains to be selected. The strategy will be to find conditions on ϕ which will allow us to deduce that Z remains positive. As long as $|\phi'| \leq 1$, this will allow us to deduce the final curvature bound. It turns out that this strategy works for the following function for some $a > 0$

$$\phi\left(\frac{l}{L}, t\right) = \frac{1}{a}\arctan\left(\frac{a}{\pi}\sin\left(\frac{\pi l}{L}\right)\right)$$

and we can deduce the following curvature bound

$$\kappa^2(x,t) \leq \left(\frac{2\pi}{L(t)}\right)^2\left(1 + \frac{2a^2}{\pi^2}e^{-8\pi^2\tau(t)}\right)$$

where

$$\tau(t) = \int_0^t \frac{1}{L^2}\,\mathrm{d}\,t.$$

We omit a proof from this point, but Grayson's theorem is much more reasonable to prove once the curvature has an upper bound only dependent on $L(t)$.

5 Boundary Conditions

Equipped with this proof strategy, we can hope to solve a wider class of problems. However, a variety of boundary conditions were already solvable via simple strategy.

Definition 8: An immersion $X : M \times [0,T) \to \mathbb{R}^2_+$ for $M \cong [0,1]$ solves Neumann-Neumann curve shortening flow if

$$\begin{cases} \frac{\partial X}{\partial t} = -\kappa \mathrm{N} \ \text{ in } M^\circ \times [0,T) \\ \langle X, \hat{\imath} \rangle = 0 \text{ on } \partial M \times [0,T) \end{cases}$$

where $\hat{\imath}$ is the standard unit vector pointing along the boundary of the half plane $\mathbb{R}^2_+$, M° is the interior of M and ∂M is the boundary of M.

A simple way to solve this flow is to consider the reflection. Let X' be the reflected flow. Then $Y = X \cup X'$ is the curve shortening flow for a closed curve, where half of the closed curve always agrees with the Neumann-Neumann curve shortening flow.

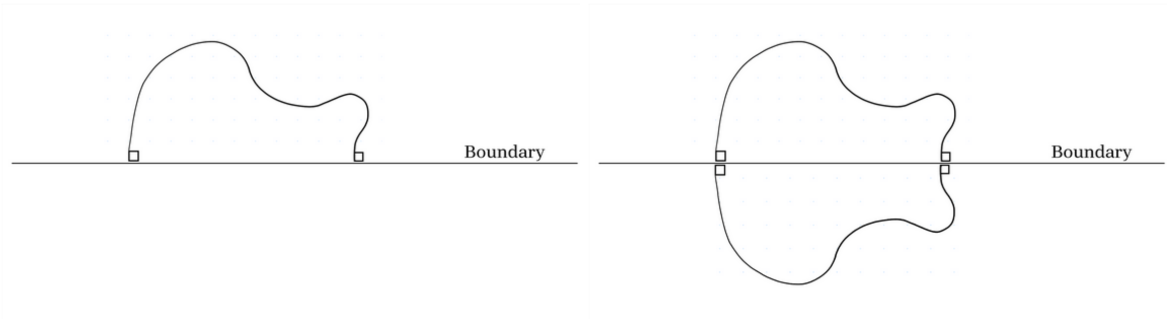

Figure 3: Neumann-Neumann Boundary Condition in Half-Plane

However, we face a problem if we slightly adjust the definition of the flow.

Definition 9: An immersion $X : M \times [0,T) \to \Omega$ for $M \cong [0,1]$ solves Neumann-Neumann curve shortening flow in a convex domain Ω if

$$\begin{cases} \frac{\partial X}{\partial t} = -\kappa \mathrm{N} \ \text{ in } M^\circ \times [0,T) \\ \langle X, \mathrm{T}^S \rangle = 0 \text{ on } \partial M \times [0,T) \end{cases}$$

where T^S is the unit tangent vector pointing along the boundary of Ω, M° is the interior of M and ∂M is the boundary of M.

A simple reflection argument is no longer possible. We would like to use a chord-arc estimate, but that required smooth second derivatives, to allow for a second derivative test at new maximums/minimums. Here, the second derivative is not smooth at the boundary. If we proceeded without considering the boundary, our chord-arc estimate would rule out singularities forming at every point on the curve except the boundary. We have to

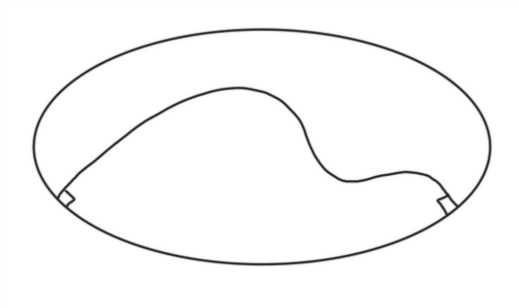

Figure 4: Neumann-Neumann Boundary Condition in Convex Domain

make an adjustment to the argument which allows us to smoothly consider the chord-arc ratio even as we move towards the boundary, and at the boundary. The method demonstrated here follows Langford and Zhu [5].

We proceed by transferring our reflection heuristic from the half-plane case. Effectively, we fold the reflected case in half to consider reflected versions of all the relevant quantities. This is shown in Figure 5.

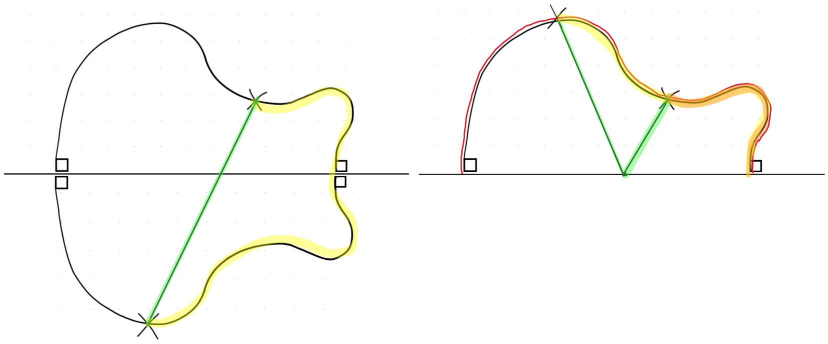

Figure 5: Neumann-Neumann Boundary Condition in Half Plane with Reflected Chord Length and Arc Length

Definition 10: The reflected chord length, $\tilde{d}$ and the reflect arc length $\tilde{l}$ are given by

$$\tilde{d}(x,y) = \min_{z\in\Omega}(|X(x) - X(z)| + |X(y) - X(z)|)$$

$$\tilde{l}(x,y) = \min_{s\in\partial M}(l(x,s) + l(y,s))$$

Hence, in proving a theorem about longtime behaviour in the convex domain case we have to consider the reflected chord-arc ratio as well.

We define $Z, \tilde{Z}$ by

$$Z(x,y,t) = d(x,y,t) - L(t)\phi\left(\frac{l(x,y,t)}{L(t)}, t\right)$$

$$\tilde{Z}(x,y,t) = \tilde{d}(x,y,t) - L(t)\phi\left(\frac{\tilde{l}(x,y,t)}{L(t)}, t\right)$$

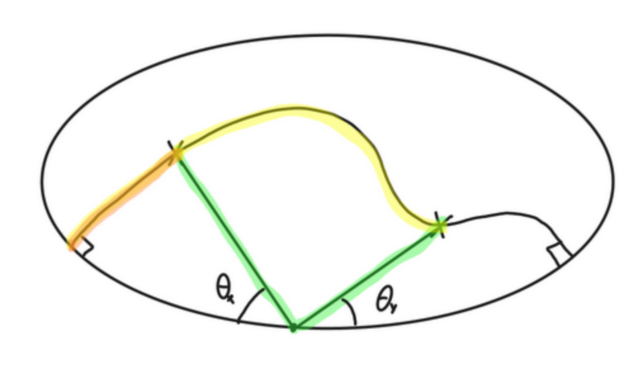

Figure 6: Heuristic for Neumann-Neumann Boundary Condition in Convex Domain

and we attempt to find a function ϕ which guarantees the positivity of $\mathbf{Z} := \min\{Z, \tilde{Z}\}$. This method will also rule out singularities forming at boundary points, since the reflected chord-arc ratio also has a lower bound. We omit the proof and technical details, which can be found in [5].

6 New Problems

Definition 11: An immersion $X : M \times [0, T) \to \Omega$ for $M \cong [0, 1]$ solves Dirichlet-Neumann curve shortening flow in a convex domain Ω if

$$\begin{cases} \frac{\partial X}{\partial t} = -\kappa \, \mathrm{N} \text{ in } M^\circ \times [0, T) \\ \langle X, \mathrm{T}^S \rangle = 0 \text{ on } \{0\} \times [0, T) \\ X = a \text{ on } \{L\} \times [0, T) \end{cases}$$

where T^S is the unit tangent vector pointing along the boundary of Ω, M° is the interior of M, $\{0, L\}$ are the boundary points of the curve X, and a is some point in Ω°.

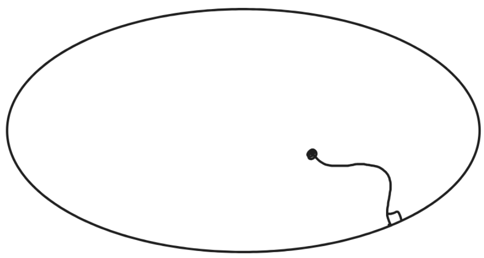

Figure 7: Dirichlet-Neumann Boundary Condition in Convex Domain

Once again, we can consider the methodology in the simplified case. For a Dirichlet-Dirichlet flow, we can rotate our initial curve by 180° to generate a new C^2 periodic curve. Since the Gage-Hamilton-Grayson theorem provides information of the flow of closed curves on Riemann surfaces, we can wrap the periodic curve on a cylinder and use the existing long-time behaviour theorems.

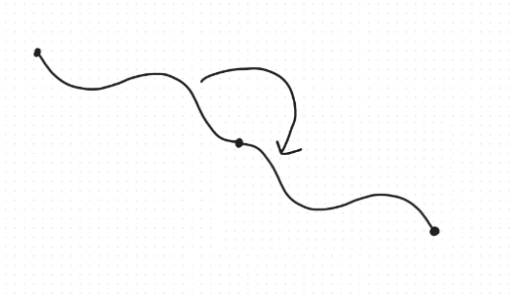

Figure 8: Dirichlet-Dirichlet Boundary Condition in $\mathbb{R}^2$

We use this heuristic to analyse a new chord-arc estimate paradigm.

Definition 12: The quantities d_1, l_1, d_2, l_2 are given by

$$\begin{aligned} d_1(x,y) &= |X(x) + X(y) - 2X(L)| \\ l_1(x,y) &= l(x,L) + l(y,L) \\ d_2(x,y) &= \tilde{d}(x,y) \\ l_2(x,y) &= l(x,0) + l(y,0) \end{aligned}$$

Note that d_2, l_2 coincide with the reflected values, but the specific reflection at the 0 boundary point, where the Neumann boundary occurs. Note additionally that d_1, l_2 coincide with the chord-arc ratios if one of the points was rotated about the point $X(L) = a$.

We define Z, Z_1, Z_2 by

$$\begin{aligned} Z(x,y,t) &= d(x,y,t) - \phi(l(x,y,t),t) \\ Z_1(x,y,t) &= d_1(x,y,t) - \phi(l_1(x,y,t),t) \\ Z_2(x,y,t) &= d_2(x,y,t) - \phi(l_2(x,y,t),t) \end{aligned}$$

Note that we no longer need to scale by $L(t)$ since this flow clearly does not have that $L(t) \to 0$. Hence we do not need to worry about scaling problems. We then attempt to find conditions on ϕ such that the positivity of $\mathbf{Z} = \min\{Z, Z_1, Z_2\}$ is preserved. This result will show preservation of embeddedness as well as rule out singularities at both boundaries.

7 Discussion

I intend on identifying the required ϕ, and deriving a curvature bound to fully describe longtime behaviour in the case of Dirichlet-Neumann boundary condition. Early results and computations are promising, since they lead to very similar inequalities. As a matter of fact, the results are convincing enough that it seems like an extremely broad class of boundary conditions can be imposed.

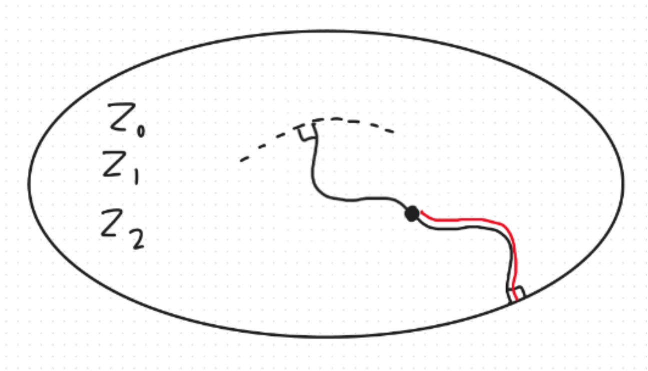

Figure 9: Heuristic for Dirichlet-Neumann Boundary Condition in Convex Domain

In work done by [5], it is shown that Neumann boundary conditions can be fully described. In the process of working through Dirichlet-Neumann boundary conditions, it becomes clear that the behaviour around the Neumann boundary is mostly described by the existing research in the Neumann-Neumann case. The existence of the Dirichlet boundary does not seem to interfere with the unfolding of the proof on one side. Hence, future proofs with mixed boundary conditions should be simple.

A potential future direction would be altering the Neumann boundary condition. We may impose any angle for the curve to meet the boundary of a convex domain, either smoothly varying along the boundary, or staying fixed. We may also adapt proofs to develop longtime existence theorems on non-flat domains as well. However, considerable work must be done to ensure that these questions are accurately formulated.

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