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Topos theory and Undirected Graphs

Alex Marciano

Supervised by Professor Finnur Lárusson & Dr David Baraglia

The University of Adelaide

AMSI AUSTRALIAN MATHEMATICAL SCIENCES INSTITUTE

Abstract

We investigate basic concepts of category and topos theory, in relation to undirected graphs. We demonstrate that undirected graphs form a valid category, with objects and morphism described. This category satisfies some of the required properties of a topos, but we prove one property is not met. Some variations of undirected graphs are also introduced, and proven to not be topoes.

1 Introduction

The notion of a topos (a.k.a elementary topos) is a fundamental concept of category theory. A topos is a category that shares several properties with the category of sets. For this reason toposes are can be thought of as 'nice places to do math'. That is, many useful mathematical constructions of familiar sets, have a direct correspondent in any topos [3]. Toposes appear in many areas of mathematics, for example geometry, topology, combinatorics, and logic.

In particular, this report focuses on reconciling aspects of topos theory with *undirected graphs*. Simply, an undirected graph is some collection of vertices and edges between them, such as below.

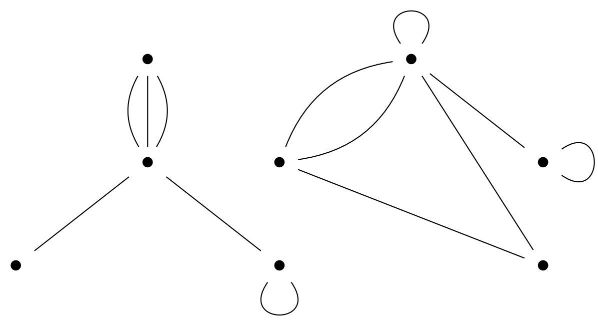


Figure 1: We represent vertices with •, and edges with lines.

Undirected graphs, and their perhaps more well known counterpart, directed graphs, have many application in areas such as computer science, optimisation, group theory and geometry. We are motivated by previous work, [1], where is was proven that *directed* graphs do indeed form a topos.

In this report we show undirected graphs do satisfy some of the required topos properties in a hands-on manner. However, we are able to *disprove* undirected graphs being a topos via counterexample, which is the main result. Furthermore, we discuss several variations of directed graphs, by placing some extra restrictions on their structure. Each of these variations is shown to not be a topos as well.

1.1 Statement of Authorship

Under direct supervision of Professor Lárusson and Dr Baraglia, all proofs of results and report structure authored by Marciano. Category theoretic definitions are assumed to be understood from *Basic Category*

Theory [2]. Some research focus inspired by previous AMSI student, Jiang [1], who focused on category of directed graphs.

2 Some definitions

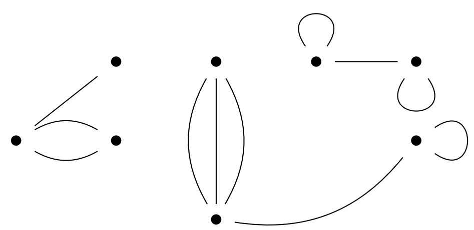
Definition 1. A **topos** is a category with

- a terminal object
- a subobject classifier
- products of any two objects (and hence all finite products)
- all exponentials

Remark 2. A topos is a category which behaves similarly to the category of sets and set functions, **Set**. We will briefly explain the preceding topos properties and what they mean in **Set**.

- A terminal object is an object T of a category which admits precisely 1 map from every other object. In **Set** the terminal object is the single object set, say $T = \{1\}$. Terminal objects are always unique up to isomorphism.
- A subobject classifier is an object Ω, such that for any object A, $\{\text{subobjects of } A\} \cong \text{map}(A, \Omega)$. In **Set**, the subobject classifier is the 2 object set, say $\Omega = \{0, 1\}$. Subobject classifiers are unique up to isomorphism.
- The categorical product is a generalisation of the familiar Cartesian product in **Set**. It must satisfy a universal property (more on this later), which relies on being able to project a product back to its factors.
- An exponential object A^B is a generalisation of functional spaces in **Set** which must satisfy a natural isomorphism. That is, an exponential A^B in **Set**, is the set of functions $B \to A$. Exponentials existing also require the existence of products in the category (again, more on this later).

Definition 3. An **undirected graph** is a set of vertices (or nodes) V and set of edges E between vertices. We will permit multiple edges between vertices and loops on vertices. Also note V and E need not be finite. For example, we may have an undirected graph as below



For every undirected graph we also have a function $\varepsilon : E \to V^2/\sim$, which maps an edge to the vertices it connects, defining undirected graphs as triples (V, E, ε). Note $\sim$ is an equivalence relation on V^2 where $(x, y) \sim (y, x)$ for all $x, y \in V$, so $V^2/\sim$ is the set of all *unordered* pairs of vertices.

Definition 4 (alternative to def. 3)**.** We may also speak of an **undirected graph**, $\mathcal{G}$, as a vertex set V, and for any pair of vertices $a, b \in V$, an edge set $\mathcal{G}(a, b)$, of the edges between a and b. We must have the requirement that $\mathcal{G}(a, b) = \mathcal{G}(b, a)$, due to the undirected nature of the edges. There is also (the perhaps superfluous) property that $\mathcal{G}(a, b) \cap \mathcal{G}(a', b') = \varnothing$, unless there is equality of sets $\{a, b\} = \{a', b'\}$. This alternative definition will mainly be useful when discussing the product of undirected graphs.

Definition 5. Further, undirected graphs form a category, which we denote $\mathscr{U}$. The objects of $\mathscr{U}$ are undirected graphs. Now say we have $A, B \in \mathscr{U}$, with $A = (A_0, A_1, \varepsilon_A)$ and $B = (B_0, B_1, \varepsilon_B)$. A morphism in $\mathscr{U}(A, B)$ is a pair (f, g) of vertex and edge maps; $f : A_0 \to B_0$ and $g : A_1 \to B_1$. Then f always induces a function on the unordered pairs, which we denote

$$f^2_\sim : (A_0)^2/\sim \to (B_0)^2/\sim$$
$$[v_1, v_2] \mapsto [f(v_1), f(v_2)]$$

(the square bracket notation [ ] refers equivalence class of pairs, with respect to $\sim$). Importantly, for any such morphism we require the condition: $\varepsilon_B \circ g = f^2_\sim \circ \varepsilon_A$, which ensures edges are preserved. Now for composition, say $A, B, C \in \mathscr{U}$, with $(f, g) \in \mathscr{U}(B, C)$ and $(f', g') \in \mathscr{U}(A, B)$. Then define the composition $(f, g) \circ (f', g') = (f \circ f', g \circ g')$ as a map in $\mathscr{U}(A, C)$. Lastly, there is an identity on any undirected graph, simply map every vertex to itself and every edge to itself.

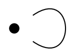
3 Is the category of undirected graphs a topos?

Proposition 6. *The category $\mathscr{U}$ of undirected graphs has a terminal object, subobject classifier, and products of any two objects.*

Proof.

Terminal Object

A terminal object of $\mathscr{U}$ is a graph $T \in \mathscr{U}$ such that for all $A \in \mathscr{U}$, there is exactly one map $A \to T$. The graph with a single vertex and loop is the terminal object in $\mathscr{U}$, that is

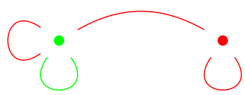
$T =$ 

This is because, for any arbitrary graph $A \in \mathscr{U}$ there is only one map $A \to T$: take all the vertices of A to the vertex of T, and all the edges of A to the loop of T, which is always a valid morphism of graphs.

Subobject Classifier

A subobject of a graph is a selection of vertices and edges, such that whenever an edge is selected, the vertices it connects must also be selected.

The subobject classifier in $\mathscr{U}$ is

$\Omega =$ 

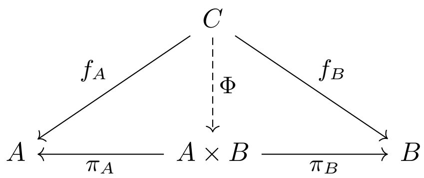
where

- a selected vertex is mapped to green vertex
- an unselected vertex is mapped to red vertex
- a selected edge is mapped to green loop
- an unselected edge, between selected vertices, is mapped to left red loop
- an unselected edge, between unselected vertices, is mapped to right red loop
- an unselected edge between selected and unselected vertices, is mapped to red edge between vertices.

then for any graph $A \in \mathscr{U}$ the one-to-one correspondence holds: $\{\text{subobjects of } A\} \cong \mathscr{U}(A, \Omega)$.

Product Objects

Let $A, B \in \mathscr{U}$ be undirected graphs. We want to show there is a well defined product graph $A \times B \in \mathscr{U}$, which satisfies a universal property. There must also exist projection maps π_A, π_B, which map the product graph back to each of its respective factors. Now for the universal property, for any $C \in \mathscr{U}$ which has maps $f_A : C \to A$ and $f_B : C \to B$, there must exist a *unique* $\Phi : C \to A \times B$ such that the diagram



commutes. We are guided by this universal property in how to define the product and projections. Firstly, say A_0, B_0 are the vertex sets of A and B respectively. Then for the vertex set of the product, we say $(A \times B)_0 = A_0 \times B_0$. That is, vertices in the product are ordered pairs of vertices from A and B. Then, let $a_1, a_2 \in A_0$ and $b_1, b_2 \in B_0$ be vertices, and define $(A \times B)((a_1, b_1), (a_2, b_2)) = A(a_1, a_2) \times B(b_1, b_2)$, as the set of edges between (a_1, b_1) and (a_2, b_2). That is, we can think of edges in the product as ordered pairs also. However, note well that we must consider the sets $(A \times B)((a_1, b_1), (a_2, b_2))$ and $(A \times B)((a_1, b_2), (a_2, b_1))$ to be *disjoint*, unless $\{(a_1, b_1), (a_2, b_1)\} = \{(a_1, b_2), (a_2, b_1)\}$, in which case they really are the same edge set. Without this condition, we run into problems with identification of the edges. There are perhaps many artificial ways we could impose this condition, but we will simply assert it.

Now for the projection maps, consider the projection to the first component, $\pi_A : A \times B \to A$. We have vertex projections,

$$(A \times B)_0 \to A_0$$
$$(a, b) \mapsto a,$$

and edge projections

$$(A \times B)((a_1, b_1), (a_2, b_2)) \to A(a_1, a_2)$$
$$(\alpha, \beta) \mapsto \alpha.$$

There is of course an analogous definition for $\pi_B : A \times B \to B$, where we just project to the second factor.

Then for the universal property, let $\Phi = (\Phi_0, \Phi_1)$ where Φ_0 maps the vertices and Φ_1 maps the edges, and similarly denote $f_A = (f_0^A, f_1^A)$, $f_B = (f_0^B, f_1^B)$. Then the apparent map of vertices is $\Phi_0 = (f_0^A, f_0^B)$, meaning

$$\begin{aligned} \Phi_0 : C_0 &\to (A \times B)_0 \\ c &\mapsto (f_0^A(c), f_0^B(c)). \end{aligned}$$

Next, the edge map is a bit more complicated, as we can not easily write down an explicit rule. Consider a single edge $e \in C$, connecting vertices X and Y: $X \overset{e}{\text{———}} Y$ (where we may have $X = Y$). Then Φ_1 maps e to the pair $(f_1^A(e), f_1^B(e))$, specifically viewed as an element of $(A \times B)\left(\left(f_0^A(X), f_0^B(X)\right), \left(f_0^A(Y), f_0^B(Y)\right)\right)$, rather than the alternative $(A \times B)\left(\left(f_0^A(X), f_0^B(Y)\right), \left(f_0^A(Y), f_0^B(X)\right)\right)$. In this way $\Phi_1(e)$ joins $\Phi_0(X)$ and $\Phi_0(Y)$, making Φ a valid graph morphism. Also, defining Φ in this way satisfies the commutativity: $\pi_A \circ \Phi = f_A$ and $\pi_B \circ \Phi = f_B$.

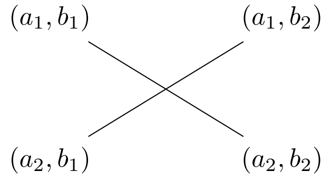
It should be clear that if we had any other map which was a valid morphism of graphs and satisfied the commutativity triangle above, it must map vertices and edges is the same way to Φ. That is, our Φ is unique, and the universal property is satisfied. □

Example 7 (product graphs)**.** We present some examples of undirected graphs and their products, as per the above discussion. First, let

$$A = \ a_1 \text{ ——— } a_2 \qquad\qquad B = \ b_1 \text{ ——— } b_2 \ ,$$

then

$A \times B =$

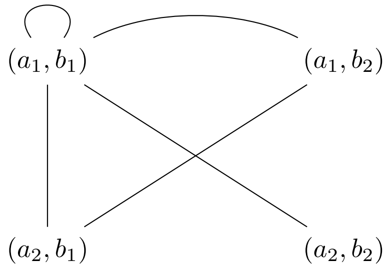


This is quite an informative example, and will be used in the next proof for exponentials. Observe edges are in some sense 'duplicated' in the product.

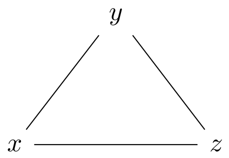
Next we check how loops operate in the product. For a similar example as before, let

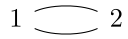
then

$A \times B =$



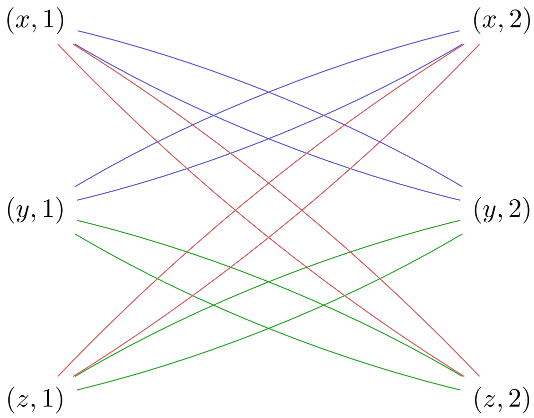
Note that loops are not 'duplicated' in the same way as regular edges. For the last product example, let

$A =$ 

$B =$ 

,

and then

$A \times B =$ 

Now with the first three topos properties satisfied, we might expect the forth to follow, making $\mathscr{U}$ a topos. However we will see a counterexample disproves all exponential objects existing.

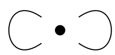
Theorem 8. *The category $\mathscr{U}$ of undirected graphs does not have all exponentials, hence is not a topos.*

Proof. Let $C, B \in \mathscr{U}$, then if we assume $\mathscr{U}$ has all exponentials, there is an object $C^B \in \mathscr{U}$ such that for all $A \in \mathscr{U}$ the natural isomorphism,

$$\mathscr{U}(A \times B, C) \cong \mathscr{U}(A, C^B),$$

holds. Our strategy is to set some particular B, C and choose different A to determine properties C^B must satisfy, eventually leading to contradiction. So we set

$B =$

$C =$ 

and our first choice of A is a simply a single vertex,

$A_1 =$

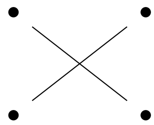
Then for the LHS of the isomorphism, we must compute $A_1 \times B$. Since A_1 is a single vertex, the number of vertices in the product will be the number of vertices of B, and since A_1 has no edges, neither will the product. So we will get the vertex set of B:

$A_1 \times B =$

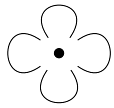
The number of maps $A_1 \times B \to C$ is 1, since there is only 1 choice of vertex in C. Now for the RHS, we note maps $A_1 \to C^B$ are equinumerous to the number of vertices of C^B. So if C^B exists and satisfies the isomorphism, it must have precisely 1 vertex. Next, we want to determine the edges of C^B, so choose

$A_2 =$

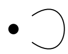
Then for the product $A_2 \times B$ we have a graph with 4 vertices and 2 edges as follows (we discussed this product earlier)

$A_2 \times B =$ 

The number of maps $A_2 \times B \to C$ is 4, as each of the 2 edges in the product have 2 choices of edge to be mapped to in C. Then for the RHS we consider maps $A_2 \to C^B$, given that C^B has a single vertex. For there to be the required 4 maps, C^B must be the graph with a single vertex and 4 loops:

$C^B =$ 

Now for the last choice of A, we use a single vertex and loop

$A_3 =$ 

Again we must compute the product $A_3 \times B$. This product will be same graph as B itself; the single vertex of A_3 'maintains' the vertices of B, and similarly for the single loop.

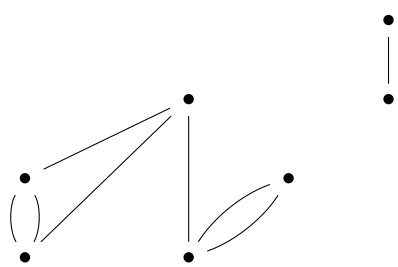
$A_3 \times B =$

Then we see the number of maps $A_3 \times B \to C$ is 2, since there is only 1 choice for mapping the vertices, and 2 choices of edge mapping. So, to satisfy the isomorphism, we require exactly 2 maps $A \to C^B$. However, the previous choices of A established C^B is a single vertex with 4 loops, which has 4 ways of being mapped to from A_3 (send the single loop of A_3 to any of the 4 loops). This is a contradiction, meaning the exponential can not exist for this particular B and C. Further, this proves $\mathscr{U}$ is *not* a topos category. $\square$

4 Variations of Undirected graphs

In this section we examine some variations of undirected graphs, and check whether they form topos categories.

Definition 9. Let $\mathscr{U}_\varnothing$ be the category of undirected graphs *without loops*. This is a valid category, as we can just adapt definitions 3 and 5 with the requirement that every edge connects distinct vertices. This category still allows graphs with multiple edges, so an object of $\mathscr{U}_\varnothing$ may look like



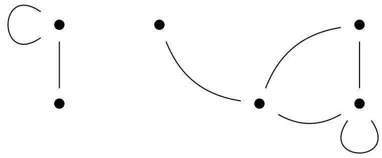
Theorem 10. *The category $\mathscr{U}_\varnothing$, of undirected graphs without loops, is not a topos.*

Proof. For a category to not be a topos, we just need to find one of the 4 topos properties that is not satisfied. The simplest property which fails for this category is the existence of a terminal object. Recall for the category $\mathscr{U}$, we found the terminal object T was the graph with a single vertex and single loop. This ensured there was precisely one map to T from any other graph. However, for this category we can not use the same object, since loops are disallowed. Instead, assume $\mathscr{U}_\varnothing$ has a terminal object, say $T_\varnothing$. Then for the single vertex graph $G = \bullet$, there must be precisely one map $G \to T_\varnothing$. This implies $T_\varnothing$ must have a single vertex; any more and there are multiple maps $G \to T_\varnothing$. In this category, a graph with single vertex can not have any edges (at least 2 vertices required for an edge). Hence, if it exists, $T_\varnothing = \bullet$. Next, if we let

$$G' = \bullet \text{ ——— } \bullet$$

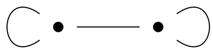
and consider maps $G' \to T_\varnothing$, we have a contradiction. There are no maps from G', since there is nowhere to send the edge. So we can negate our assumption, and conclude $\mathscr{U}_\varnothing$ does not have a terminal object. □

Definition 11. Let $\mathscr{U}_{\neq}$ be the category of undirected graphs *without multiple edges* between vertices. That is, for any two vertices, there is at most one edge connecting them. This applies to loops also, we allow at most one loop on any vertex. Again, we can see this a valid category from the same reasoning in definitions 3, 5. An example of some graph in $\mathscr{U}_{\neq}$ is



Theorem 12. *The category $\mathscr{U}_{\neq}$, of undirected graphs without multiple edges between vertices, is not a topos.*

Proof. Similar to the previous proof, we find one topos property that is not satisfied. Note for this category, the terminal object does exist, simply use the same single vertex loop from $\mathscr{U}$. Instead, consider the existence of subobject classifier Ω. We note that a graph which is a subobject classifier must have precisely 2 vertices, to distinguish between selected and unselected subobject vertices. In our category $\mathscr{U}_{\neq}$, the maximum edges we can equip a graph with 2 vertices is



This is a problem, since we have only 3 edges, and there are 4 distinct types of edges which need to be classified:

- a selected edge
- an unselected edge, between selected vertices
- an unselected edge, between unselected vertices

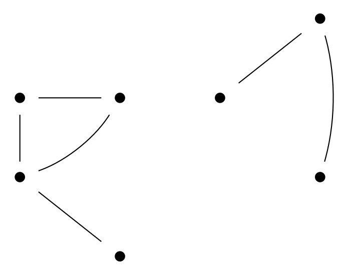
- an unselected edge between selected and unselected vertices

hence such a graph cannot be a subobject classifier. □

Remark 13. Note, if we alter the definition of $\mathscr{U}_{\neq}$ to allow for multiple loops on a single vertex, then the category has both terminal object and subobject classifier. Products could also be define analogously to $\mathscr{U}$, however this category would still not have all exponentials. We could just present the same counterexample as earlier for $\mathscr{U}$, which involved a double loop.

We now present one more variation of $\mathscr{U}$, by combining both of the previous.

Definition 14. Let $\mathscr{G}$ be the category of undirected graphs without loops, and without multiple edges between vertices. Such graphs are sometimes called **simple graphs**. Such a simple graph may look like



Refer to initial definitions as to why we can think of these graphs as objects of a category.

Remark 15. The categories of variations of undirected graphs we introduced, $\mathscr{U}_{\varnothing}$, $\mathscr{U}_{\neq}$ and $\mathscr{G}$, are all subcategories of our initial $\mathscr{U}$. This is because the objects of each category are just a subclass of the objects of $\mathscr{U}$. Further, each of the variations are *full subcategories* of $\mathscr{U}$ since mappings between graphs in any of the subcategories are exactly mappings between the same graphs in $\mathscr{U}$.

Corollary 16. *The category $\mathscr{G}$ of simple graphs is not a topos.*

Proof. Since $\mathscr{G}$ has graphs with neither loops nor multiple edges, we use the previous proofs to say it can not have a terminal object, and it can not have a subobject classifier. Hence the category is not a topos. □

5 Conclusion

We introduced and studied the category of undirected graphs $\mathscr{U}$ (with loops and multiple edges allowed). Our main focus was to check this category against each of the topos properties, to determine whether they all held. We found $\mathscr{U}$ has terminal object, subobject classifier and products of any two objects, however a counterexample showed exponentials do not always exist. Following, 3 subcategories of undirected graphs were defined, and each proven to not be topoes.

Future direction for research may be to look for other subcategories of undirected graphs that satisfy all the topos properties. Perhaps a stronger result would be to prove/disprove the existence of an appropriate topos category of undirected graphs.

6 Acknowledgements

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7 References

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