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Categories of directed and undirected graphs

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Abstract

This project mainly investigate the connection between the category of directed graphs and category of undirected graph, by finding adjunctions to the forgetful functor. In addition, some various definitions of graphs are introduced and relations between those definitions are discussed. The major result of this project is answering the existence of adjunctions to the forgetful functor when different definitions of graphs are adopted.

Contents

1 Introduction

Intuitively, one can obtain an undirected graph by forgetting how the edges of a directed graph is directed. The first question come into mind is to find a reasonable way to undo this process. However, it is impossible to exactly recover the original directed graph once you forget the orientation of each edges. The main theme of this project is to find a way to convert an undirected graph to a directed graph which interact well with the forgetting process.

Statement of Authorship

The results of this report are developed and written by Chun Hei Lee under the guidance provided by Professor Finnur Larusson and Dr David Baraglia.

2 Categories of graphs and forgetful functor

In this section, we will introduce the category of directed graphs **DG**, category of undirected graphs **UG** and the forgetful functor U.

2.1 Directed graphs

Definition 2.1. A *directed graph* A is a 4-tuple (V, E, s, t) where V, E are sets and $s, t : E \to V$ are functions.

- An element $v \in V$ is called a *vertex*.
- An element $e \in E$ is called an *edge*.
- The function s maps an edge e to a vertex $s(e)$ called the *source* of that edge.
- The function t maps an edge e to a vertex $t(e)$ called the *target* of that edge.

We say an edge is oriented from its source to its target. In addition, we can characterise the vertices and edges of a graph by the following properties.

Definition 2.2. Let $A = (V, E, s, t)$ be a directed graph. An edge e is called a *loop* if $s(e) = t(e)$, a *line* otherwise. We say A is *loopless* if $\{e \in E : s(e) = t(e)\} = \varnothing$. A vertex is called *isolated vertex* if it is neither a source nor a target of any edge, denote the set of isolated vertex as V_{iso}, i.e. $V_{\text{iso}} := V \setminus (s(E) \cup t(E))$.

Definition 2.3. Let $A_1 = (V_1, E_1, s_1, t_1)$ and $A_2 = (V_2, E_2, s_2, t_2)$ be directed graphs. A *directed graph homomorphism* $\alpha : A_1 \to A_2$ consists of a pair of maps $\langle \alpha_V : V_1 \to V_2, \alpha_E : E_1 \to E_2 \rangle$ which satisfy

$$s_2\alpha_E = \alpha_V s_1 \quad \text{and} \quad t_2\alpha_E = \alpha_V t_1 \tag{2.1}$$

Remark 2.4. Condition (2.1) is called *graph homomorphism property*. It requires α to preserve the orientation of edges, hence is a structure preserving map. When we write the pairs of maps of α, we use angle bracket

$\langle \alpha_V, \alpha_E \rangle$ to emphasis it is a directed graph homomorphism rather than functions with ordered pairs as output. We always put the vertex map on the first entry and the edge map on the second entry.

Proposition 2.5. *A loop cannot be map into a line through directed graph homomorphism.*

Proof. Let E_1 be a edge set of directed graph A_1 and $e \in E_1$ be a loop. By way of contradiction, assume there exist a graph homomorphism α maps e to a line $\alpha_E(e) \in E_2$. Since e is a loop, $s_1(e) = t_1(e)$. Apply α_V on both side give $\alpha_V(s_1(e)) = \alpha_V(t_1(e))$ and (2.1) imply $s_2(\alpha_E(e)) = t_2(\alpha_E(e))$. On the other hand, $\alpha_E(e)$ is a line yields $s_2(\alpha_E(e)) \neq t_2(\alpha_E(e))$, this is absurd. $\square$

Remark 2.6. Now if we see directed graphs as objects and graph homomorphisms as morphisms, a category of directed graphs **DG** is formed. A natural way to define the composition of morphisms is to compose the maps termwise, in particular, suppose $\alpha_1 : A_1 \to A_2$, $\alpha_2 : A_2 \to A_3$ and $\alpha_1 = \langle \alpha_{1V}, \alpha_{1E} \rangle$, $\alpha_2 = \langle \alpha_{2V}, \alpha_{2E} \rangle$, we have

$$\alpha_2 \circ \alpha_1 : A_1 \to A_3 \quad \text{and} \quad \alpha_2 \circ \alpha_1 := \langle \alpha_{2V} \circ \alpha_{1V}, \alpha_{2E} \circ \alpha_{1E} \rangle.$$

In this manner, the identity morphism is the identity map on vertex and edge. The associativity and unity of graph homomorphisms follow through the associativity and unity of ordinary functions. To be rigorous, we should verify the composed morphism $\alpha_2 \circ \alpha_1$ satisfy (2.1). We prove this on the source function and the target function is analogous. Suppose $A_1 = (V_1, E_1, s_1, t_1)$, $A_2 = (V_2, E_2, s_2, t_2)$ and $A_3 = (V_3, E_3, s_3, t_3)$. Then apply graph homomorphism property and associativity repeatedly yields

$$s_3 \circ (\alpha_{2E} \circ \alpha_{1E}) = (s_3 \circ \alpha_{2E}) \circ \alpha_{1E} = \alpha_{2V} \circ s_2 \circ \alpha_{1E} = \alpha_{2V} \circ (\alpha_{1V} \circ s_1) = (\alpha_{2V} \circ \alpha_{1V}) \circ s_1.$$

Hence, we have defined the category of directed graphs **DG**.

2.2 Quotient and universal property

Intuitively, an undirected graph is a graph whose edges have no preferred orientation. We can always get an undirected graph from a directed graph by forgetting the orientation of every edge. In fact, this can be made precise into a mathematical statement. Let us take a step back and develop some theory on equivalence relation and quotient sets.

Definition 2.7. Let $\sim$ be an equivalence relation on a set X. Denote $q : X \to X/_\sim$ to be the *canonical surjection*, i.e. $x \mapsto [x]_\sim$.

Theorem 2.8 (universal property of quotients)**.** *Let $\sim$ be an equivalence relation on a set X. If $f : X \to Y$ satisfy $x_1 \sim x_2 \implies f(x_1) = f(x_2)$ for any $x_1, x_2 \in X$, then there exist unique $\overline{f} : X/_\sim \to Y$ such that $f = \overline{f}q$.*

Proof. We claim $\overline{f} : X/_\sim \to Y$ is defined by $[x]_\sim \mapsto f(x)$. To show existence, we have to justify this map is well-defined. Note $x_1, x_2 \in [x]_\sim \implies x_1 \sim x_2 \implies f(x_1) = f(x_2)$, hence any representative of the equivalence class give the same output. Simple calculation shows $f = \overline{f}q$. To see uniqueness, suppose $f' : X/_\sim \to Y$ is an arbitrary map satisfy $f = f'q$, then for any $x \in X$, we have $f(x) = f'(q(x)) = f'([x]_\sim)$. In other words, $f' : [x]_\sim \mapsto f(x)$. $\square$

Next, we are going to define *the* equivalence relation to be used in this report.

Definition 2.9. Given a set X, we define the equivalence relation on X^2 by $(x_1, x_2) \sim (x_2, x_1)$.

Remark 2.10.
1. Instead of writing the equivalence class $[(x_1, x_2)]_\sim$ in full terms, $[x_1, x_2]$ is used to represent the set of equivalent class. Square brackets will only be used to denote equivalence class in this report.
2. For any $x_1, x_2 \in X$, follow thought $(x_1, x_2) \sim (x_2, x_1)$, we have $\#[x_1, x_2]$ be either 1 or 2. If $\#[x_1, x_2] = 1$, then $x_1 = x_2$ and $[x_1, x_2] = \{(x_1, x_1)\}$. Else if $\#[x_1, x_2] = 2$, then $x_1 \neq x_2$ and $[x_1, x_2] = \{(x_1, x_2), (x_2, x_1)\}$.

Since $\sim$ is defined on X^2, it is expected that we want to extend our maps from taking in one element to a pair of elements, a shorthand notation is introduced below.

Definition 2.11. For any $f : X \to Y$, define $f^\dagger : X^2 \to Y^2$ by $f^\dagger = (f, f)$. i.e. $f^\dagger : (x_1, x_2) \mapsto (f(x_1), f(x_2))$.

Corollary 2.12. *Let X, Y be sets. Given $f : X \to Y$ and obtain $f^\dagger : X^2 \to Y^2$ from the definition above, there exist a unique $\overline{qf^\dagger} : X^2/_\sim \to Y^2/_\sim$ such that $qf^\dagger = \overline{qf^\dagger}q$.*

Proof. Observe that any $x, x' \in X^2$ satisfy $x \sim x' \implies qf^\dagger(x) = qf^\dagger(x')$. Apply theorem 2.8 to $qf^\dagger$ yields the claim. ☐

Remark 2.13.
1. The expression $\overline{qf^\dagger}$ is quite bulky, we simplify it to $f^\ddagger$. In short, if $f : x \mapsto f(x)$, then $f^\ddagger : [x_1, x_2] \mapsto [f(x_1), f(x_2)]$.
2. Technically, $(-)^\ddagger$ is a functor from **Set** to **Set** that send a set X to $X^\ddagger = X^2/_\sim$ and send function $f : X \to Y$ to the unique function $f^\ddagger : X^\ddagger \to Y^\ddagger$. We will use this notation from now on, reader may fill in the details for this functor.

2.3 Undirected graphs and forgetful functor

Definition 2.14. An undirected graph B is a triple (V, E, u) where V, E are sets and $u : E \to V^\ddagger$ is a function.

- An element $v \in V$ is called a *vertex*.
- An element $e \in E$ is called an *(undirected) edge*.
- The function u maps an undirected edge e to an equivalence class of vertex pairs $u(e)$, we say $u(e)$ is *connected* by e.

Remark 2.15. For any undirected graph $B = (V, E, u)$ and $e \in E$, from remark 2.10, $\#u(e)$ is either 1 or 2. An edge e is a *loop* if $\#u(e) = 1$, a *line* if $\#u(e) = 2$. So, B is *loopless* if $\#u(e) = 2$ for all $e \in E$.

Definition 2.16. Let $B_1 = (V_1, E_1, u_1)$ and $B_2 = (V_2, E_2, u_2)$ be undirected graphs. An *undirected graph homomorphism* $\beta : B_1 \to B_2$ consists of a pair of maps $\langle \beta_V : V_1 \to V_2, \beta_E : E_1 \to E_2 \rangle$ which satisfy

$$u_2 \beta_E = \beta_V^\ddagger u_1. \tag{2.2}$$

Remark 2.17. 1. Condition (2.2) requires β to preserve adjacency relations. Similar to directed graph homomorphisms, we use angle bracket $\langle \beta_V, \beta_E \rangle$ to write the pairs of map defining the undirected graph homomorphisms β.

2. Likewise, the category of undirected graphs **UG** has undirected graphs as objects and undirected graph homomorphisms as morphisms. The composition of morphisms is defined as termwise composition and the identity morphism is just identity maps in both vertex and edge. Checking the composition respect (2.2) and the associativity, unity axioms are left as an exercise for the reader.

Definition 2.18. The forgetful functor $U : \mathbf{DG} \to \mathbf{UG}$ consists of:

- function on objects:

$$\mathrm{ob}(\mathbf{DG}) \to \mathrm{ob}(\mathbf{UG}), \qquad A = (V, E, s, t) \mapsto U(A) = (V, E, q \circ (s, t)) \tag{2.3}$$

  where $q \circ (s, t) : E \to V^{\ddagger}$ by $e \mapsto q(s(e), t(e))$;

- any objects $A_1, A_2 \in \mathrm{ob}(\mathbf{DG})$, a function on morphisms:

$$\hom_{\mathbf{DG}}(A_1, A_2) \to \hom_{\mathbf{UG}}(U(A_1), U(A_2)), \qquad \alpha \mapsto U(\alpha) = \alpha. \tag{2.4}$$

Loosely speaking, the forgetful functor only change the way to relate edges and vertices by forgetting which vertex is the source and which vertex is the target by merging them together as an equivalence class. Despite of the ease of understanding the forgetful functor, reader should check the definition above indeed define a functor.

Proposition 2.19. *U is faithful but not full.*

Proof. Let $A_1, A_2 \in \mathrm{ob}(\mathbf{DG})$, the function on morphisms $\hom_{\mathbf{DG}}(A_1, A_2) \to \hom_{\mathbf{UG}}(U(A_1), U(A_2))$ given by U is the identity function and hence injective, proves the faithfulness. Now fix $A_1 = \underset{v_1}{\bullet} \overset{e_1}{\underset{e_2}{\rightrightarrows}} \underset{v_2}{\bullet}$, $A_2 = \underset{w_1}{\bullet} \overset{d_1}{\underset{d_2}{\rightleftarrows}} \underset{w_2}{\bullet}$. Then, $U(A_1) = \underset{v_1}{\bullet} \overset{e_1}{\underset{e_2}{=\!=}} \underset{v_2}{\bullet}$, $U(A_2) = \underset{w_1}{\bullet} \overset{d_1}{\underset{d_2}{=\!=}} \underset{w_2}{\bullet}$. Consider $\lambda \in \hom_{\mathbf{UG}}(U(A_1), U(A_2))$ does the following: $(v_1 \mapsto w_1, v_2 \mapsto w_2, e_1 \mapsto d_1, e_2 \mapsto d_2)$. However, λ fails to preserve orientation imply $\lambda \notin \hom_{\mathbf{DG}}(A_1, A_2)$. The surjectivity for the map of a particular pair of object is disproved and therefore U is not full. $\square$

This says that U is not an equivalence because a functor is an equivalence if and only if it is full, faithful and essentially surjective on objects. However, in some sense, there is a weak form of equivalence, namely adjunction. So, it guides us to the main goal of this project, finding an (left or right) adjoint to U.

3 Adjunctions of the forgetful functor

We will first give the definition of adjunction between functors and determine whether U has a left or right adjoint. The definition of adjunction is adopted from Leinster's book, *Basic Category Theory*, Chapter 2 [1].

3.1 Definition of adjunction

Definition 3.1. Let $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$ be categories and functors. F is a *left adjoint* to G, and G is a *right adjoint* to F, and write $F \dashv G$ if

$$\hom_{\mathcal{B}}(F(A), B) \cong \hom_{\mathcal{A}}(A, G(B)) \tag{3.1}$$

for all $A \in \mathrm{ob}(\mathcal{A})$ and $B \in \mathrm{ob}(\mathcal{B})$ such that the *naturality axiom* is satisfied.

We will state the *naturality axiom* shortly. Beforehand, some notations and terminology are introduced to make the work precise. The bijection between hom-sets in (3.1) can be described by a pair of mutually inverse functions, $\mathrm{tp} : \hom_{\mathcal{B}}(F(A), B) \to \hom_{\mathcal{A}}(A, G(B))$ and $\mathrm{tp}^* : \hom_{\mathcal{A}}(A, G(B)) \to \hom_{\mathcal{B}}(F(A), B)$. Those functions are called *transpose*, by mutually inverse, we meant that $\mathrm{tp}^* \circ \mathrm{tp} = \mathrm{id}_{\hom_{\mathcal{B}}(F(A),B)}$ and $\mathrm{tp} \circ \mathrm{tp}^* = \mathrm{id}_{\hom_{\mathcal{A}}(A,G(B))}$.

Definition 3.2. The *naturality axiom* demands for any $F(A_1) \xrightarrow{\gamma} B_1$, $A_1 \xrightarrow{\delta} G(B_1)$, $A_2 \xrightarrow{\alpha} A_1$ and $B_1 \xrightarrow{\beta} B_2$, where $A_1, A_2 \in \mathrm{ob}(\mathcal{A})$ and $B_1, B_2 \in \mathrm{ob}(\mathcal{B})$ that

$$\mathrm{tp}(\beta \circ \gamma) = G(\beta) \circ \mathrm{tp}(\gamma) \tag{3.2}$$

and

$$\mathrm{tp}^*(\delta \circ \alpha) = \mathrm{tp}^*(\delta) \circ F(\alpha). \tag{3.3}$$

Remark 3.3.
1. The name "naturality axiom" comes from the fact that $F \dashv G$ if and only if the bifunctors $\hom_{\mathcal{B}}(F(-), -)$ and $\hom_{\mathcal{A}}(-, G(-))$ are naturally isomorphic.
2. Whenever an adjoint exist, it must be unique up to isomorphism. This is a consequence of the Yoneda lemma.

3.2 Left adjoint of U

One may guess U has a left adjoint since many forgetful functors of algebraic structures have a left adjoint. Unluckily, this is not the case for graphs.

Lemma 3.4. *Let $A_1 = (V_1, E_1, s_1, t_1)$ and $A_2 = (V_2, \{e\}, s_2, t_2)$ be directed graphs which satisfy $s_1(E_1) \cap t_1(E_1) = \varnothing$. Then*

$$\hom_{\mathbf{DG}}(A_1, A_2) \cong \hom_{\mathbf{Set}}(V_{1\,\mathrm{iso}}, V_2).$$

Proof. Recall $V_1 = \big(s_1(E_1) \cup t_1(E_1)\big) \cup V_{1\,\mathrm{iso}}$ and by definition $\big(s_1(E_1) \cup t_1(E_1)\big) \cap V_{1\,\mathrm{iso}} = \varnothing$. Suppose $\alpha = \langle \alpha_V, \alpha_E \rangle \in \hom_{\mathbf{DG}}(A_1, A_2)$, then α_E have no choice but the constant map $E_1 \to \{e\}$. Thus, to satisfy the graph homomorphism property (2.1), α_V must have

$$v \in s_1(E_1) \Longrightarrow \alpha_V : v \mapsto s_2(e) \quad \text{and} \quad v \in t_1(E_1) \Longrightarrow \alpha_V : v \mapsto t_2(e).$$

On the other hand, $v \in V_{1\,\mathrm{iso}}$ can be mapped to any $w \in V_2$ without violating (2.1), let $\eta : V_{1\,\mathrm{iso}} \to V_2$ be such a map. In this manner, there is a one to one correspondence between η and α, hence proven the claim. $\square$

Theorem 3.5. *U does not have a left adjoint.*

The strategy for the proof is to find a particular undirected graph B and deduce that the existence of $L(B)$ (here we denote the left adjoint to be L) will violate (3.1). This method will be used repeatedly in this report to disprove the existence of adjunction.

Proof. Assume otherwise U has a left adjoint L. Fix $B = \bullet \text{———} \bullet$ and denote $L(B) = (V, E, s, t)$.

Pick 1. $A_1 = \underset{v_1}{\bullet} \longrightarrow \underset{v_2}{\bullet}$, the forgetful functor give $U(A_1) = \underset{v_1}{\bullet} \text{———} \underset{v_2}{\bullet}$. Note $\#\,\mathrm{hom}_{\mathbf{UG}}(B, U(A_1)) = 2$ imply $\#\,\mathrm{hom}_{\mathbf{DG}}(L(B), A_1) = 2$. Here $v_1 \neq v_2 \Rightarrow s(E) \cap t(E) = \varnothing$, apply lemma 3.4 on directed graphs $L(B)$ and A_1 yields $\mathrm{hom}_{\mathbf{DG}}(L(B), A_1) \cong \mathrm{hom}_{\mathbf{Set}}(V_{\text{iso}}, \{v_1, v_2\})$. Therefore,

$$2 = \#\,\mathrm{hom}_{\mathbf{DG}}(L(B), A_1) = \#\,\mathrm{hom}_{\mathbf{Set}}(V_{\text{iso}}, \{v_1, v_2\}) = (\#\{v_1, v_2\})^{(\# V_{\text{iso}})} = 2^{(\# V_{\text{iso}})}.$$

The equality above requires V_{iso} to be an one element set, name that element as v^*. As a result, we can express the vertex set of $L(B)$ as three mutually disjoint union, $V = s(E) \cup t(E) \cup \{v^*\}$.

Pick 2. $A_2 = \underset{v_1}{\bullet} \xrightarrow[e_1]{} \underset{v_2 v_3}{\bullet\bullet} \xrightarrow[e_2]{} \underset{v_4}{\bullet}$, $U(A_2) = \underset{v_1}{\bullet} \overset{}{\underset{e_1}{\text{———}}} \underset{v_2 v_3}{\bullet\bullet} \underset{e_2}{\text{———}} \underset{v_4}{\bullet}$ and $\#\,\mathrm{hom}_{\mathbf{UG}}(B, U(A_2)) = 4$. Yet, we will show $\#\,\mathrm{hom}_{\mathbf{DG}}(L(B), A_2) \geq 8$ by writing out at least 8 maps between $L(B)$ and A_2:

$$E \to \{e_1\}, \quad s(E) \to \{v_1\}, \quad t(E) \to \{v_2\}, \quad \{v^*\} \to \{v_1, v_2, v_3, v_4\}$$

give four distinct map and

$$E \to \{e_2\}, \quad s(E) \to \{v_3\}, \quad t(E) \to \{v_4\}, \quad \{v^*\} \to \{v_1, v_2, v_3, v_4\}$$

give another four. In summary, we deduced $\#\,\mathrm{hom}_{\mathbf{DG}}(L(B), A_2) > \#\,\mathrm{hom}_{\mathbf{UG}}(B, U(A_2))$ which contradicts (3.1).

□

Actually this proof says something more than what we stated in theorem 3.5, we will come back to this point later. To move on, it is time for us to consider the right adjoint of U.

3.3 Right adjoint of U

Definition 3.6. Let $B = (V, E, u)$ be a undirected graph, define the *double edge set*

$$E' = \{(e, v, w) \in E \times V^2 : (v, w) \in u(e)\}. \tag{3.4}$$

Remark 3.7. Note $(e, v, w) \in E' \Rightarrow (e, w, v) \in E'$. If B is loopless (and E is finite), then $\#E' = \#E + \#E$. This explains the word "double" in its name.

Definition 3.8. The *i-th projection map* π_i extract the i-th entry from an ordered n-tuple.

Definition 3.9. Let $R : \mathbf{UG} \to \mathbf{DG}$ be a functor which contain the following:

- functions on objects:

$$\mathrm{ob}(\mathbf{UG}) \to \mathrm{ob}(\mathbf{DG}), \qquad B = (V, E, u) \mapsto R(B) = (V, E', \pi_2, \pi_3) \tag{3.5}$$

- any objects $B_1, B_2 \in \mathrm{ob}(\mathbf{UG})$, a function on morphisms:

$$\hom_{\mathbf{UG}}(B_1, B_2) \to \hom_{\mathbf{DG}}(R(B_1), R(B_2)), \qquad \beta = \langle \beta_V, \beta_E \rangle \mapsto R(\beta) = \langle R(\beta)_V, R(\beta)_E \rangle \tag{3.6}$$

where $R(\beta)_V = \beta_V$ and $R(\beta)_E = (\beta_E \circ \pi_1, \beta_V \circ \pi_2, \beta_V \circ \pi_3)$.

Remark 3.10. 1. The definition above may be complicated to understand at the first glance, however, the picture behind it is relatively simple. Basically it does the following:

$$R\Big(\underset{v_1}{\bullet} \overset{e}{\text{———}} \underset{v_2}{\bullet}\Big) = \underset{v_1}{\bullet} \underset{(e,v_2,v_1)}{\overset{(e,v_1,v_2)}{\rightleftarrows}} \underset{v_2}{\bullet}\ , \quad R\Big(\underset{v}{\bullet} \circlearrowright e\Big) = \underset{v}{\bullet} \circlearrowleft (e,v,v)\ , \quad R(\ \bullet\ ) = \bullet$$

these three undirected graphs can be seen as the basic building blocks of undirected graphs, every undirected graph is in a sense a combination of those three graphs.

2. From definition 3.9, every $R(B)$ share the same source function (target function) which is the projection map. For example, $e' = (e, v, w) \in E'$, the source of e' is $\pi_2(e') = v$ and the target of e' is $\pi_3(e') = w$. So, most of the information is loaded in the edge set and the source function (target function) is kept as simple as possible.

3. Since the element of E' is an ordered triple, the function $R(\beta)_E : E_1' \to E_2'$ maps a triple to a triple. For example, $R(\beta)_E = (\beta_E \circ \pi_1, \beta_V \circ \pi_2, \beta_V \circ \pi_3)$ and $e' = (e, v, w) \in E_1'$. Then,

$$R(\beta)_E(e') = \big(\beta_E(\pi_1(e, v, w)), \beta_V(\pi_2(e, v, w)), \beta_V(\pi_3(e, v, w))\big) = \big(\beta_E(e), \beta_V(v), \beta_V(w)\big).$$

4. In contrast to the (un)directed graph homomorphisms, the way to compose $R(\beta_1)$ and $R(\beta_2)$ is **NOT** obtained by composing the function termwise, it is not meaningful to compose it termwise. For completeness, we will recall the composition rule of n-tuple valued functions.

Definition 3.11. Let X be a set, Y be set of n-tuples and Z be set of m-tuples. Given $f : X \to Y$ and $g : Y \to Z$. Denote $f_i = \pi_i \circ f$ to be the *i-th component function* of f, write $f = (f_1, \cdots, f_n)$. Similarly, $g = (g_1, \cdots, g_m)$. Then,

$$g \circ f := (g_1 \circ f, \cdots g_m \circ f) = \big(g_1 \circ (f_1, \cdots, f_n), \cdots, g_m \circ (f_1, \cdots, f_n)\big), \tag{3.7}$$

which is $(g \circ f)_i = g_i \circ f$.

Remark 3.12. In the context of multivariable calculus, X, Y, Z will be $\mathbb{R}^n$ for some $n \in \mathbb{N}$, the component function are scalar-valued function and f, g are vector-valued function/vector field.

Proposition 3.13. $R : \mathbf{UG} \to \mathbf{DG}$ *satisfy the axioms of functor, hence R is a functor.*

Proof. First, we will show the functions on morphisms are well defined. That is, a undirected graph homomorphism β being mapped to $R(\beta)$ is indeed a directed graph homomorphism. This is obvious from the definition, consider

$$\pi_2 \circ R(\beta)_E = \pi_2 \circ (\beta_E \circ \pi_1, \beta_V \circ \pi_2, \beta_V \circ \pi_3) = \beta_V \circ \pi_2 = R(\beta)_V \circ \pi_2.$$

It is analogous for the target function π_3. This shows $R(\beta)$ is a directed graph homomorphism. Showing R preserve identity is elementary, we move on to show R preserve composition. Suppose $\beta_1 : B_1 \to B_2$, $\beta_2 : B_2 \to B_3$ and $\beta_1 = \langle \beta_{1V}, \beta_{1E} \rangle$, $\beta_2 = \langle \beta_{2V}, \beta_{2E} \rangle$, then

$$\begin{aligned}
R(\beta_2) \circ R(\beta_1) &= \big\langle \beta_{2V}, (\beta_{2E} \circ \pi_1, \beta_{2V} \circ \pi_2, \beta_{2V} \circ \pi_3) \big\rangle \circ \big\langle \beta_{1V}, (\beta_{1E} \circ \pi_1, \beta_{1V} \circ \pi_2, \beta_{1V} \circ \pi_3) \big\rangle \\
&= \big\langle \beta_{2V} \circ \beta_{1V}, (\beta_{2E} \circ \pi_1, \beta_{2V} \circ \pi_2, \beta_{2V} \circ \pi_3) \circ (\beta_{1E} \circ \pi_1, \beta_{1V} \circ \pi_2, \beta_{1V} \circ \pi_3) \big\rangle \\
&= \big\langle \beta_{2V} \circ \beta_{1V}, (\beta_{2E} \circ \beta_{1E} \circ \pi_1, \beta_{2V} \circ \beta_{1V} \circ \pi_2, \beta_{2V} \circ \beta_{1V} \circ \pi_3) \big\rangle \\
&= \big\langle (\beta_2 \circ \beta_1)_V, ((\beta_2 \circ \beta_1)_E \circ \pi_1, (\beta_2 \circ \beta_1)_V \circ \pi_2, (\beta_2 \circ \beta_1)_V \circ \pi_3) \big\rangle \\
&= R(\beta_2 \circ \beta_1)
\end{aligned}$$

Despite it is a little bit messy to read, every steps unpack the definition of composition for the one highlighted in red. Identifying what objects are being composed helps reading the proof (and all the proofs later on). In the first step we compose undirected graphs morphism; the second step we compose n-tuple valued functions; and the third step we use the composition rule of undirected graph homomorphism again. ☐

Definition 3.14. Let $A \in \text{ob}(\mathbf{DG})$, $B \in \text{ob}(\mathbf{UG})$. Define a pair of transposes

$$\begin{aligned}
\text{tp} : \hom_{\mathbf{UG}}(U(A), B) &\longrightarrow \hom_{\mathbf{DG}}(A, R(B)) \\
\gamma = \langle \gamma_V, \gamma_E \rangle &\mapsto \text{tp}(\gamma) = \langle \gamma_V, (\gamma_E, \gamma_V \circ s, \gamma_V \circ t) \rangle
\end{aligned}$$

and

$$\begin{aligned}
\text{tp}^* : \hom_{\mathbf{DG}}(A, R(B)) &\longrightarrow \hom_{\mathbf{UG}}(U(A), B) \\
\delta = \langle \delta_V, \delta_E \rangle &\mapsto \text{tp}^*(\delta) = \langle \delta_V, \pi_1 \circ \delta_E \rangle.
\end{aligned}$$

Remark 3.15. Both transpose did nothing to the vertex map, this is expected as the vertex map and vertex set are invariant under U and R.

Lemma 3.16. *Let $A = (V_1, E_1, s, t) \in \text{ob}(\mathbf{DG})$ and $B \in \text{ob}(\mathbf{UG})$. Suppose $\langle \delta_V, \delta_E \rangle \in \hom_{\mathbf{DG}}(A, R(B))$, then $\delta_E = (\pi_1 \circ \delta_E, \delta_V \circ s, \delta_V \circ t)$.*

Proof. Let $B = (V_2, E_2, u)$ and thus $R(B) = (V_2, E_2', \pi_2, \pi_3)$. Consider $\delta_E : E_1 \to E_2'$ and

$$\begin{aligned}
e \mapsto \delta_E(e) &= (\pi_1(\delta_E(e)), \pi_2(\delta_E(e)), \pi_3(\delta_E(e))) \\
&= (\pi_1(\delta_E(e)), \delta_V(s(e)), \delta_V(t(e))).
\end{aligned}$$

The last line follows from the graph homomorphism property (2.1) as we are in **DG**. This hold for any $e \in E_1$ and hence $\delta_E = (\pi_1 \circ \delta_E, \delta_V \circ s, \delta_V \circ t)$. ☐

Without exaggeration, the lemma we proved above will be the least obvious thing in the proof of $U \dashv R$. With that being said, the notation in the following proof is quite heavy and tedious. Anyway, we have already established enough tools for the next theorem.

Theorem 3.17. *$U \dashv R$.*

Proof. It is sufficient to show tp and tp* defined in definition 3.14 satisfy (3.1), (3.2) and (3.3).

Step 1. We begin by defining all the notations needed for this proof. Let $A_1, A_2 \in \mathrm{ob}(\mathbf{DG})$ and $B_1, B_2 \in \mathrm{ob}(\mathbf{UG})$, we want to explicitly write $A_1 = (V_1, E_1, s, t)$, $B_1 = (V_2, E_2, u)$ and thereby $R(B_1) = (V_2, E_2', \pi_2, \pi_3)$. Next, denote $\langle \alpha_V, \alpha_E \rangle \in \hom_{\mathbf{DG}}(A_2, A_1)$, $\langle \beta_V, \beta_E \rangle \in \hom_{\mathbf{UG}}(B_1, B_2)$, $\langle \gamma_V, \gamma_E \rangle \in \hom_{\mathbf{UG}}(U(A_1), B_1)$ and $\langle \delta_V, \delta_E \rangle \in \hom_{\mathbf{DG}}(A_1, R(B_1))$.

Step 2. Consider

$$\begin{aligned}(\mathrm{tp}^* \circ \mathrm{tp})\langle \gamma_V, \gamma_E \rangle &= \mathrm{tp}^*\langle \gamma_V, (\gamma_E, \gamma_V \circ s, \gamma_V \circ t)\rangle \\ &= \langle \gamma_V, \pi_1 \circ (\gamma_E, \gamma_V \circ s, \gamma_V \circ t)\rangle \\ &= \langle \gamma_V, \gamma_E \rangle \\ &= \mathrm{id}\langle \gamma_V, \gamma_E \rangle\end{aligned}$$

this shows $\mathrm{tp}^* \circ \mathrm{tp} = \mathrm{id}_{\hom_{\mathbf{UG}}(U(A_1), B_1)}$. Hereafter, we will show $\mathrm{tp} \circ \mathrm{tp}^* = \mathrm{id}_{\hom_{\mathbf{DG}}(A_1, R(B_1))}$,

$$\begin{aligned}(\mathrm{tp} \circ \mathrm{tp}^*)\langle \delta_V, \delta_E \rangle &= \mathrm{tp}\langle \delta_V, \pi_1 \circ \delta_E \rangle \\ &= \langle \delta_V, (\pi_1 \circ \delta_E, \delta_V \circ s, \delta_V \circ t)\rangle \\ &= \langle \delta_V, \delta_E \rangle \\ &= \mathrm{id}\langle \delta_V, \delta_E \rangle\end{aligned}$$

note the second last line uses lemma 3.16. At this stage we proved tp and tp* are mutually inverse.

Step 3. Consider

$$\mathrm{tp}\langle \beta_V \circ \gamma_V, \beta_E \circ \gamma_E \rangle = \langle \beta_V \circ \gamma_V, (\beta_E \circ \gamma_E, \beta_V \circ \gamma_V \circ s, \beta_V \circ \gamma_V \circ t)\rangle$$

meanwhile

$$\begin{aligned}R\langle \beta_V, \beta_E \rangle \circ \mathrm{tp}\langle \gamma_V, \gamma_E \rangle &= \langle \beta_V, (\beta_E \circ \pi_1, \beta_V \circ \pi_2, \beta_V \circ \pi_3)\rangle \circ \langle \gamma_V, (\gamma_E, \gamma_V \circ s, \gamma_V \circ t)\rangle \\ &= \langle \beta_V \circ \gamma_V, (\beta_E \circ \gamma_E, \beta_V \circ \gamma_V \circ s, \beta_V \circ \gamma_V \circ t)\rangle.\end{aligned}$$

Together imply $\mathrm{tp}(\beta \circ \gamma) = R(\beta) \circ \mathrm{tp}(\gamma)$. Next, we have

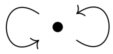
$$\begin{aligned}\mathrm{tp}^*\langle \delta_V \circ \alpha_V, \delta_E \circ \alpha_E \rangle &= \langle \delta_V \circ \alpha_V, \pi_1 \circ \delta_E \circ \alpha_E \rangle \\ &= \langle \delta_V, \pi_1 \circ \delta_E \rangle \circ \langle \alpha_V, \alpha_E \rangle \\ &= \mathrm{tp}^*\langle \delta_V, \delta_E \rangle \circ U\langle \alpha_V, \alpha_E \rangle\end{aligned}$$

which shown $\mathrm{tp}^*(\delta \circ \alpha) = \mathrm{tp}^*(\delta) \circ U(\alpha)$.

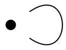
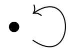
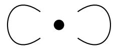
□

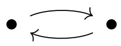
Does R itself has a right adjoint? The answer is no.

Theorem 3.18. *R does not have a right adjoint.*

Proof. Assume otherwise R has a right adjoint S. We fix $A =$  and show the existence of $S(A)$ contradicts (3.1), i.e. $\hom_{\mathbf{DG}}(R(B), A) \cong \hom_{\mathbf{UG}}(B, S(A))$ for all $A \in \mathrm{ob}(\mathbf{DG})$ and $B \in \mathrm{ob}(\mathbf{UG})$ is violated.

Pick 1. $B_1 = \bullet$, we have $R(B_1) = \bullet$. Note $\# \hom_{\mathbf{DG}}(R(B_1), A) = 1$ conclude $\# \hom_{\mathbf{UG}}(B_1, S(A)) = 1$. This imply $S(A)$ has exactly one vertex.

Pick 2. $B_2 =$  and thus $R(B_2) =$ . Observe $\# \hom_{\mathbf{DG}}(R(B_2), A) = 2$ and hence $\# \hom_{\mathbf{UG}}(B_2, S(A)) = 2$. We already know $S(A)$ has one vertex, to obtain two maps for $\hom_{\mathbf{UG}}(B_2, S(A))$, we need two edges. This completely characterise $S(A)$, in particular, $S(A) =$ .

Pick 3. $B_3 = \bullet$ —— $\bullet$, $R(B_3) =$ . Under explicit counting, we get $\# \hom_{\mathbf{DG}}(R(B_3), A) = 4$ but $\# \hom_{\mathbf{UG}}(B_3, S(A)) = 2$. This is a contradiction. Thus, no such functor S exist.

□

4 Various types of graphs

So far, we have been working on the categories of (un)directed graphs that allow multiple edges and loops. In practical, there are some alternative definitions of graphs that prohibit multiple edges and/or loops. Those graphs and their respective graph homomorphisms also form categories. Thus, we can ask the same question, do the forgetful functor of those categories have an adjoint? It turns out the answer varies very differently when we adopt different definition for graphs.

4.1 Restricted (un)directed graphs and forgetful functors

Definition 4.1. Let $A = (V_1, E_1, s, t)$ be a directed graph and $B = (V_2, E_2, u)$ be an undirected graph. Then, A is said to have *multiple edges* if there exist $v, w \in V_1$ such that

$$\#\{e \in E : q\big(s(e), t(e)\big) = [v, w]\} > 1.$$

Similarly, B is said to have *multiple edges* if there exist $[v, w] \in V_2^{\ddagger}$ such that

$$\#\{e \in E : u(e) = [v, w]\} > 1.$$

Definition 4.2. Let $\mathcal{A}$ be a category. $\mathcal{B}$ is a *full subcategory* of $\mathbf{A}$ if $\mathrm{ob}(\mathcal{B})$ is a subset (subclass) of $\mathrm{ob}(\mathcal{A})$ and $\hom_{\mathcal{B}}(A_1, A_2) = \hom_{\mathcal{A}}(A_1, A_2)$ for all $A_1, A_2 \in \mathrm{ob}(\mathcal{B})$. Then, we define

- The category of (un)directed graphs without multiple edges, $\mathbf{UG^1}$ and $\mathbf{DG^1}$, to be the full subcategory of (un)directed graphs such that every (un)directed graphs does not have multiple edges.
- The categories of (un)directed graphs without loops, $\mathbf{UG}_\varnothing$ and $\mathbf{DG}_\varnothing$, to be the full subcategory of (un)directed graphs such that every (un)directed graphs is loopless.
- The categories of (un)directed graphs without multiple edges and loops, $\mathbf{UG}^\mathbf{1}_\varnothing$ and $\mathbf{DG}^\mathbf{1}_\varnothing$, to be the full subcategory of (un)directed graphs such that every (un)directed graphs does not have multiple edges and is loopless.

Remark 4.3. 1. The definition of loopless (un)directed graph is given in definition 2.2 and remark 2.15.

2. We use a superscript **1** to indicate at most one edge between any vertices. Next, a visualized notation $\varnothing$ is placed in the subscript to represent loops is not allowed.
3. $\mathbf{DG}^\mathbf{1}_\varnothing$ is a subcategory of $\mathbf{DG^1}$ and $\mathbf{DG}_\varnothing$. Likewise, $\mathbf{UG}^\mathbf{1}_\varnothing$ is a subcategory of $\mathbf{UG^1}$ and $\mathbf{UG}_\varnothing$.

Recall from definition 2.18, the forgetful functor $U : \mathbf{DG} \to \mathbf{UG}$ consists of a function on objects and a family of functions on morphisms for any pair of objects. In some sense, we can impose a suitable restriction to U such that it become a functor from a subcategory of directed graphs to the corresponding subcategory of undirected graphs we defined above. Let us make this idea precise by the following definition.

Definition 4.4. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. Suppose $\mathcal{S}$ is a subcategory of $\mathcal{A}$. Then, *restriction of* F *to* $\mathcal{S}$, is a functor $F|_\mathcal{S} : \mathcal{S} \to \mathcal{B}$ consists of:

- function on objects:
$$\mathrm{ob}(\mathcal{S}) \to \mathrm{ob}(\mathcal{B}), \qquad A \mapsto F|_\mathcal{S}(A) = F(A),$$
- any object $A_1, A_2 \in \mathrm{ob}(\mathcal{S})$, a function on morphisms:
$$\hom_\mathcal{S}(A_1, A_2) \to \hom_\mathcal{B}(F|_\mathcal{S}(A_1), F|_\mathcal{S}(A_2)) = \hom_\mathcal{B}(F(A_1), F(A_2)), \qquad f \mapsto F|_\mathcal{S}(f) = F(f).$$

Remark 4.5. Continue from the last definition, if we further assume $\mathcal{T}$ is a full subcategory of $\mathcal{B}$ and $F(A) \in \mathrm{ob}(\mathcal{T})$ for all $A \in \mathrm{ob}(\mathcal{S})$. Then for any object $A_1, A_2 \in \mathrm{ob}(\mathcal{S})$, $\hom_\mathcal{B}(F(A_1), F(A_2)) = \hom_\mathcal{T}(F(A_1), F(A_2))$. Thence, we obtain another functor $F|^\mathcal{T}_\mathcal{S} : \mathcal{S} \to \mathcal{T}$, named as the restriction of F to $\mathcal{S}$ and $\mathcal{T}$. Obviously, we define $F|^\mathcal{T}_\mathcal{S}(A) = F|_\mathcal{S}(A)$ for all $A \in \mathrm{ob}(\mathcal{S})$ and $F|^\mathcal{T}_\mathcal{S}(f) = F|_\mathcal{S}(f)$ for all $f \in \hom_\mathcal{S}(A_1, A_2)$.

Definition 4.6. Recall the forgetful functor $U : \mathbf{DG} \to \mathbf{UG}$ from definition 2.18, it follows immediately from the definition that U maps loopless directed graph to loopless undirected graph, and maps directed graph without multiple edges to undirected graph without multiple edges. Therefore, we define

- $U^\mathbf{1}$ to be the restriction of U to $\mathbf{DG^1}$ and $\mathbf{UG^1}$, i.e. $U^\mathbf{1} : \mathbf{DG^1} \to \mathbf{UG^1}$.
- $U_\varnothing$ to be the restriction of U to $\mathbf{DG}_\varnothing$ and $\mathbf{UG}_\varnothing$, i.e. $U_\varnothing : \mathbf{DG}_\varnothing \to \mathbf{UG}_\varnothing$.

- $U^{\mathbf{1}}_{\varnothing}$ to be the restriction of U to $\mathbf{DG}^{\mathbf{1}}_{\varnothing}$ and $\mathbf{UG}^{\mathbf{1}}_{\varnothing}$, i.e. $U^{\mathbf{1}}_{\varnothing} : \mathbf{DG}^{\mathbf{1}}_{\varnothing} \to \mathbf{UG}^{\mathbf{1}}_{\varnothing}$.

With all the pre-work being done, we are ready to answer the question we asked in the beginning of this section.

4.2 Forgetful functor theorem of graphs

Theorem 4.7. *$U_{\varnothing}$, $U^{\mathbf{1}}$ and $U^{\mathbf{1}}_{\varnothing}$ do not have a left adjoint.*

Proof. This follows from the proof of theorem 3.5. Recall we assumed a left adjoint L exist and used $B = \bullet \text{———} \bullet$, $A_1 = \underset{v_1}{\bullet} \longrightarrow \underset{v_2}{\bullet}$ and $A_2 = \underset{v_1}{\bullet} \xrightarrow{e_1} \underset{v_2}{\bullet}\underset{v_3}{\bullet} \xrightarrow{e_2} \underset{v_4}{\bullet}$ to deduce the existence of the L contradicts (3.1). Because $B \in \operatorname{ob}(\mathbf{UG}^{\mathbf{1}}_{\varnothing})$, $A_1, A_2 \in \operatorname{ob}(\mathbf{DG}^{\mathbf{1}}_{\varnothing})$ and $\mathbf{UG}^{\mathbf{1}}_{\varnothing}, \mathbf{DG}^{\mathbf{1}}_{\varnothing}$ are full subcategories, their respective hom-sets remain unchanged. Consequently, we can use the exact same pick to show the existence of the left adjoint of $U^{\mathbf{1}}_{\varnothing}$ will lead to a contradiction. Mutatis mutandis, this argument also apply to $U_{\varnothing}$ and $U^{\mathbf{1}}$. □

Theorem 4.8. *$U^{\mathbf{1}}$ does not have a right adjoint.*

Proof. Assume otherwise $U^{\mathbf{1}}$ has a right adjoint $R^{\mathbf{1}} : \mathbf{UG}^{\mathbf{1}} \to \mathbf{DG}^{\mathbf{1}}$. Fix $B = \bullet \text{———} \bullet$ and consider

Pick 1. $A_1 = \bullet\circlearrowleft$, $U^{\mathbf{1}}(A_1) = \bullet\supset$. Note $\#\hom_{\mathbf{UG}^{\mathbf{1}}}(U^{\mathbf{1}}(A_1), B) = 0$ and hence $\#\hom_{\mathbf{DG}^{\mathbf{1}}}(A_1, R^{\mathbf{1}}(B)) = 0$. This imply $R^{\mathbf{1}}(B)$ is loopless.

Pick 2. $A_2 = \bullet$, $U^{\mathbf{1}}(A_2) = \bullet$. Note $\#\hom_{\mathbf{UG}^{\mathbf{1}}}(U^{\mathbf{1}}(A_2), B) = 2$ imply $\#\hom_{\mathbf{DG}^{\mathbf{1}}}(A_2, R^{\mathbf{1}}(B)) = 2$. Hence $R^{\mathbf{1}}(B)$ has 2 vertices. Since $R^{\mathbf{1}}(B) \in \operatorname{ob}(\mathbf{DG}^{\mathbf{1}})$ is not allowed to have multiple edges and we know $R^{\mathbf{1}}(B)$ is loopless, $R^{\mathbf{1}}(B)$ can only either be $\bullet \quad\quad \bullet$ or $\bullet \longrightarrow \bullet$.

Pick 3. $A_3 = \bullet \longrightarrow \bullet$, $U^{\mathbf{1}}(A_3) = \bullet \text{———} \bullet$. Note $\#\hom_{\mathbf{UG}^{\mathbf{1}}}(U^{\mathbf{1}}(A_3), B) = 2$. However, neither possible candidate of $R^{\mathbf{1}}(B)$ give a correct number of maps,

$$\#\hom_{\mathbf{DG}^{\mathbf{1}}}(A_3, \bullet \longrightarrow \bullet) = 1 \neq 2$$

and

$$\#\hom_{\mathbf{DG}^{\mathbf{1}}}(A_3, \bullet \quad\quad \bullet) = 0 \neq 2.$$

This is absurd, no such $R^{\mathbf{1}}$ exist.

□

Corollary 4.9. *$U^{\mathbf{1}}_{\varnothing}$ does not have a right adjoint.*

Proof. Again, assume otherwise $U^{\mathbf{1}}_{\varnothing}$ has a right adjoint $R^{\mathbf{1}}_{\varnothing} : \mathbf{UG}^{\mathbf{1}}_{\varnothing} \to \mathbf{DG}^{\mathbf{1}}_{\varnothing}$. Fix $B = \bullet \text{———} \bullet$, observe that the first pick A_1 in the previous proof is to show $R^{\mathbf{1}}(B)$ is loopless. Yet, we cannot do such pick here and we do not need to do it since $R^{\mathbf{1}}_{\varnothing}(B) \in \operatorname{ob}\mathbf{DG}^{\mathbf{1}}_{\varnothing}$ already imply $R^{\mathbf{1}}_{\varnothing}(B)$ is loopless. In this manner, follow the second and third pick from the previous proof lead us to the same contradiction because $A_2, A_3 \in \operatorname{ob}(\mathbf{UG}^{\mathbf{1}}_{\varnothing})$ and $\mathbf{DG}^{\mathbf{1}}_{\varnothing}, \mathbf{UG}^{\mathbf{1}}_{\varnothing}$ are full subcategories. Therefore, we can draw the same conclusion, no such $R^{\mathbf{1}}_{\varnothing}$ exist. □

Lemma 4.10. *Let $B = (V, E, u) \in \text{ob}(\mathbf{UG})$. B is loopless if and only if $R(B)$ is loopless.*

Proof. $(\Rightarrow)$: Suppose B is loopless. By definition, $\#u(e) = 2$ for all $e \in E$. Recall a direct graph $A = (V, E, s, t)$ is loopless if $\{e \in E : s(e) = t(e)\} = \varnothing$. Now consider $R(B) = (V, E', \pi_2, \pi_3)$ and let $e' = (e, v, w) \in E'$. Since $(v, w) \in u(e)$ and $\#u(e) = 2$, we have $v \neq w$. In other words, $\{e' \in E' : \pi_2(e') = \pi_3(e')\} = \varnothing$. Thus $R(B)$ is loopless.

$(\Leftarrow)$: Suppose $R(B)$ is loopless. Then $\{e' \in E' : \pi_2(e') = \pi_3(e')\} = \varnothing$, which is $\pi_2(e') \neq \pi_3(e')$ for all $e' \in E'$. In addition, from the construction of E', $(\pi_2(e'), \pi_3(e')) \in u(e)$ for some $e \in E$ if and only if $(\pi_3(e'), \pi_2(e')) \in u(e)$. Together with $\pi_2(e') \neq \pi_3(e')$ give $\#u(e) = 2$, this hold for any $e \in E$. As a result, B is loopless. □

Theorem 4.11. *Let $R_\varnothing$ be the restriction of R to $\mathbf{UG}_\varnothing$ and $\mathbf{DG}_\varnothing$, then $U_\varnothing \dashv R_\varnothing$.*

Proof. Using the $(\Rightarrow)$ direction of lemma 4.10 shows $R(B) \in \text{ob}(\mathbf{DG}_\varnothing)$ whenever $B \in \text{ob}(\mathbf{UG}_\varnothing)$. This justify $R_\varnothing : \mathbf{UG}_\varnothing \to \mathbf{DG}_\varnothing$ is well defined. Next, follow everything we did in theorem 3.17 and make necessary change to the categories yields the claim. □

At the end of section 3.3, we asked if R has a right adjoint and the answer is no. Yet, opposite to R, the functor $R_\varnothing$ has a right adjoint. As always, to show the existence of an adjoint, our procedure will be first creating the candidate functor, follow by defining a pair of mutually inverse transpose and proof the naturality axiom is satisfied at last.

Definition 4.12. Let $A = (V, E, s, t) \in \text{ob}(\mathbf{DG})$ and $v, w \in V$. Define the *enhanced edge set*

$$E_{v,w} = \{(e, v, w) \in E \times V^2 : v = s(e), w = t(e)\}. \tag{4.1}$$

Remark 4.13.

1. If $A \in \text{ob}(\mathbf{DG}_\varnothing)$, the set $\{e \in E : s(e) = t(e)\} = \varnothing$. Thereby, $E_{v,v} = \varnothing$ for any $v \in V$.
2. This set is call enhanced because it carry more information than a ordinary edge set, note that

$$E \cong \bigcup_{v,w \in V} E_{v,w} \quad \text{and} \quad A \cong (V, \bigcup_{v,w \in V} E_{v,w}, \pi_2, \pi_3)$$

this hold generally for any directed graph $A \in \text{ob}(\mathbf{DG})$

Definition 4.14. Let $A = (V, E, s, t) \in \text{ob}(\mathbf{DG}_\varnothing)$ and $v, w \in V$. Define the *component of the pairwise edge set*

$$\hat{E}_{[v,w]} = \big((E_{v,w} \times E_{w,v}) \cup (E_{w,v} \times E_{v,w})\big)\big/_{\sim}$$

and define the *pairwise edge set*

$$\hat{E} = \bigcup_{[v,w] \in V^\ddagger} \hat{E}_{[v,w]}. \tag{4.2}$$

Remark 4.15.

1. Suppose $\hat{e} \in \hat{E}$, then $\hat{e}$ must have the following form

$$\hat{e} = [(e_1, v, w), (e_2, w, v)]$$

for some $e_1, e_2 \in E$ and $v, w \in V$ such that $v = s(e_1) = t(e_2)$ and $w = t(e_1) = s(e_2)$. The set $[(e_1, v, w), (e_2, w, v)]$ is called the *equivalence class form* of $\hat{e}$.

2. Note $\hat{E} \subset (E \times V^2)^\ddagger$, so $\hat{e} = [(e_1, v, w), (e_2, w, v)]$ is the set $\Big\{\big((e_1, v, w), (e_2, w, v)\big), \big((e_2, w, v), (e_1, v, w)\big)\Big\}$.

3. If either $E_{v,w}$ or $E_{w,v}$ is empty, then $\hat{E}_{[v,w]} = \varnothing$. The element of $\hat{E}_{[v,w]}$ can be seen as the product of edge sets where one set is oriented from v to w and the other set is oriented from w to v. We do not want it to be an ordered pair for technical reason, thus we use the quotient to make it become an unordered pair.

Definition 4.16. Let $S_\varnothing : \mathbf{DG}_\varnothing \to \mathbf{UG}_\varnothing$ be a functor which contain the following:

- functions on objects:

$$\mathrm{ob}(DG_\varnothing) \to \mathrm{ob}(UG_\varnothing), \qquad A = (V, E, s, t) \mapsto S_\varnothing(A) = (V, \hat{E}, \pi_2^\ddagger) \tag{4.3}$$

- any objects $A_1, A_2 \in \mathrm{ob}(DG_\varnothing)$, a function on morphisms:

$$\hom_{DG_\varnothing}(A_1, A_2) \to \hom_{UG_\varnothing}(S_\varnothing(A_1), S_\varnothing(A_2)), \qquad \alpha = \langle \alpha_V, \alpha_E \rangle \mapsto S_\varnothing(\alpha) = \langle S_\varnothing(\alpha)_V, S_\varnothing(\alpha)_E \rangle \tag{4.4}$$

where $S_\varnothing(\alpha)_V = \alpha_V$ and $S_\varnothing(\alpha)_E = (\alpha_E \circ \pi_1, \alpha_V \circ \pi_2, \alpha_V \circ \pi_3)^\ddagger$.

Remark 4.17. 1. Let $A = \underset{v_1}{\bullet} \overset{(n)}{\underset{(m)}{\rightleftarrows}} \underset{v_2}{\bullet}$, where (n) denote the number of edge oriented from v_1 to v_2 and (m) denote the number of edge oriented from v_2 to v_1. Then $S_\varnothing(A) = \underset{v_1}{\bullet} \overline{\ (mn)\ } \underset{v_2}{\bullet}$, meaning that there are $m \times n$ edges connecting $[v_1, v_2]$. If $m = 0$ or $n = 0$, then $mn = 0$. This is consistent with remark 4.15.3. In fact, this property of multiplying number of edges motivate how $\hat{E}$ is defined.

2. Suppose $\hat{e} = [(e_1, v, w), (e_2, w, v)] \in \hat{E}$. Note that $\pi_2^\ddagger(\hat{e}) = [v, w]$. At the same time, one may find $\pi_3^\ddagger(\hat{e}) = [w, v] = [v, w] = \pi_2^\ddagger(\hat{e})$. So, $S_\varnothing(A)$ can be equally defined as $(V, \hat{E}, \pi_3^\ddagger)$.

3. What will be the composition of $S_\varnothing(\alpha_1)$ and $S_\varnothing(\alpha_2)$? Suppose $\alpha_1 : A_1 \to A_2$, $\alpha_2 : A_2 \to A_3$ and $\alpha_1 = \langle \alpha_{1V}, \alpha_{1E} \rangle$, $\alpha_2 = \langle \alpha_{2V}, \alpha_{2E} \rangle$, then for the edge part of $S_\varnothing(\alpha_2 \circ \alpha_1)$,

$$\begin{aligned} S_\varnothing(\alpha_2)_E \circ S_\varnothing(\alpha_1)_E &= (\alpha_{2E} \circ \pi_1, \alpha_{2V} \circ \pi_2, \alpha_{2V} \circ \pi_3)^\ddagger \circ (\alpha_{1E} \circ \pi_1, \alpha_{1V} \circ \pi_2, \alpha_{1V} \circ \pi_3)^\ddagger \\ &= \Big((\alpha_{2E} \circ \pi_1, \alpha_{2V} \circ \pi_2, \alpha_{2V} \circ \pi_3) \circ (\alpha_{1E} \circ \pi_1, \alpha_{1V} \circ \pi_2, \alpha_{1V} \circ \pi_3)\Big)^\ddagger \\ &= (\alpha_{2E} \circ \alpha_{1E} \circ \pi_1, \alpha_{2V} \circ \alpha_{1V} \circ \pi_2, \alpha_{2V} \circ \alpha_{1V} \circ \pi_3)^\ddagger \\ &= S_\varnothing(\alpha_2 \circ \alpha_1)_E. \end{aligned}$$

Note the second line uses $(-)^\ddagger$ is a functor, the third line uses the composition rule of n-tuple valued functions. Together with $S_\varnothing(\alpha_2)_V \circ S_\varnothing(\alpha_1)_V = \alpha_{2V} \circ \alpha_{1V} = S_\varnothing(\alpha_2 \circ \alpha_1)_V$ proves S preserve composition. In addition, showing $S_\varnothing$ preserve identity is elementary whilst $S_\varnothing$ preserve graphs homomorphism property is built in into its definition, we did a similar proof for R. These suggest $S_\varnothing$ satisfy axioms of functor.

Let us make a small recap to make sure we are still on the same page. We attempt to show $R_\varnothing$ has a right adjoint and we have created a candidate functor $S_\varnothing$. Thereby, the next step will be defining a pair of transpose between hom-sets $\hom_{\mathbf{DG}_\varnothing}(R_\varnothing(B), A)$ and $\hom_{\mathbf{UG}_\varnothing}(B, S_\varnothing(A))$ for all $A \in \mathrm{ob}(\mathbf{DG}_\varnothing), B \in \mathrm{ob}(\mathbf{UG}_\varnothing)$. Due to some technical difficulties, instead of writing functional equations for transpose as in section 3, we track the effect of transpose element-wise to make everything as clear as possible.

Definition 4.18. Let $A = (V_1, E_1, s, t) \in \mathrm{ob}(\mathbf{DG}_\varnothing)$ and $B = (V_2, E_2, u) \in \mathrm{ob}(\mathbf{UG}_\varnothing)$. Define

$$\mathrm{tp} : \hom_{\mathbf{DG}_\varnothing}(R_\varnothing(B), A) \to \hom_{\mathbf{UG}_\varnothing}(B, S_\varnothing(A)), \qquad \gamma = \langle \gamma_V, \gamma_E \rangle \mapsto \mathrm{tp}(\gamma) = \langle \mathrm{tp}(\gamma)_V, \mathrm{tp}(\gamma)_E \rangle$$

by $\mathrm{tp}(\gamma)_V = \gamma_V$, whilst the effect on γ_E is expressed element-wise. Suppose $e' \in E_2'$, then $e' = (x, w_1, w_2)$ for some $x \in E_2$ and $[w_1, w_2] = u(x)$. Furthermore, we must have $(x, w_2, w_1) \in E_2'$. The transpose tp on γ_E is

$$\left(\gamma_E : \begin{array}{l} (x, w_1, w_2) \mapsto e_1 \\ (x, w_2, w_1) \mapsto e_2 \end{array} \right) \longmapsto \left(\mathrm{tp}(\gamma)_E : x \mapsto \big[(e_1, \gamma_V(w_1), \gamma_V(w_2)), (e_2, \gamma_V(w_2), \gamma_V(w_1)) \big] \right). \tag{4.5}$$

Definition 4.19. Let $A = (V_1, E_1, s, t) \in \mathrm{ob}(\mathbf{DG}_\varnothing)$ and $B = (V_2, E_2, u) \in \mathrm{ob}(\mathbf{UG}_\varnothing)$. Define

$$\mathrm{tp}^* : \hom_{\mathbf{UG}_\varnothing}(B, S_\varnothing(A)) \to \hom_{\mathbf{DG}_\varnothing}(R_\varnothing(B), A), \qquad \delta = \langle \delta_V, \delta_E \rangle \mapsto \mathrm{tp}(\delta) = \langle \mathrm{tp}(\delta)_V, \mathrm{tp}(\delta)_E \rangle$$

by $\mathrm{tp}^*(\delta)_V = \delta_V$, whilst the effect on δ_E is expressed element-wise. Suppose $x \in E_2$ and denote $u(x) = [w_1, w_2]$, then the transpose tp^* on δ_E is

$$\left(\delta_E : x \mapsto \hat{e} := \big[(e_1, \delta_V(w_1), \delta_V(w_2)), (e_2, \delta_V(w_2), \delta_V(w_1)) \big] \right) \longmapsto \left(\mathrm{tp}^*(\delta_E) : \begin{array}{l} (x, w_1, w_2) \mapsto e_1 \\ (x, w_2, w_1) \mapsto e_2 \end{array} \right). \tag{4.6}$$

Remark 4.20.

1. First we have to justify tp and tp^* are well-defined, in other words, showing what we claim in (4.5) and (4.6) really exist. For (4.5), given $e_1, e_2 \in E_1$ and γ is a directed graph homomorphism, $s(e_1) = \gamma_V(\pi_2(e')) = w_1$ and $t(e_1) = \gamma_V(\pi_3(e')) = w_2$. Likewise, $t(e_2) = w_1$ and $s(e_2) = w_2$. Therefore, $\big[(e_1, \gamma_V(w_1), \gamma_V(w_2)), (e_2, \gamma_V(w_2), \gamma_V(w_1)) \big] \in \hat{E}_1$ and thus $\mathrm{tp}(\gamma)_E$ is well-defined. Using this argument and invoke the undirected graph homomorphism property explains why we can write $\hat{e} = \big[(e_1, \delta_V(w_1), \delta_V(w_2)), (e_2, \delta_V(w_2), \delta_V(w_1))$ in (4.6).

2. Once understanding (4.5) and (4.6), it is straight forward that tp and tp^* are mutually inverse of each other. There are many layers of maps here, to be specific, we talk through one of it as an example. For any $\gamma \in \hom_{\mathbf{DG}_\varnothing}(R_\varnothing(B), A)$, we have $\mathrm{tp}^*(\mathrm{tp}(\gamma)) = \gamma$. This can be examined by seeing that for all $e' \in E_2$, $\mathrm{tp}^*(\mathrm{tp}(\gamma))_E$ applied on e' is equal to $\gamma_E(e')$ and for all $v \in V_2$, $\mathrm{tp}^*(\mathrm{tp}(\gamma))_V$ applied on v is equal to $\gamma_V(v)$.

Theorem 4.21. $R_\varnothing \dashv S_\varnothing$.

Proof. We have to show the pair of transpose tp and tp^* we defined satisfy naturality axiom. Suppose $A_1, A_2 \in \mathrm{ob}(\mathbf{DG}_\varnothing)$ and $B_1, B_2 \in \mathrm{ob}(\mathbf{UG}_\varnothing)$ and let $\alpha = \langle \alpha_V, \alpha_E \rangle \in \hom_{\mathbf{DG}_\varnothing}(A_1, A_2)$, $\beta = \langle \beta_V, \beta_E \rangle \in \hom_{\mathbf{UG}_\varnothing}(B_2, B_1)$, $\gamma = \langle \gamma_V, \gamma_E \rangle \in \hom_{\mathbf{DG}_\varnothing}(R_\varnothing(B_1), A_1)$ and $\delta = \langle \delta_V, \delta_E \rangle \in \hom_{\mathbf{UG}_\varnothing}(B_1, S_\varnothing(A_1))$. Recall our goal is to proof

$$\mathrm{tp}(\alpha \circ \gamma) = S_\varnothing(\alpha) \circ \mathrm{tp}(\gamma) \tag{4.7}$$

and

$$\mathrm{tp}^*(\delta \circ \beta) = \mathrm{tp}^*(\delta) \circ R_\varnothing(\beta). \tag{4.8}$$

The maps of vertices remain unchanged at the time, so there is nothing to show for maps of vertices. Hence, we are left with showing the maps of edges satisfy (4.7) and (4.8). To make the things clean, the proof will be split into two parts, the first part will be proving (4.7) on map of edges.

Part 1. To begin with, write $B_1 = (V, E, u)$ and thus $R_\varnothing(B_1) = (V, E', \pi_2, \pi_3)$. The idea of the proof is to start with some arbitrary elements and track the effect on them by every map. Let $e' \in E'$ and express $e' = (x, w_1, w_2)$ for some $x \in E$ and $[w_1, w_2] = u(x)$. At the same time, $(x, w_2, w_1) \in E'$. Given γ_E and α_E, we denote

$$\left(\gamma_E : \begin{matrix} (x, w_1, w_2) \mapsto e_1 \\ (x, w_2, w_1) \mapsto e_2 \end{matrix}\right) \quad \text{and} \quad \left(\alpha_E : \begin{matrix} e_1 \mapsto \alpha_E(e_1) \\ e_2 \mapsto \alpha_E(e_2) \end{matrix}\right).$$

Composing them to get $(\alpha \circ \gamma)_E$,

$$\left((\alpha \circ \gamma)_E : \begin{matrix} (x, w_1, w_2) \mapsto \alpha_E(e_1) \\ (x, w_2, w_1) \mapsto \alpha_E(e_2) \end{matrix}\right).$$

According to (4.5), transpose yields

$$\left(\text{tp}(\alpha \circ \gamma)_E : x \mapsto \Big[\big(\alpha_E(e_1), \alpha_V(\gamma_V(w_1)), \alpha_V(\gamma_V(w_2)\big), \big(\alpha_E(e_2), \alpha_V(\gamma_V(w_2)), \alpha_V(\gamma_V(w_1)\big)\Big]\right).$$

Meanwhile, on the other side, transposing γ_E is

$$\left(\text{tp}(\gamma)_E : x \mapsto [(e_1, \gamma_V(w_1), \gamma_V(w_2)), (e_2, \gamma_V(w_2), \gamma_V(w_1))]\right)$$

and using $S_\varnothing(\alpha)_E = (\alpha_E \circ \pi_1, \alpha_V \circ \pi_2, \alpha_V \circ \pi_3)^\ddagger$ on $\big[(e_1, \gamma_V(w_1), \gamma_V(w_2)), (e_2, \gamma_V(w_2), \gamma_V(w_1))\big]$ give

$$\left(S_\varnothing(\alpha)_E : \Big[\big(e_1, \gamma_V(w_1), \gamma_V(w_2)\big), \big(e_2, \gamma_V(w_2), \gamma_V(w_1)\big)\Big] \right.$$
$$\left. \mapsto \Big[\big(\alpha_E(e_1), \alpha_V(\gamma_V(w_1)), \alpha_V(\gamma_V(w_2))\big), \big(\alpha_E(e_2), \alpha_V(\gamma_V(w_2)), \alpha_V(\gamma_V(w_1))\big)\Big]\right)$$

Therefore,

$$\left(\big(S_\varnothing(\alpha) \circ \text{tp}(\gamma)\big)_E : x \mapsto \Big[\big(\alpha_E(e_1), \alpha_V(\gamma_V(w_1)), \alpha_V(\gamma_V(w_2))\big), \big(\alpha_E(e_2), \alpha_V(\gamma_V(w_2)), \alpha_V(\gamma_V(w_1))\big)\Big]\right).$$

This show $\big(S_\varnothing(\alpha) \circ \text{tp}(\gamma)\big)_E = \text{tp}(\alpha \circ \gamma)_E$ for a particular $x \in E$ where x depends on the choice of $e' \in E'$ at the beginning. In fact, from how E' is constructed, for all $x \in E$, denote $u(x) = [w_1, w_2]$, there exist (x, w_1, w_2) and $(x, w_2, w_1) \in E'$. In this manner, given any $x \in E$, choose $(x, w_1, w_2), (x, w_2, w_1) \in E'$ and restart the whole argument, we have $\big(S_\varnothing(\alpha) \circ \text{tp}(\gamma)\big)_E = \text{tp}(\alpha \circ \gamma)_E$, this proves (4.7) on map of edges.

Part 2. We are going to use the same trick. Write $B_2 = (V_2, E_2, u_2)$, $B_1 = (V_1, E_1, u_1)$ and $A_1 = (V, E, s, t)$. So, $S_\varnothing(A_1) = (V, \hat{E}, \pi_2^\ddagger)$. Let $x \in E_2$ and denote $u_2(x) = [w_1, w_2]$. Given β_E and δ_E, we have

$$\beta_E : x \mapsto \beta_E(x) \quad \text{and} \quad \delta_E : \beta_E(x) \mapsto \delta_E(\beta_E(x)).$$

Since $\delta : B_1 \to S_\varnothing(A_1)$ is a undirected graph homomorphism, it must obey $\pi_2^\ddagger \delta_E = \delta_V^\ddagger u_1$. Apply this on $\beta_E(x)$ give

$$\pi_2^\ddagger\Big(\delta_E(\beta_E(x)\Big) = \delta_V^\ddagger\Big(u_1(\beta_E(x))\Big)$$

and the undirected graph homomorphism property of $\beta : B_2 \to B_1$ tell $u_1(\beta_E(x)) = [\beta_V(w_1), \beta_V(w_2)]$. Therefore, we deduced that

$$\pi_2^{\ddagger}\Big(\delta_E(\beta_E(x)\Big) = \delta_V^{\ddagger}\Big([\beta_V(w_1), \beta_V(w_2))]\Big) = [\delta_V(\beta_V(w_1)), \delta_V(\beta_V(w_2)]$$

and this determine the equivalence class form of $\delta_E(\beta_E(x)) \in \hat{E}$, i.e.

$$\delta_E(\beta_E(x)) = \Big[\big(e_1, \delta_V(\beta_V(w_1)), \delta_V(\beta_V(w_2))\big), \big(e_2, \delta_V(\beta_V(w_2)), \delta_V(\beta_V(w_1))\big)\Big]$$

for some suitable $e_1, e_2 \in E$. So, the map δ_E is

$$\delta_E : \beta_E(x) \mapsto \Big[\big(e_1, \delta_V(\beta_V(w_1)), \delta_V(\beta_V(w_2))\big), \big(e_2, \delta_V(\beta_V(w_2)), \delta_V(\beta_V(w_1))\big)\Big]$$

whilst

$$(\delta \circ \beta)_E : x \mapsto \Big[\big(e_1, \delta_V(\beta_V(w_1)), \delta_V(\beta_V(w_2))\big), \big(e_2, \delta_V(\beta_V(w_2)), \delta_V(\beta_V(w_1))\big)\Big].$$

Invoke (4.6), the transpose is

$$\left(\mathrm{tp}^*(\delta \circ \beta)_E : \begin{array}{l} (x, w_1, w_2) \mapsto e_1 \\ (x, w_2, w_1) \mapsto e_2 \end{array} \right).$$

Simultaneously, apply transpose on δ_E produce

$$\left(\mathrm{tp}^*(\delta)_E : \begin{array}{l} (\beta_E(x), \beta_V(w_1), \beta_V(w_2)) \mapsto e_1 \\ (\beta_E(x), \beta_V(w_2), \beta_V(w_1)) \mapsto e_2 \end{array} \right).$$

Next, apply $R_\varnothing(\beta)_E = (\beta_E \circ \pi_1, \beta_V \circ \pi_2, \beta_V \circ \pi_3)$ on (x, w_1, w_2) and (x, w_2, w_1) give

$$\left(R_\varnothing(\beta)_E : \begin{array}{l} (x, w_1, w_2) \mapsto (\beta_E(x), \beta_V(w_1), \beta_V(w_2)) \\ (x, w_2, w_1) \mapsto (\beta_E(x), \beta_V(w_2), \beta_V(w_1)) \end{array} \right)$$

combine the last two result, we arrive at

$$\left(\big(\mathrm{tp}^*(\delta) \circ R_\varnothing(\beta)\big)_E : \begin{array}{l} (x, w_1, w_2) \mapsto e_1 \\ (x, w_2, w_1) \mapsto e_2 \end{array} \right).$$

Again, this show $\big(\mathrm{tp}^*(\delta) \circ R_\varnothing(\beta)\big)_E = \mathrm{tp}^*(\delta \circ \beta)_E$ for a two particular element $(x, w_1, w_2), (x, w_2, w_1) \in E_2'$. Yet, we can reach every element of $e' \in E_2'$ by starting with $x = \pi_1(e') \in E_2$ in this proof. Thus, $\big(\mathrm{tp}^*(\delta) \circ R_\varnothing(\beta)\big)_E = \mathrm{tp}^*(\delta \circ \beta)_E$ hold for any $e' \in E_2$ and proves (4.6) on maps of edges.

□

Theorem 4.22. *$S_\varnothing$ does not have a right adjoint.*

Proof. Assume otherwise $S_\varnothing$ has a right adjoint $T_\varnothing$. We fix $B = \bullet \!-\!\!-\!\!-\! \bullet$ and show the existence of $T_\varnothing(B)$ violate the bijection between hom-sets $\hom_{\mathbf{UG}_\varnothing}(S_\varnothing(A), B)$ and $\hom_{\mathbf{DG}_\varnothing}(A, T_\varnothing(B))$ for some $A \in \mathrm{ob}(\mathbf{DG}_\varnothing)$, $B \in \mathrm{ob}(\mathbf{UG}_\varnothing)$.

Pick 1. $A_1 = \bullet$, then $S_\varnothing(A_1) = \bullet$. Consider $\# \hom_{\mathbf{UG}_\varnothing}(S_\varnothing(A_1), B) = 2$ and thus $\# \hom_{\mathbf{DG}_\varnothing}(A_1, T_\varnothing(B)) = 2$. This imply $T_\varnothing(B)$ has two vertices.

Pick 2. $A_2 = \bullet \longrightarrow \bullet$ imply $S_\varnothing(A_2) = \bullet \qquad \bullet$. Counting the number of maps give $\# \hom_{\mathbf{UG}_\varnothing}(S_\varnothing(A_2), B) = 4$, thence $\# \hom_{\mathbf{DG}_\varnothing}(A_2, T_\varnothing(B)) = 4$. Notice loops are not allowed here, edges can only be connected by two distinct vertices. To obtain 4 maps from A_2 to $T_\varnothing(B)$, $T_\varnothing(B)$ must have 4 edges. At this stage, we know $T_\varnothing(B)$ will be either of the following:

$$G_1 = \bullet \overset{\rightrightarrows}{\rightrightarrows} \bullet \quad \text{or} \quad G_2 = \bullet \overset{\leftarrow}{\underset{\rightrightarrows}{\rightarrow}} \bullet \quad \text{or} \quad G_3 = \bullet \overset{\leftleftarrows}{\rightrightarrows} \bullet .$$

Pick 3. $A_3 = \bullet \rightleftarrows \bullet$ and hence $S_\varnothing(A_3) = \bullet \text{———} \bullet$. Note that $\# \hom_{\mathbf{UG}_\varnothing}(S_\varnothing(A_3), B) = 2$ but none of G_1, G_2 or G_3 give the correct cardinality for the hom-set, $\hom_{\mathbf{UG}_\varnothing}(A_3, T_\varnothing(B))$. In particular,

$$\# \hom_{\mathbf{UG}_\varnothing}(A_3, G_1) = 0, \quad \# \hom_{\mathbf{UG}_\varnothing}(A_3, G_2) = 6 \quad \text{and} \quad \# \hom_{\mathbf{UG}_\varnothing}(A_3, G_3) = 8.$$

This is absurd. The right adjoint of $S_\varnothing$ does not exist.

□

We end this report by summarising all the result we have establish so far into one big theorem.

Theorem 4.23. *(forgetful functor theorem of graphs). Let* **DG** *be the category of directed graphs,* **UG** *be the category of undirected graphs. Then, the chain of adjunction for the following functor:*

1. $U : \mathbf{DG} \to \mathbf{UG}$ *is* $U \dashv R$.
2. $U^{\mathbf{1}} : \mathbf{DG}^{\mathbf{1}} \to \mathbf{UG}^{\mathbf{1}}$ *is empty, it has no adjunction on both sides.*
3. $U^{\mathbf{1}}_\varnothing : \mathbf{DG}^{\mathbf{1}}_\varnothing \to \mathbf{UG}^{\mathbf{1}}_\varnothing$ *is empty, it has no adjunction on both sides.*
4. $U_\varnothing : \mathbf{DG}_\varnothing \to \mathbf{UG}_\varnothing$ *is* $U_\varnothing \dashv R_\varnothing \dashv S_\varnothing$.

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