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Random Walks and Martingales

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1 Prelude

1.1 Abstract

We study Random Walks and Martingales, their uses and how they can be applied. We expand on our knowledge of these topics by introducing Renewal Processes and Markov Chains to formalise certain ideas and to prove interesting properties about specific random walks.

1.2 Introduction

There are many problems in physics, economics, and probability which can be modeled by random walks. How particles move and stock-market prices vary can be modelled using coin-flipping.

Some questions that arise when studying random walks includes; Does this walk return to where it started? If so, how often does it return, and how many steps do we expect to take before returning to our starting point?

For these questions we study a special class of processes called Martingales, which can be described as being fair games. In particular we have studied the Harris Martingale, named after the American mathematician Ted Harris [13] who was one of the early developers of some fields of probability, e.g. interacting particle systems and percolation [10]. We use the Harris Martingale to show that some random walks return to their starting point infinitely many times, and some never return.

We also use other processes such as Renewal Processes and Markov Chains to formalise some of the ideas about random walks. In this report we have included a number of well-known theorems, with their proofs. All which we will use to prove a nice result about a random walk on a chess board.

1.3 Statement of Authorship

The theorems and proofs in this report can be found in the literature referenced in the bibliography. Certain proofs have been reworded to adhere to a broader audience, and points in proofs have been elaborated on to make them clearer to readers. This has been done by Ms Tucker with the help of her supervisors A/Prof Collevecchio and Prof Hamza.

Any images not referenced in the bibliography have been generated by Python code written by Ms Tucker, based on the lecture [16].

2 Random Walks

A random walk is a random process describing a 'walk' in a mathematical space made up of random steps.

2.1 Simple Random Walk

A **simple random walk** takes place in a 1-dimensional space. We start our walk at the origin. With equal probability we take one step up or down, that is, with equal probability we add 1 or we subtract 1 from our current position.

More formally this can be described as a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ where $X_0 = 0$ and

$$X_t = \sum_{i=1}^{t} Z_i$$

where

$$Z_i = \begin{cases} +1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2, \end{cases}$$

and $(Z_i)_i$ is a sequence composed by independent random variables. By definition $X_0 = 0$, as this is our starting point.

Then X_1 can take values $+1$ or -1. X_2 can take values $-2, 0, +2$.

Plotting our position X_t against 'time' t we can produce some nice plots of our walks seen in figure 1.

2.2 Random Walks in Higher Dimensions

We can extend the notion of a 'simple' random walk into higher dimensions by letting Z_i take values of $\pm$ the unit vectors which form the standard base of Euclidean spaces.

For ease of visualisation let us consider the 2-dimensional case. X_n is defined the same as for a simple random walk:

$$X_t = \sum_{i=1}^{t} Z_i.$$

In this context, Z_i take values as follows:

$$Z_i = \begin{cases} (0,1) & \text{with probability } 1/4 \\ (0,-1) & \text{with probability } 1/4 \\ (1,0) & \text{with probability } 1/4 \\ (-1,0) & \text{with probability } 1/4 \end{cases}$$

This can be extended to even higher dimensions by adding more vectors to Z_i and adjusting the probabilities.

Figure 2 and 3 show some random walks on the 2d-cartesian plane using a step size of 1. Note that there is no time-axis on these plots.

Random walks such as these can be used to approximate Brownian motion. First described by Robert Brown in 1827 [5], Brownian Motion can be used to model how particles move in a gas or liquid. It can also be used to model stock and asset prices. A proof of the existence of Brownian motion was first provided by Norber Wiener.

Brownian motion is defined as a random walk with many infinitesimally short steps in a set time interval. Figure 4 shows a random walk of 1000 steps of length 0.1 taken every 0.01 time units. Figure 5 shows a random walk of 250000 steps of length 5. Common practice is to use time intervals of t with steps of length $\sqrt{t}$ [7].

2.3 Knight's Walk

Originating from chess, a knights walk is a walk on a 2-dimensional plane where a step is defined as those a knight can take in chess. Figure 6 shows the possible moves of a knight on a chess board.

If we think of these 'steps' as vectors in the 2-dimensional plane we find that Z_i can take the values $\{(2,1),(1,2),(-1,2),(-2,1),(-2,-1),(-1,-2),(1,-2),(2,-1)\}$. For example, moving from the space b8 to c6 is equivalent to $Z_i = (-2,1)$.

If we let each possible step be taken with equal probability ($p = \frac{1}{8}$) then we can define the random walk

$$X_t = \sum_{i=1}^{t} Z_i$$

with $X_0 = 0$ just as previously. This can be done on a board of limited size, such as a chess board, or on an infinite plane. If done with restricted boundaries, the probabilities must be adjusted to account for illegal moves.

Figure 7 and 8 show some random knights walks on 10 steps in the unrestricted 2-dimensional plane. These walks have not been restricted by any 'boardsize', that is there is no upper and lower limit on the knight's allowed x and y-position.

3 Martingales

3.1 Definition

A **martingale** is a sequence of random variables such that at any time the expected value of the next variable given the knowledge of all past variables is equal to the value of the current variable. A martingale also satisfies that the expected absolute value is finite at any time.

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables. Then $(X_n)_{n\in\mathbb{N}}$ is a martingale if it satisfies both:

$$\mathbb{E}(|X_n|) < \infty \qquad \forall n \in \mathbb{N}$$
$$\mathbb{E}(X_{n+1}|X_0, \cdots X_n) = X_n \qquad \forall n \in \mathbb{N}$$

We commonly denote $\mathcal{F}_n$ a filtration of $(X_n)_{n\in\mathbb{N}}$, and we write the expected value as $\mathbb{E}(X_{n+1}|\mathcal{F}_n)$. A rigorous definition of a filtration is given in [25] (p. 191), but for our uses we will think of it as the 'past' of our process up until time n.

3.2 Doob's Optional Stopping and Convergence Theorems

Joseph L Doob was an American mathematician who started his career in complex variables but later moved on to probability theory and proved some of the most important convergence theorems in martingale theory [12].

Doob's Optional Stopping Theorem makes great use of Martingale properties to show that given certain conditions for a stopping time, the expected value of the martingale at the stopping time is equal to the expected value at time 0.

Definition 3.2.1 (Stopping Time)**.** *A random variable T in $\mathbb{Z}_+ \cup \{\infty\}$ is a stopping time for a filtration $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots$ if for every $n \in \mathbb{N}$, the event $\{T = n\}$ is an element of the σ-algebra $\mathcal{F}_n$.*

What this means is that the decision to stop at a certain time $T = n$ must be able to be based only on information already known, not on any future information.

Examples of stopping times include "the first time the sequence reaches 0", or any constant time $T = m$ for a fixed m. A random time that is not a stopping time could be "the time at which the value of the sequence is smaller out of either $T = m_1$ or $T = m_2$", since at $T = m_1$ we do not know what the value of the sequence at $T = m_2$ is and therefore we cannot say if the value at $T = m_1$ is smaller and whether we should stop or not.

Theorem 3.2.2 (Doob's Optional Stopping Theorem)**.** *Let $(M_n)_{n\in\mathbb{N}}$ be a martingale and T a stopping time relative to a filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$. Assume that one of the following holds:*

(1) the stopping time T is almost surely bounded

(2) $\mathbb{E}[T] < \infty$ and there exists some fixed $c \in \mathbb{R}$ such that $\mathbb{E}[X_{t+1} - X_t|\mathcal{F}_t] \leq c$ almost surely when $T > t$

(3) there exists some fixed $c \in \mathbb{R}$ such that $X_{t\wedge T} \leq c$ almost surely ($\wedge$ denotes the minimum operator)

Then X_T is almost surely well defined and $\mathbb{E}[X_T] = \mathbb{E}[X_0]$

Another important theorem is the following:

Theorem 3.2.3 (Doob's Convergence Theorem)**.** *Consider a martingale $(M_n)_{n\in\mathbb{N}}$ such that $\sup_n \mathbb{E}[|M_n|] < \infty$. Then $\lim_{n\to\infty} M_n = M_\infty$ exists almost surely and $\mathbb{E}[|M_\infty|] < \infty$.*

Corollary 3.2.3.1. *If $(M_n)_{n\in\mathbb{N}}$ is a non-negative martingale, then $\lim_{n\to\infty} M_n = M_\infty$ exists almost surely and $\mathbb{E}[|M_\infty|] < \infty$.*

3.3 Uses and Examples

Martingales are very useful in probability, analysis, and finance for many reasons. They can be used to analyse games, stock prices, gambling, random walks, etc.

Definition 3.3.1 (Doob Martingale)**.** *Let X be any random variable such that $\mathbb{E}[|X|] < \infty$. Let $\{Y_n : n \geq 0\}$ be any stochastic process on the same probability space as X. Let $\mathcal{F}_n = \sigma\{Y_0, \cdots Y_n\}$. Then the following is a martingale:*

$$X_n := \mathbb{E}[X|\ \mathcal{F}_n]$$

This martingale is used for example to prove McDiarmids Inequality [3] used in probability theory to estimate bounded functions of independent random variables [6].

It is also possible to prove Wald's Equation using martingales.

Theorem 3.3.2 (Wald's Equation)**.** *Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables. Let T be a stopping time for $(X_i)_{i\in\mathbb{N}}$. Then*

$$\mathbb{E}\left[\sum_{i=1}^{T} X_i\right] = \mathbb{E}[T]\mathbb{E}[X_1]$$

provided that $\mathbb{E}[X_i], \mathbb{E}[T] < \infty$

Proof. Let $\mu = \mathbb{E}[X_i]$. We begin by defining the sequence $(Y_i)_{i\in\mathbb{N}}$ such that

$$Y_i = \left(\sum_{j=1}^{i} X_j\right) - \mu i$$

We show that $(Y_i)_{i\in\mathbb{N}}$ is a martingale.

$$\mathbb{E}[|Y_i|] = \mathbb{E}[|\left(\sum_{j=1}^{i} X_j\right) - \mu i|] \leq \mathbb{E}\left[\left|\sum_{j=1}^{i} X_j\right|\right] + \mathbb{E}\left[|\mu i|\right] \leq \sum_{j=1}^{i} \mathbb{E}\left[|X_j|\right] + \mathbb{E}[|\mu i|] < \infty$$

Now let $\mathcal{F}_i$ be a filtration for $(Y_i)_{i\in\mathbb{N}}$.

$$\begin{aligned}
\mathbb{E}[Y_i|\mathcal{F}_{i-1}] &= \mathbb{E}[\left(\sum_{j=1}^{i} X_j\right) - \mu i|\mathcal{F}_{i-1}] \\
&= \left(\sum_{j=1}^{i} \mathbb{E}[X_j|\mathcal{F}_{i-1}]\right) - \mu i \\
&= \left(\sum_{i=1}^{i-1} X_j\right) + \mathbb{E}[X_i|\mathcal{F}_{i-1}] - \mu i \\
&= \left(\sum_{i=1}^{i-1} X_j\right) + \mu - \mu i \\
&= \left(\sum_{i=1}^{i-1} X_j\right) - \mu(i-1) \\
&= Y_{i-1}
\end{aligned}$$

So $(Y_i)_{i\in\mathbb{N}}$ is a martingale with respect to the filtration $\mathcal{F}_i$.

Then since T is a stopping time for $(X_i)_{i\in\mathbb{N}}$ we can determine T based on the 'past': $X_0, \cdots, X_t$. But if we know $Y_0, \cdots, Y_T$ we can deduce $X_0, \cdots, X_T$ and therefore we can decide T based on $Y_0, \cdots, Y_T$. So we see that T is also a stopping time for $(Y_i)_{i\in\mathbb{N}}$. And we can see that

$$\mathbb{E}[Y_{t+1} - Y_t|\mathcal{F}_t] = \mathbb{E}[\left(\sum_{j=1}^{t} X_j\right) - \mu i - \left(\sum_{j=1}^{t-1} X_j\right) + \mu(t-1)|\mathcal{F}_t] = \mathbb{E}[X_{t+1} - \mu|\mathcal{F}_t] = 0$$

Then by Doob's Optional Stopping Theorem (2) (3.2.2) we have that $\mathbb{E}[Y_T] = \mathbb{E}[Y_1]$. Therefore:

$$\begin{aligned}
\mathbb{E}[Y_T] &= \mathbb{E}[Y_1] \\
\mathbb{E}\left[\left(\sum_{i=1}^{T} X_i\right) - \mu T\right] &= \mathbb{E}\left[\left(\sum_{i=1}^{1} X_i\right) - \mu\right] \\
\mathbb{E}\left[\sum_{i=1}^{T} X_i\right] - \mathbb{E}[\mu T] &= \mathbb{E}[X_1] - \mu \\
\mathbb{E}\left[\sum_{i=1}^{T} X_i\right] &= \mathbb{E}[\mu T] \\
\mathbb{E}\left[\sum_{i=1}^{T} X_i\right] &= \mathbb{E}[T]\mathbb{E}[X_1]
\end{aligned}$$

□

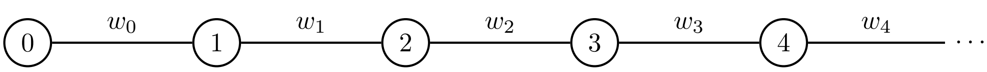
A nice consequence of Wald's Theorem is the fact that for a simple random walk, the expected time it takes to reach $+1$ is infinite. Let T_1 be the time at which we first reach $+1$. For sake of contradiction, assume that $\mathbb{E}[T_1] < \infty$. We know that the steps Z_n (taking values $+1, -1$ with probability $= 1/2$ each) of a simple random walk are i.i.d random variables and $\mathbb{E}[Z_n] = 0$. By Wald's equation we obtain $\sum_{i=0}^{T_1} Z_n = \mathbb{E}[Z_n]\mathbb{E}[T_1]$. Now $\sum_{i=0}^{T_1} Z_n = 1$ (by definition of T_1) and $\mathbb{E}[Z_n]\mathbb{E}[T_1] = 0$. But then $1 = 0$ which is a contradiction, hence our assumption must be false. And so $\mathbb{E}[T_1] = \infty$. We have therefore shown that the expected time it takes for a simple random walk to reach $+1$ is infinite.

3.4 Harris Martingale

Named after mathematician Theodore Edward Harris, this martingale is a stochastic process on the positive integer line $\mathbb{Z}^+$ where at each integer we are varyingly likely to progress to a higher number or a lower number based on weights assigned to the edges between adjacent integers.

Consider a graph with vertices for each integer in $\mathbb{Z}^+ \cup \{0\}$ and edges between two vertices $i, j \in \mathbb{Z}^+ \cup \{0\}$ if $i = j + 1$, for all $i \geq 1, j \geq 0$.

We define weights w_i for each edge, such that the edge $(i, i+1)$ has the weight w_i and each $w_i > 0$.

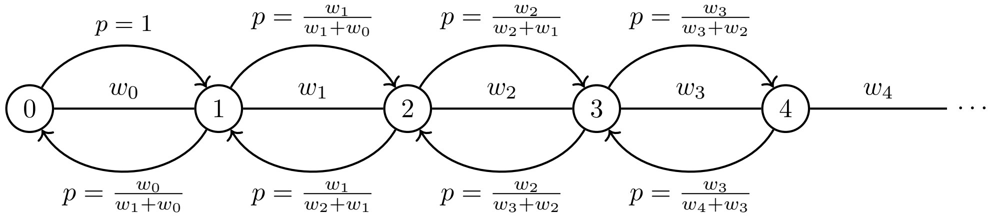


The weight w_i corresponds to how likely we are to move from i to $j = i + 1$ in relation to w_{i-1}.

Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of random variables, with $X_0 = 1$, and determined by the probabilites:

$$\begin{aligned}
&\mathbb{P}(X_{n+1} = i + 1 |\ X_n = i) = \frac{w_i}{w_i + w_{i-1}} && \text{for } i \geq 1 \\
&\mathbb{P}(X_{n+1} = i - 1 |\ X_n = i) = \frac{w_{i-1}}{w_i + w_{i-1}} && \text{for } i \geq 1 \\
&\mathbb{P}(X_{n+1} = 1 |\ X_n = 0) = 1
\end{aligned}$$

Visualising this on the graph:



We define the filter $\mathcal{F}_n = \{X_0, \cdots, X_n\}$. We define the stopping time $\tau := \inf\{t : X_t = 0\}$.

Definition 3.4.1. *The Harris Martingale* $(M_n)_{n\in\mathbb{N}}$ *is defined as follows:*

$$M_n = \sum_{i=0}^{X_{\tau\wedge n}-1} \frac{1}{w_i}$$

We will now prove that the Harris Martingale is in fact a martingale.

First we show that $\mathbb{E}[|M_n|] < \infty$ for all $n \in \mathbb{N}$.

$$\mathbb{E}[|M_n|] = \mathbb{E}\left[| \sum_{i=0}^{X_{\tau\wedge n}-1} \frac{1}{w_i} | \right] \leq \sum_{i=0}^{n-1} \mathbb{E}\left[\frac{1}{w_i}\right] < \infty$$

Now we will show that $\mathbb{E}[M_n|\mathcal{F}_{n-1}] = M_{n-1}$.

First we consider the case where $\tau \leq n-1$:

$$\begin{aligned}
\mathbb{E}[M_n|\mathcal{F}_{n-1}] &= \mathbb{E}[\sum_{i=0}^{X_{\tau\wedge n}-1} \frac{1}{w_i} |\mathcal{F}_{n-1}] \\
&= \mathbb{E}[\sum_{i=0}^{X_{\tau}-1} \frac{1}{w_i} |\mathcal{F}_{n-1}] \\
&= \mathbb{E}[\sum_{i=0}^{-1} \frac{1}{w_i} |\mathcal{F}_{n-1}] \\
&= 0 \\
&= M_{n-1}
\end{aligned}$$

Then we consider the case where $\tau \geq n$:

$$\begin{aligned}
\mathbb{E}[M_n|\mathcal{F}_{n-1}] &= \mathbb{E}[\sum_{i=0}^{X_{\tau\wedge n}-1} \frac{1}{w_i} |\mathcal{F}_{n-1}] \\
&= \mathbb{E}[\sum_{i=0}^{X_{n}-1} \frac{1}{w_i} |\mathcal{F}_{n-1}]
\end{aligned}$$

Now we have again two cases. Either $X_{n-1} = X_n - 1$ or $X_{n-1} = X_n + 1$.

If $X_{n-1} = X_n - 1$:

$$\begin{aligned}
\mathbb{E}[M_n|\mathcal{F}_{n-1}] &= \mathbb{E}[\sum_{i=0}^{X_{\tau\wedge n}-1} \frac{1}{w_i}|\mathcal{F}_{n-1}] \\
&= \mathbb{E}[\sum_{i=0}^{X_n-1} \frac{1}{w_i}|\mathcal{F}_{n-1}] \\
&= \sum_{i=0}^{X_{n-1}} \mathbb{E}[\frac{1}{w_i}|\mathcal{F}_{n-1}] \\
&= M_{n-1} + \frac{1}{w_{X_{n-1}}} \\
&= M_{n-1} + \frac{1}{w_{X_n-1}}
\end{aligned}$$

If $X_{n-1} = X_n + 1$:

$$\begin{aligned}
\mathbb{E}[M_n|\mathcal{F}_{n-1}] &= \mathbb{E}[\sum_{i=0}^{X_{\tau\wedge n}-1} \frac{1}{w_i}|\mathcal{F}_{n-1}] \\
&= \mathbb{E}[\sum_{i=0}^{X_n-1} \frac{1}{w_i}|\mathcal{F}_{n-1}] \\
&= \sum_{i=0}^{X_{n-1}-2} \mathbb{E}[\frac{1}{w_i}|\mathcal{F}_{n-1}] \\
&= M_{n-1} - \frac{1}{w_{X_{n-1}-1}} \\
&= M_{n-1} - \frac{1}{w_{X_n}}
\end{aligned}$$

Now let p be the probability that $X_{n-1} = X_n - 1$. Then the probability that $X_{n-1} = X_n + 1$ is $1 - p$. Therefore

$$\mathbb{E}[M_n|\mathcal{F}_{n-1}] = p\left(M_{n-1} + \frac{1}{w_{X_n-1}}\right) + (1-p)\left(M_{n-1} + \frac{1}{w_{X_n}}\right)$$

This gives us

$$\begin{aligned}
\mathbb{E}[M_n|\mathcal{F}_{n-1}] &= p\left(M_{n-1} + \frac{1}{w_{X_n-1}}\right) + (1-p)\left(M_{n-1} - \frac{1}{w_{X_n}}\right) \\
&= M_{n-1} + \frac{p}{w_{X_n-1}} + \frac{p-1}{w_{X_n}} \\
&= M_{n-1} + \frac{(p-1)w_{X_n-1} + pw_{X_n}}{w_{X_n-1} + w_{X_n}} \\
&= M_{n-1} + p\frac{w_{X_n-1} + w_{X_n}}{w_{X_n-1} + w_{X_n}} - \frac{w_{X_n-1}}{w_{X_n-1} + w_{X_n}} \\
&= M_{n-1} + p - \frac{w_{X_n-1}}{w_{X_n-1} + w_{X_n}}
\end{aligned}$$

And we also know that $p = \frac{w_{X_n-1}}{w_{X_n} + w_{X_n-1}}$ from the definition of our graph and our weights.

Therefore

$$\begin{aligned}\mathbb{E}[M_n|\mathcal{F}_{n-1}] &= M_{n-1} + p - \frac{w_{X_n-1}}{w_{X_n-1}+w_{X_n}} \\ &= M_{n-1} + \frac{w_{X_n-1}}{w_{X_n}+w_{X_n-1}} - \frac{w_{X_n-1}}{w_{X_n-1}+w_{X_n}} \\ &= M_{n-1}\end{aligned}$$

And therefore M_n is a martingale with respect to $\mathcal{F}$.

3.5 Transience and Recurrence

Two important properties of stochastic processes are transience and recurrence. Simply put, a sequence of random variables is transient if it visits each possible value a finite number of times. It is recurrent if it returns to its initial value infinitely often[1].

Definition 3.5.1 (Recurrence)**.** *A random walk $(X_n)_{n\in\mathbb{N}}$ on a graph is recurrent if*

$$\mathbb{P}(X_n = X_0 \text{ for some } n \geq 1) = 1$$

Definition 3.5.2 (Transience)**.** *If $(X_n)_{n\in\mathbb{N}}$ is not recurrent, then it is transient.*

We can use the Harris Martingale (3.4.1) to prove transience and recurrence for some sequences of random variables $(X_n)_{n\in\mathbb{N}}$.

Theorem 3.5.3. *As $n \to \infty$ either $M_n \to 0$ or $M_n \to \sum_{i=0}^{\infty} \frac{1}{w_i}$.*

The proof can be found in the appendix.

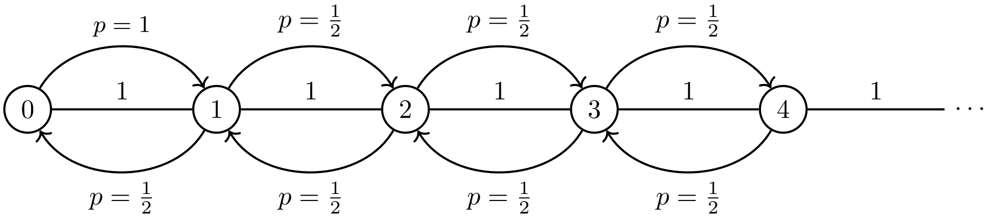
Now we will use Theorem 3.5.3 to prove recurrence and transience properties of the Harris Martingale.

Theorem 3.5.4. *If $\sum_{i=0}^{\infty} \frac{1}{w_i} = \infty$ then $(X_n)_{n\in\mathbb{N}}$ is recurrent.*

Proof. By Theorem 3.5.3 we know that $(M_n)_{n\in\mathbb{N}}$ converges almost surely to either 0 or $\sum_{i=0}^{\infty} \frac{1}{w_i}$. Since $(M_n)_{n\in\mathbb{N}}$ is a non-negative martingale, we know that $\mathbb{E}[|M_\infty|] < \infty$ by Corollary 3.2.3.1.

Therefore $M_\infty \neq \sum_{i=0}^{\infty} \frac{1}{w_i}$, so $M_\infty = 0$ which implies that $\mathbb{P}(\tau < \infty) = 1$. □

We can use these theorems to show, for example, that a simple random walk is recurrent. A simple random walk can be modelled by a Harris Martingale where each weight $w_i = 1$. A simple random walk starts at 0 and then either jumps to $+1$ or -1. If our walk starts at $+1$ then we model our Harris Martingale on the positive number line with $X_0 = 1$, and if it starts at -1 we can model it on the negative number line with $X_0 = -1$. The two are symmetrical.



or equivalently Then to show that a simple random walk is recurrent we simply find that

$$\sum_{i=0}^{\infty} \frac{1}{w_i} = \sum_{i=0}^{\infty} 1 = \infty$$

and so $(X_n)_{n\in\mathbb{N}}$ is recurrent. What this means is that the simple random walk returns to 0 with probability 1.

4 Renewal Processes

A renewal process is a generalisation of the Poisson process, and is a process where over the course of time 'renewals' happen at identically and independently distributed intervals of time.

4.1 Definition

Definition 4.1.1 (Renewal Process)**.** *Let $(X_n)_{n\in\mathbb{Z}^+}$ be a sequence of identically and independently distributed non-negative random variables with a common distribution F. We interpret X_n as the time interval between the $(n-1)$th and nth renewal.*

(We consider only the case where $\mathbb{P}(X_n = 0) < 1$.)

Let $\mu = \mathbb{E}[X]$ be the expected time between events. Since $X_i \geq 0$ for all $i \in \mathbb{Z}^+$ and since $\mathbb{P}(X_n = 0) < 1$ we know that $0 < \mu \leq \infty$.

Now we define

$$S_0 = 0, \qquad S_n = \sum_{i=1}^{n} X_i, \quad \text{for } n \geq 1$$

Then S_n is the time of the nth event.

Next we define the number of renewals before time t:

$$N(t) = \sup\{n : S_n \leq t\}$$

We let $m(t) = \mathbb{E}[N(t)]$ this is called the renewal function.

4.2 Distribution of $N(t)$

To better understand the distribution of $N(t)$ we have some simple theorems that will be useful later, when proving the reward-renewal theorem.

Theorem 4.2.1. *Let* $N(\infty) = \lim_{t\to\infty} N(t)$ *then* $N(\infty) = \infty$ *with probability 1.*

The proof can be found in the appendix.

Using this proof we can prove a nice theorem about the limiting distribution of $N(t)$

Theorem 4.2.2. *With probability 1:*

$$\frac{N(t)}{t} \to \frac{1}{\mu} \qquad \text{as } t \to \infty$$

The proof can be found in the appendix.

4.3 Elementary Renewal Theorem

Corollary 4.3.0.1. *if* $\mu < \infty$ *then*

$$\mathbb{E}[S_{N(t)+1}] = \mu(m(t) + 1)$$

This is a corollary of Wald's Equation (3.3.2), and the proof can be found in appendix.

We will use this to prove the Elementary Renewal Theorem:

Theorem 4.3.1 (Elementary Renewal Theorem)**.**

$$\frac{m(t)}{t} \to \frac{1}{\mu} \qquad \text{as } t \to \infty$$

The proof can be found in the appendix.

4.4 Reward Renewal Theorem

We can consider renewal processes where at each renewal we obtain a reward. We let R_n be the reward obtained at the nth renewal. In our case we let R_n depend on X_n, but the sequence of pairs (X_n, R_n) is identically and independently distributed.

Definition 4.4.1. *We define the total reward earned by time* t*:*

$$R(t) = \sum_{i=0}^{N(t)} R_i$$

We let $\mathbb{E}[X] = \mathbb{E}[X_n]$ and $\mathbb{E}[R] = \mathbb{E}[R_n]$.

Theorem 4.4.2 (Reward Renewal Theorem)**.** *If* $\mathbb{E}[R] < \infty$ *and* $\mathbb{E}[X] < \infty$, *then*

(I) with probability 1,

$$\frac{R(t)}{t} \to \frac{\mathbb{E}[R]}{\mathbb{E}[X]}$$

(II)

$$\frac{\mathbb{E}[R(t)]}{t} \to \frac{\mathbb{E}[R]}{\mathbb{E}[X]}$$

The proof can be found in the appendix.

If we think of a random walk on a graph, where we get a reward based on how often we return to a certain vertex v in a certain time interval X_i, then this theorem says that if we let our walk go on for a long time ($t \to \infty$), then the proportion of time we spend at vertex v equals the expected number of times we return to v in the time X_i divided by the expected length of X_i.

5 Markov Chains

A Markov Chain is a sequence of states where the next state depends only on the previous state, not any earlier information. For example a genealogical tree where the offsprings' genes depend only on those of the parents can be modelled as a Markov Chain.

5.1 Definition

Definition 5.1.1 (Markov Chain)**.** *A Markov chain is a random process $(X_n)_{n\in\mathbb{N}}$ that satisfies*

$$\mathbb{P}(X_n|X_0, \cdots, X_{n-1}) = \mathbb{P}(X_n|X_{n-1})$$

for all $n \geq 2$.

The values that X_n can take are called 'states' and are denoted $E_1, E_2, \cdots$. If the number of states is finite, then the Markov Chain is finite.

We define some terms to help us work with Markov Chains.

Definition 5.1.2 (Absolute Probability)**.** *The probability that at time n we are in state i is called the absolute probability and we denote it:*

$$p_i^{(n)} = \mathbb{P}(X_n = E_i)$$

We can see that $\sum_i p_i^{(n)} = 1$ as at any time we must be in some state.

Definition 5.1.3 (Transition Probability)**.** *The probability of moving from state i to state j is called a transition probability and is defined as follows:*

$$p_{ij} = \mathbb{P}(X_n = E_j|X_{n-1} = E_i)$$

Similarly, we can define a transition probability in r steps as follows

$$p_{ij}^{(r)} = \mathbb{P}(X_n = E_j|X_{n-r} = E_i)$$

Definition 5.1.4 (Transition Matrix)**.** *The transition matrix is the matrix of all transition probabilities.*

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \\ \vdots & & \ddots \end{bmatrix}$$

Then we define the matrix $P^{(r)} = (p_{ij}^{(r)})$, and by induction it can be shown that $P^{(r)} = P^r$.

5.2 Stationary Distribution

Definition 5.2.1 (Initial Distribution)**.** *We suppose that the process at time $t = n$ is in a state E_i with probability $p_i^{(n)}$. We define the row-vector*

$$p^{(n)} = (p_1^{(n)}, p_2^{(n)}, p_3^{(n)}, \cdots)$$

. $p^{(0)}$ is called the initial distribution.

Theorem 5.2.2. $p^{(n)} = p^{(0)}P^n$

The proof can be found in the appendix.

Definition 5.2.3 (Stationary Distribution)**.** *Let $\pi = (\pi_1, \pi_2, \cdots)$ be a probability vector. We start a process by letting $p^{(0)} = \pi$. If $p^{(n)} = \pi$ for all $n = 1, 2, \cdots$ then π is a stationary distribution.*

If we use the fact that

$$p^{(n)} = p^{(0)}P^n$$

with $n = 1$ we see that the stationary distribution π must satisfy

$$\pi = \pi P$$

5.3 Limiting Distribution

Definition 5.3.1. *Consider a homogeneous Markov Chain. If*

- *$p^{(n)} \to \pi$ as $n \to \infty$, for all j, and*
- *$\sum_j \pi_j = 1$, and*
- *π is independent of $p^{(0)}$,*

then the Markov chain is said to have a limiting distribution π. Notice that we have by default that $\pi_j \geq 0$ for all j.

Theorem 5.3.2. *If a Markov chain has a limiting distribution π, then it satisfies $\pi = \pi P$.*

The proof can be found in the appendix.

We can see that the stationary distribution and the limiting distribution have similar definitions, but they are not always the same. A stationary distribution may exist without the Markov chain having a limiting distribution. Moreover, we could have infinitely many stationary distributions for a markov chain, while the limiting distribution is unique.

5.4 Expected Returns

The stationary distribution is very useful, it is used in many other theorems regarding Markov chains, specifically the theorem deciding the expected first return to the initial state.

Theorem 5.4.1 (Expected number of visits before stopping)**.** *Consider the chain started at state E_i. Let $0 < S < \infty$ be a stopping time such that $X_S = E_i$ and $\mathbb{E}(S|X_0 = E_i) < \infty$. Let E_j be an arbitrary state. Then*

$$\mathbb{E}(\textit{number of visits to } E_j \textit{ before time } S|X_0 = E_i) = \pi_j \mathbb{E}(S|X_0 = E_i)$$

The proof can be found in the appendix.

Now we can prove a very powerful theorem:

Theorem 5.4.2 (Expected first return to initial state)**.**

$$\mathbb{E}(\min \geq 1 : X_t = E_i\}|X_0 = E_i) = \frac{1}{\pi_i}$$

This says that the expected time at which we return to our initial state E_i given that we started at E_i equals one over π_i

Proof. By Theorem 5.4.1 which we proved above, we have that

$$\mathbb{E}(\text{number of visits to } E_j \text{ before time } S|X_0 = E_i) = \pi_j \mathbb{E}(S|X_0 = E_i)$$

Letting $S = \min\{t \geq 1 : X_t = E_i\}$ and $E_j = E_i$ we obtain:

$$\mathbb{E}(\text{number of visits to } E_i \text{ before time } \min\{t \geq 1 : X_t = E_i\}|X_0 = E_i) = \pi_j \mathbb{E}(\min\{t \geq 1 : X_t = E_i\}|X_0 = E_i)$$

The right-hand side described the expected number of visits to E_i before the first time we return to E_i. This expected value must therefore equal 1 (since $X_0 = E_i$).

Then

$$1 = \pi_j \mathbb{E}(\min\{t \geq 1 : X_t = E_i\}|X_0 = E_i),$$

which we can rearrange to find

$$\mathbb{E}(\min\{t \geq 1 : X_t = E_i\}|X_0 = E_i) = \frac{1}{\pi_i}$$

□

Another proof of this theorem may be found in [24] (p. 11) where they utilise the fact that for an irreducible and aperiodic Markov chain there exists a unique stationary distribution. Therefore if we find a vector g such that $gP = g$, all entries of g are non-negative, and $\sum_i g = 1$ then g is the stationary distribution.

5.5 Random Walk on a Graph as a Markov Chain

The unbiased random walk on an unweighted finite graph G can be modelled by a finite Markov chain with the transition probabilities

$$p_{ij} = \mathbb{P}(X_n = E_j | X_{n-1} = E_i) = \begin{cases} \frac{1}{\deg(E_i)} & \text{if } E_iE_j \text{ is an edge of G} \\ 0 & \text{if } E_iE_j \text{ is not an edge of G} \end{cases}$$

Then we want to find a stationary distribution of our random walk on G, so that we can apply Theorem 5.4.1.

If we think of how likely we would be to end up at a vertex E_i we could imagine that this would depend on the vertex-degree of E_i in relation to the sum of all degrees.

We define $\pi = (\pi_1, \pi_2, \cdots)$ with

$$\pi_i = \frac{\deg(E_i)}{\sum_j \deg(E_j)}$$

By Handshaking Lemma we know that $\sum_j \deg(E_j) = 2|E(G)|$, so

$$\pi_i = \frac{\deg(E_i)}{\sum_j \deg(E_j)} = \frac{\deg(E_i)}{2|E(G)|}$$

Theorem 5.5.1. *For a finite, undirected, connected, simple graph G. The stationary distribution of an unbiased random walk on this graph is given by the vector $\pi = (\pi_1, \pi_2, \cdots)$ where*

$$\pi_i = \frac{\deg(E_i)}{2|E(G)|}$$

The proof can be found in the appendix.

5.6 Knight's Return

Let us think again about the Knight's walk. This time we will restrict the Knight to the chessboard (8x8).

We can model all the possible paths of the Knight on the chessboard as a graph G where each vertex is a square, and an edge exists between two vertices if the Knight can move between the corresponding squares in one move.

If you play chess, you (hopefully) know that the Knight can reach any square on the chess board in a finite number of moves, therefore G is connected. This graph has 64 vertices and 168 edges.

Now imagine we place our Knight in the top corner of our chess board, and then we start randomly walking, by at each step choosing one of the possible legal moves each with equal probability. This can be modelled as a random walk on the graph G. A natural question to ask is: in how many steps do we expect to return to where we started? We can answer this question by using Markov chains.

Theorem 5.6.1 (Knight's Return)**.** *If we start a Knight in a corner square of a chess board, the expected number of steps taken to return to the same corner square is* 168.

Proof. We model the Knight's walk on the chess board as an unbiased random walk on a connected, finite graph G.

Let the vertex set and edge set be denoted by:

$$V(G) = \{E_1, E_2, \cdots, E_{64}\}$$

$$E(G) = \{E_iE_j : E_i \text{ can be reached from } E_j \text{ by a Knight in one move}\}$$

And our transition matrix P be defined as $P = (p_{ij})$ where

$$p_{ij} = \mathbb{P}(X_n = E_j | X_{n-1} = E_i) = \begin{cases} \frac{1}{\deg(E_i)} & \text{if } E_iE_j \text{ is an edge of G} \\ 0 & \text{if } E_iE_j \text{ is not an edge of G} \end{cases}$$

By Theorem 5.5.1 we know that the stationary distribution is $\pi = (\pi_1, \pi_2, \cdots)$ where

$$\pi_i = \frac{\deg(E_i)}{2|E(G)|}$$

Then by Theorem 5.4.2 we know that

$$\mathbb{E}(\min\{t \geq 1 : X_t = E_i\} | X_0 = E_i) = \frac{1}{\pi_i}$$

If we denote our initial corner square by E_1 we have that

$$\mathbb{E}(\min\{t \geq 1 : X_t = E_1\} | X_0 = E_1) = \frac{1}{\pi_1}$$

Which gives us

$$\frac{1}{\pi_1} = \frac{2|E(G)|}{\deg(E_1)} = \frac{2|E(G)|}{2} = |E(G)| = 168$$

So we expect to return to E_1 (our corner square) in 168 steps. □

6 Discussion and Conclusion

In this report we have explored some examples of random walks and their applications. We showed that the Harris Martingale is a martingale, and that it can be used to prove recurrence and transience of some processes, which we illustrated by shownig that the simple random walk is recurrent.

We have explored the concepts of Renewal Theory to form an understanding of stationary and limiting distributions of Markov chains, and the expected first return to an initial state of a Markov process.

These results together form the foundation for the proof of the expected time to return to a corner square when walking randomly as a knight on a chess board.

For future research it would be interesting to look at more martingales to find ways of proving recurrence and transience for other types of random walks than those described by the Harris Martingale. Additionally, researching the limiting and stationary distributions of other random walks than that of the knight to understand the asymptotic behaviour of said walks would be a compelling project.

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7 Appendix

These plots represent simple random walks of 10 steps. The vertical axis shows the position and the horizontal axis shows time.

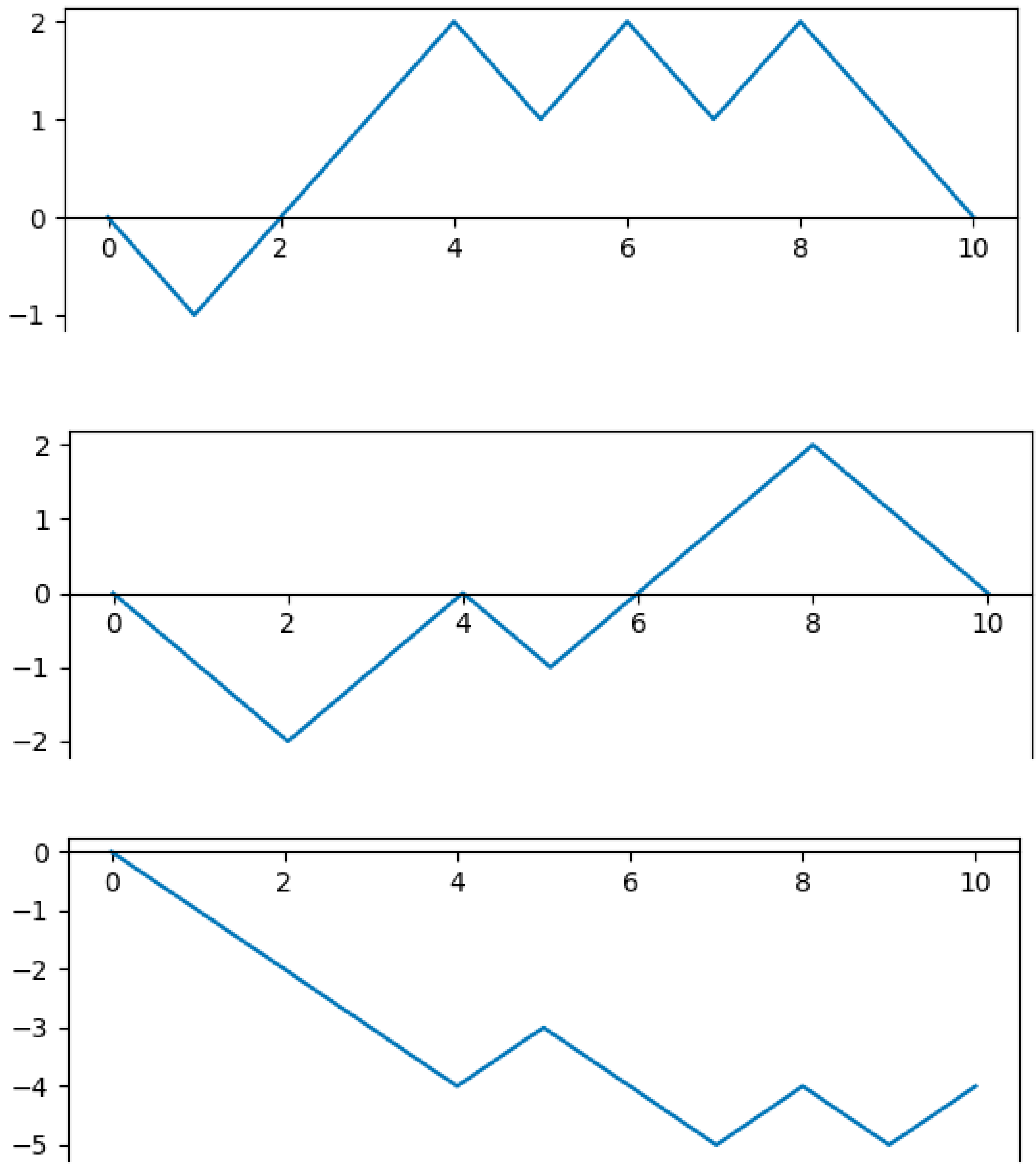

Figure 1: Three 10-step simple random walks

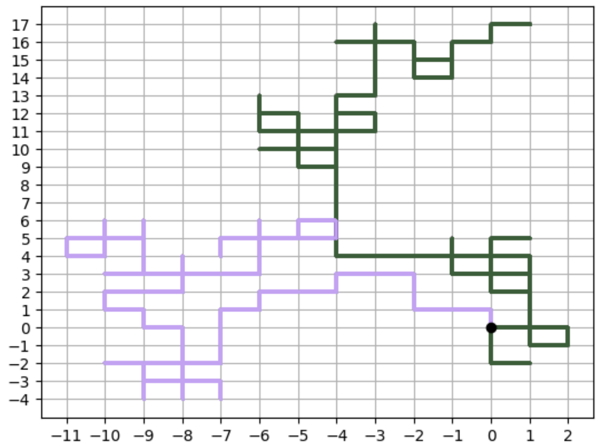

Figure 2: two 100-step random walks

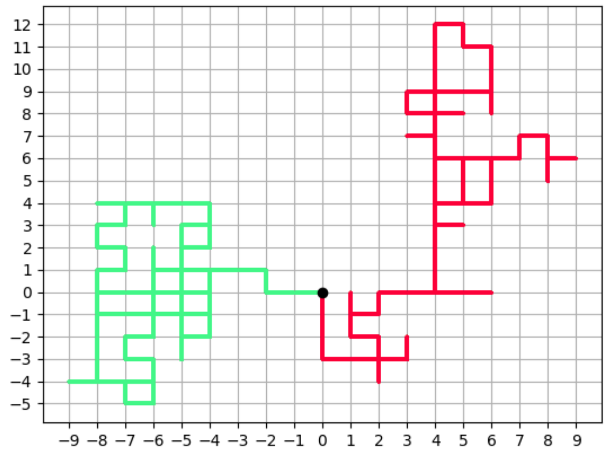

Figure 3: two 100-step random walks

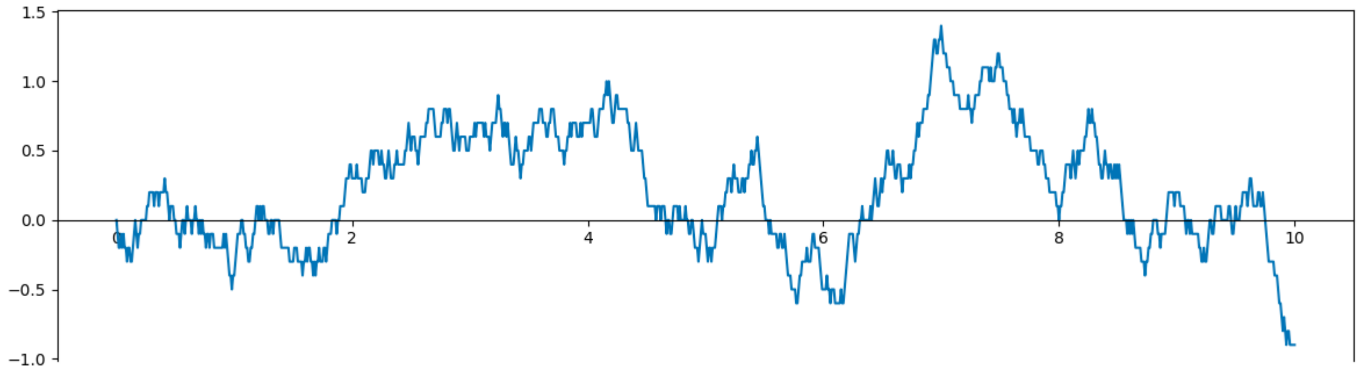

Figure 4: Simulation of Brownian Motion in 1 Dimension

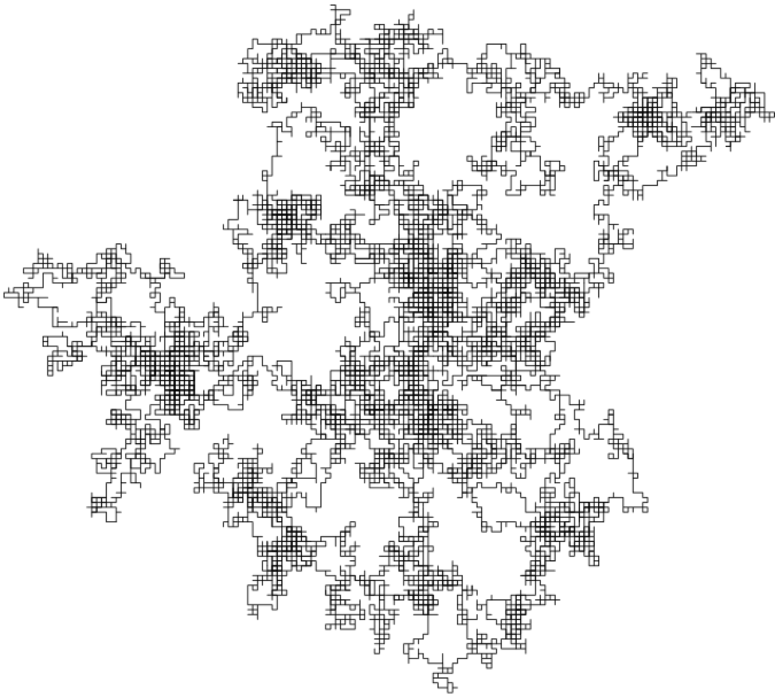

Figure 5: Simulation of Brownian Motion in 2 Dimensions[15]

7.1 Martingales

Proof of Corollary 3.2.3.1. Since $(M_n)_{n\in\mathbb{N}}$ is non-negative, it is bounded below by 0. Then

$$\sup_n \mathbb{E}[|M_n|] \leq \sup_n \mathbb{E}[M_n]$$

Since $(M_n)_{n\in\mathbb{N}}$ is a martingale, and any fixed $n \in \mathbb{N}$ is a stopping time, by Doob's Optional Stopping Theorem (3.2.2)

$$\mathbb{E}(M_0) = \mathbb{E}(M_n)$$

Therefore

$$\sup_n \mathbb{E}[M_n] = \mathbb{E}[M_0]$$

Thus we can apply Doob's Convergence Theorem (3.2.3) and so $\lim_{n\to\infty} M_n = M_\infty$ exists almost surely and $\mathbb{E}[|M_\infty|] < \infty$. □

Proof of Theorem 3.5.3. $(M_n)_{n\in\mathbb{N}}$ is a non-negative martingale, and thus by Corollary 3.2.3.1, $\lim_{n\to\infty} M_n = M_\infty$ exists almost surely and $\mathbb{E}[|M_\infty|] < \infty$.

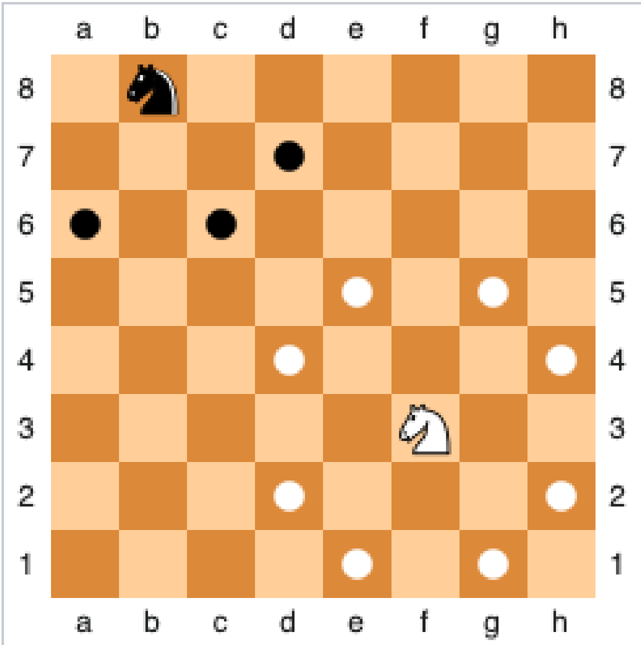

Figure 6: a Knight's movement on a chess board[8]

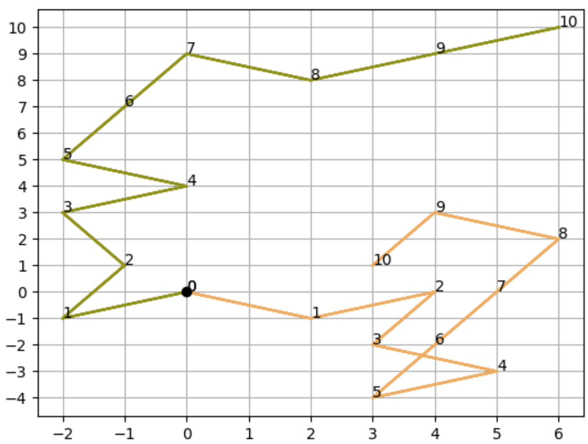

Figure 7: two 10-step knights walks

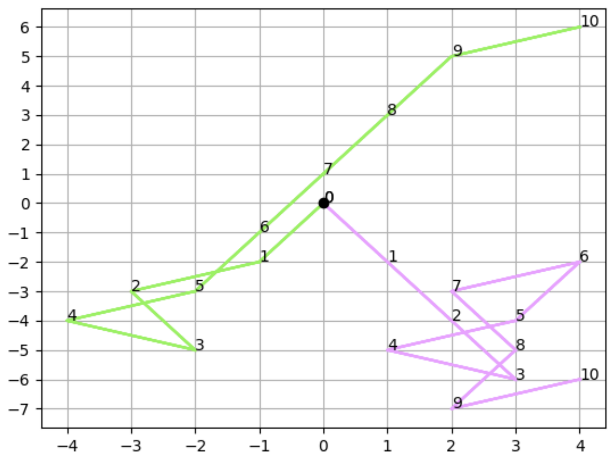

Figure 8: two 10-step knights walks

Suppose, by way of contradiction, that $M_\infty = c$, where $c \neq 0$ and $c \neq \sum_{i=0}^{\infty} \frac{1}{w_i}$.

Then

$$c = \lim_{n\to\infty} \sum_{i=0}^{X_{\tau\wedge n}-1} \frac{1}{w_i} = \sum_{i=0}^{k-1} \frac{1}{w_i}$$

for some $k \in \mathbb{N}$

This implies that $\mathbb{P}(X_{\tau\wedge n} = k) = 1$ which in turn implies that $\lim_{n\to\infty} X_{\tau\wedge n} = k$

But this implies that

$$\lim_{n\to\infty} |X_{n+1} - X_n| = |k - k| = 0$$

which is a contradiction since $|X_{n+1} - X_n| = 1$.

Therefore, as $n \to \infty$ either $M_n \to 0$ or $M_n \to \sum_{i=0}^{\infty} \frac{1}{w_i}$. □

Proof of the Doob Martingale 3.3.1. By definition $\mathbb{E}[|X|] < \infty$. Now we show that $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$

$$\begin{aligned}\mathbb{E}[X_{n+1}|\mathcal{F}_n] &= \mathbb{E}[\mathbb{E}[X|\,\mathcal{F}_n]|\mathcal{F}_n] \\ &= \mathbb{E}[X|\,\mathcal{F}_n] && \text{(Law of Total Expectation)} \\ &= X_n\end{aligned}$$

□

7.2 Renewal Processes

Proof of Theorem 4.2.1.

$$\begin{aligned}\mathbb{P}(N(\infty) < \infty) &= \mathbb{P}(X_n = \infty \quad \text{for some } n) \\ &= \mathbb{P}(\bigcup_{n=1}^{\infty} (X_n = \infty)) \\ &\leq \sum_{n=1}^{\infty} \mathbb{P}(X_n = \infty) \\ &= 0\end{aligned}$$

So we see that $\mathbb{P}(N(\infty) < \infty) = 0$, therefore $N(\infty) = \infty$ with probability 1. □

Proof of Theorem 4.2.2. ***DO*** □

Proof of Corollary 4.3.0.1. This is a corollary of Wald's Equation (3.3.2).

We can see that $N(t)+1$ is a stopping time for our sequence of inter-renewal times (X_n). Then by Wald's Equation:

$$\begin{aligned}\mathbb{E}[S_{N(t)+1}] &= \mathbb{E}\left[\sum_{i=1}^{N(t)+1} X_i\right] \\ &= \mathbb{E}[X]\mathbb{E}[N(t)+1] \\ &= \mu(m(t)+1)\end{aligned}$$

□

Proof of Theorem 4.3.1 (Elementary Renewal Theorem). Suppose first that $\mu < \infty$.

We will prove a lower bound of our limit first. We know that $S_{N(t)} \leq t < S_{N(t)+1}$ by the definition of $N(t)$. Using Corollary 4.3.0.1 we have that:

$$\begin{aligned}\mathbb{E}[S_{N(t)+1}] &> \mathbb{E}[t] \\ \mu(m(t)+1) &> t\end{aligned}$$

We rearrange:

$$\frac{m(t)+1}{t} > \frac{1}{\mu}$$

This implies that

$$\lim_{t\to\infty} \frac{m(t)+1}{t} > \lim_{t\to\infty} \frac{1}{\mu}$$

Simplifying, we obtain:

$$\liminf_{t\to\infty} \frac{m(t)}{t} \geq \frac{1}{\mu}$$

So we have proven the lower bound of our limit.

Now we prove the upper bound. We define a new 'truncated' renewal process. Fix a constant M and define a renewal process by the inter-renewal times:

$$X_n^* = \begin{cases} X_n & \text{if } X_n \leq M \\ M & \text{if } X_n > M \end{cases}$$

Then let $S_n^* = \sum_{i=1}^{n} X_i^*$ and $N^*(t) = \sup\{n : S_n^* \leq t\}$.

We can see that the inter-renewal times for this truncated renewal process are bounded by M. That is, we have that

$$S_{N(t)+1}^* = \sum_{i=1}^{N(t)+1} X_i^* \leq \sum_{i=1}^{N(t)+1} M \leq M + t$$

If we let $m^*(t)$ be the renewal function of (X_n^*) and μ^* the expected value of X_n^*. Then we have that

$$\mu^*(m^*(t)+1) \le t + M$$

by the same argument as above.

By rearranging we find that

$$\frac{m^*(t)+1}{t+M} \le \frac{1}{\mu^*}$$

which implies that

$$\limsup_{t\to\infty} \frac{m^*(t)}{t} \le \frac{1}{\mu^*}$$

Letting M go to infinity we find that

$$\limsup_{t\to\infty} \frac{m(t)}{t} \le \frac{1}{\mu}$$

Therefore

$$\limsup_{t\to\infty} \frac{m(t)}{t} \le \frac{1}{\mu} \le \liminf_{t\to\infty} \frac{m(t)}{t}$$

and so

$$\frac{m(t)}{t} \to \frac{1}{\mu} \qquad \text{as } t \to \infty$$

as required.

If we consider the case where $\mu = \infty$ we use the same truncated renewal process as above, and we obtain that

$$\limsup_{t\to\infty} \frac{m^*(t)}{t} \le \frac{1}{\mu^*}$$

Now we know that $\frac{1}{\mu} = 0$ since $\mu = \infty$ and we know that as $M \to \infty$, we have that $\mu^* \to \mu$ and $m^*(t) \to m(t)$. Therefore we obtain that:

$$\limsup_{t\to\infty} \frac{m(t)}{t} \le \frac{1}{\mu} = 0$$

But we also know that $m(t)$ and t are both non-negative, therefore $0 \le \limsup_{t\to\infty} \frac{m(t)}{t} \le \frac{1}{\mu} = 0$. And so we have that

$$\frac{m(t)}{t} \to \frac{1}{\mu} \qquad \text{as } t \to \infty$$

as required.

□

Proof of Theorem 4.4.2 (Reward Renewal Theorem). We will prove the first part of the theorem. (I) $\frac{R(t)}{t} \to \frac{\mathbb{E}[R]}{\mathbb{E}[X]}$

We begin by writing $R(t)$ as a sum.

$$\frac{R(t)}{t} = \frac{\sum_{i=0}^{N(t)} R_i}{t}$$

Next we multiply by $\frac{N(t)}{N(t)}$ to obtain

$$\frac{\sum_{i=0}^{N(t)} R_i}{t} = \left(\frac{\sum_{i=0}^{N(t)} R_i}{N(t)}\right)\left(\frac{N(t)}{t}\right)$$

By the strong law of large numbers we have that

$$\lim_{t\to\infty} \frac{\sum_{i=0}^{N(t)} R_i}{N(t)} = \mathbb{E}[R]$$

By 4.2.2 we know that

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mathbb{E}[X]}$$

Therefore

$$\lim_{t\to\infty} \left(\frac{\sum_{i=0}^{N(t)} R_i}{N(t)}\right)\left(\frac{N(t)}{t}\right) = \frac{\mathbb{E}[R]}{\mathbb{E}[X]}$$

□

7.3 Markov Chains

Proof of Theorem 5.3.2. We prove it for a finite Markov chain. We know that $p^{(n)} = p^{(0)}P^n \to \pi$ as $n \to \infty$ We also know that $p^{(0)}P^{n+1} = (p^{(0)}P^n)P$. Taking the limit of both sides we obtain $\pi = \pi P$ □

Proof of Theorem 5.2.2. $p^{(n)} = (p_1^{(n)}, p_2^{(n)}, p_3^{(n)}, \cdots)$ so therefore $(p^{(n)})_j = p_j^{(n)}$

$$\begin{aligned}
(p^{(0)}P^n)_j &= \sum_i p_i^{(0)} p_{ij}^{(n)} \\
&= \sum_i \mathbb{P}(X_0 = E_i)\mathbb{P}(X_n = E_j | X_0 = E_i) \\
&= \mathbb{P}(X_n = E_j) \\
&= p_j^{(n)} \\
&= (p^{(n)})_j
\end{aligned}$$

□

Proof of Theorem 5.4.1. By Theorem 4.4.2 we know that the asymptotic time spent in a state E_j equals the expected number of visits to state E_j in the given time S divided by the expected length of S.

We also know the limiting distribution of state E_j is equal to this.

Therefore we have that

$$\pi_j = \mathbb{E}(\text{number of visits to } E_j \text{ before time } S | X_0 = E_i)\mathbb{E}(S | X_0 = E_i)$$

which we rearrange to find

$$\mathbb{E}(\text{number of visits to } E_j \text{ before time } S | X_0 = E_i) = \pi_j \mathbb{E}(S | X_0 = E_i)$$

as required. □

Proof of Theorem 5.5.1. We know that the Markov chain describing this random walk has the transition matrix $P = (p_{ij})$ where

$$p_{ij} = \begin{cases} \frac{1}{\deg(E_i)} & \text{if } E_iE_j \text{ is an edge of G} \\ 0 & \text{if } E_iE_j \text{ is not an edge of G} \end{cases}$$

We know that π must satisfy $\pi = \pi P$ to be a stationary distribution of our Markov chain.

$$\begin{aligned} (\pi P)_i &= \sum_j \pi_j p_{ij} \\ &= \sum_{j:ij \in E(G)} \pi_j p_{ij} \\ &= \sum_{j:ij \in E(G)} \frac{\deg(E_j)}{2|E(G)|} \frac{1}{\deg(E_j)} \\ &= \sum_{j:ij \in E(G)} \frac{1}{2|E(G)|} \\ &= \frac{\deg(E_i)}{2|E(G)|} \\ &= (\pi)_i \end{aligned}$$

So π is a stationary distribution, as required. □