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Mackey Functor (Co)homology of G-C.W. Complexes

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Abstract

In this paper we will extend the theory of homology and cohomology of topological spaces to spaces with a group action. This field of study is called Equivariant Algebraic Topology. We will introduce this theory and give definitions of Mackey functor homology and cohomology of $G-C.W$ complexes (C.W complexes with G action) using fixed point functors and give interesting examples of them, backing these examples with intuition. Finally, we will discuss representation spheres of finite groups, in particular p^l-cyclic groups, and prove a theorem about their $G-C.W$ decomposition. This paper assumes knowledge of ordinary homology of spheres ($\mathbb{Z}$ coefficients) and very preliminary knowledge on categories.

Statement of Authorship

This project was suggested by my supervisor Mircea Voineagu who I thank greatly for the guidance. A majority of the definitions and results were adapted or motivated from the contents of "Equivariant Stable Homotopy Theory and the Kervaire Invariant Problem" written by Douglas Ravenel, Michael J. Hopkins, and Mike Hill [3].

Contents

0 Introduction

In algebraic topology, we study various homology theories which give invariants to help distinguish between different spaces. One such theory is the cellular homology theory which gives invariants on cellular spaces, in particular a class of spaces called C.W-complexes, introduced by Whitehead [1]. A G-C.W complex modifies the notion of C.W complexes by admitting a (cellular) $G-$ action onto it's cells. This might seem like a quite artificial or random extension to the theory, however, considering a G action on spaces reveals quite a large part of the space. In particular many $C.W$ complexes are not compatible with certain groups, that is there does not exist a non trivial action preserving its cellular structure.

In classic algebraic topology we study the (cellular) homology of a C.W complex. This theory naturally extends to equivariant homology originally introduced by Bredon [2], which concerns the homology (and cohomology) of these spaces. We will discuss equivariant (Co)homology theory slightly differently to how Bredon first introduced it, using what is called a Mackey functor.

In the last section of this report, we will be looking at a particular case of $G-$C.W complexes which are known as representation spheres. Let V be a module over $\mathbb{R}[G]$ for G a finite group, then we define the representation sphere as the one point compactification of V. Every finite dimensional orthogonal linear representation sphere can be given a $G - C.W$ decomposition and we will give explicitly the decomposition in the case that $G = C_{p^l}$ with p prime and l a positive integer.

1 Preliminaries

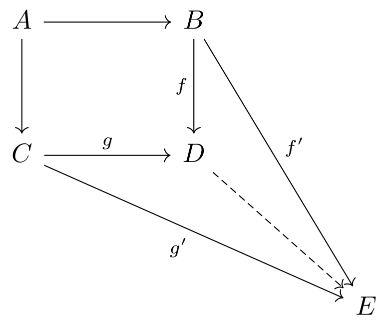
Definition 1. Let

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \\ C & & \end{array}$$

be a diagram in some category. If there is an object D such that there are morphisms f and g making

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow f \\ C & \xrightarrow[g]{} & D \end{array}$$

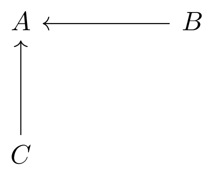
commute. And furthermore for any another object E with $f' : B \to E$ and $g' : C \to E$, there is a morphism $D \to E$ making



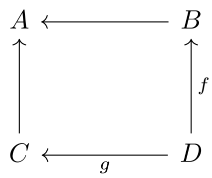
commute. We call D the **pushout** of the diagram.

We can define the dual notion which is the notion of a **pullback**

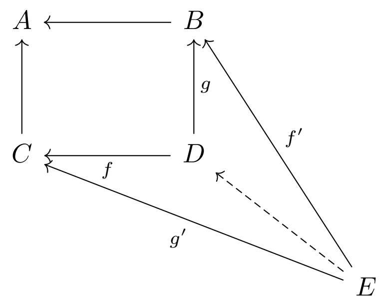
Definition 2. Let



be a diagram in some category. If there is an object D such that there are morphisms f and g making



commute. And furthermore for any other object E with $f' : E \to B$ and $g' : E \to C$, there is a morphism $E \to D$ making



commute. We call D the **pullback** of the diagram.

Definition 3. Let $A \subset X$ be closed and $f : A \to Y$ continuous. Define

$$X \cup_f Y = \frac{X \sqcup Y}{\sim}$$

where $x \sim f(x)$.

Pictorially we are 'gluing' two spaces together via a mapping f which tells us where to glue.

Definition 4. A morphism $f : X \to Y$ between G topological spaces is called **equivariant** if for all $x \in X$ and $g \in G$, $f(g \cdot x) = g \cdot f(x)$.

Definition 5. Define $D^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ and S^n its boundary.

2 G-C.W Complexes

Definition 6. Fix a group G. Start with a set of "points" of the form $G/H_\alpha \times D^0$ where H_α are subgroups. Denote this set X^0 and call these points 0-cells. We define X^n from X^{n-1} inductively. First define equivariant attaching maps $\varphi_\alpha : G/H_\alpha \times S^{n-1} \to X^{n-1}$ then X^n is obtained by the pushout diagram

$$\begin{array}{ccc} \sqcup_\alpha G/H_\alpha \times S^{n-1} & \xrightarrow{f = \sqcup_\alpha \varphi_\alpha} & X^{n-1} \\ \downarrow \iota & & \downarrow \\ \sqcup_\alpha G/H_\alpha \times \overline{D^n} & \longrightarrow & X^n \end{array}$$

Explicitly, we have

$$X^n = X^{n-1} \cup_f \sqcup_\alpha G/H_\alpha \times \overline{D^n}$$

Then we call a space of the form $X = \cup X^n$ a **G-C.W complex** and equip it with the weak topology: U is closed if and only if $U \cap X^n$ is closed for all n.

In general we call the images of $G/H_\alpha \times \overline{D^n}$ in X^n $n-$cells and call the collection of all $n-$cells the $n-$skeleton of X.

Remark. For X, Y $G-$spaces, we take diagonal action on $X \times Y$ i.e $g \cdot (x, y) = (g \cdot x, g \cdot y)$. Furthermore, take the convention that G acts trivially on $\overline{D^n}$ so for some $g \in G$ and $(g'H, x) \in G/H \times \overline{D^n}$ we have $g \cdot (g'H, x) = (gg'H, x)$, this allows us to talk about the notion of an equivariant attaching map $G/H_\alpha \times S^{n-1} \to X^{n-1}$.

Example 1. For $G = C_2 \times C_2$ with generators ϵ, σ and identity denoted e, consider the $G - C.W$ complex decomposition onto S^1 as follows. Let $X^0 = G/\langle\epsilon\rangle \times D^0 \cup G/\langle\sigma\rangle \times D^0$ where say

$$x = e\langle\sigma\rangle \times D^0$$
$$z = \epsilon\langle\sigma\rangle \times D^0$$
$$y = e\langle\epsilon\rangle \times D^0$$
$$t = \sigma\langle\epsilon\rangle \times D^0$$

and consider a single attaching map $\varphi : G/e \times S^0 \to X^0$ given by $(e, 0) \to x, (e, 1) \to y$ where the rest of the

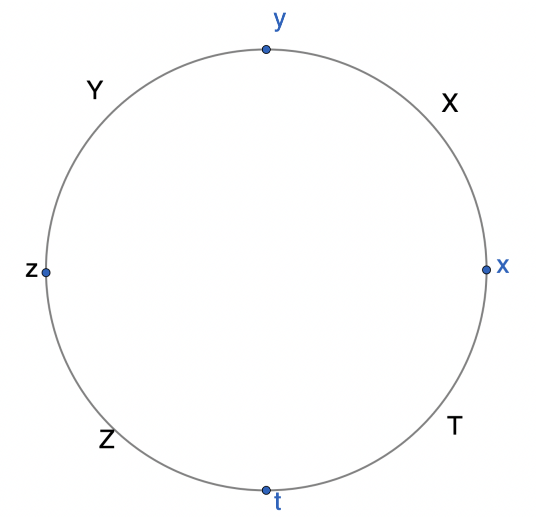


Figure 1: Example 1

map is determined uniquely via equivariance.

$$(\epsilon, 0) \to \epsilon \cdot x = z$$
$$(\epsilon, 1) \to \epsilon \cdot y = y$$
$$(\sigma, 0) \to \sigma \cdot x = x$$
$$(\sigma, 1) \to \sigma \cdot y = t$$
$$(\sigma \otimes \epsilon, 0) \to \sigma \otimes \epsilon \cdot x = z$$
$$(\sigma \otimes \epsilon, 1) \to \sigma \otimes \epsilon \cdot y = t$$

This is all that's needed to give S^1 a $G - C.W$ complex structure and can be geometrically visualised by figure 1.

3 Mackey Functors

Definition 7. Let $\mathcal{C}$ be a category. Let $(\mathcal{C})^{op}$ be the category with the same objects as $\mathcal{C}$ but $Hom_{\mathcal{C}}(A, B) = Hom_{(\mathcal{C})^{op}}(B, A)$. This is called the **opposite category**.

Notation. Let $\mathcal{F}_{\mathcal{G}}$ be the category of finite $G-$sets with morphisms G equivariant maps. Also let $\mathcal{A}b$ be the category of abelian groups.

Definition 8. A **Mackey functor** $\underline{M}$ for a finite group G is a pair of functors

$$M_* : \mathcal{F}_G \to \mathcal{A}b, M^* : (\mathcal{F}_G)^{op} \to \mathcal{A}b$$

such that:

- finite disjoint unions go to direct sums,
- $M_*(T) = M^*(T) := \underline{M}(T)$ for $T \in \mathcal{F}_G$
- for a pullback

$$\begin{array}{ccc} R & \xrightarrow{\alpha} & S \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ T & \xrightarrow[\delta]{} & U \end{array}$$

  we have $M^*(\gamma)M_*(\delta) = M_*(\alpha)M^*(\beta)$.

For subgroups $K \subseteq H \subseteq G$ with projection $p : G/K \to G/H$ let

$$Res_K^H = M^*(p), Tr_K^H = M_*(p)$$

.

Definition 9. For X a $G-$set and H a subgroup of G let

$$X^H := \{x \in X : h \cdot x = x, \text{for all } h \in H\}$$

be the **fixed point set** for the subgroup H.

Definition 10. Let M be a $\mathbb{Z}[G]-$module. Define the **fixed point Mackey functor** $\underline{M}$ by $\underline{M}(G/H) = M^H$. Note that as any G set can be decomposed into orbits of the form G/H, this definition can be extended to any G set.

For $K \subseteq H \subseteq G$ define

$$Tr_K^H : M^K \to M^H$$

by

$$Tr_K^H(x) = \sum_{\gamma K \in H/K} \gamma K(x)$$

and define

$$Res_K^H(x) = x$$

.

4 Equivariant Homology

For a $G - C.W$ Complex X define $C_n(X) = H_n(X^n, X^{n-1})$ and define boundary operator $d_n : C_n(X) \to C_{n-1}(X)$ by $d_n = \iota \circ \delta$ where δ is the connecting homomorphism from the long exact sequence of pair (X^n, X^{n-1}) and ι inclusion. Note that we can define $(C_*(X), d_*)$ as the cellular chain complex of X and define homology accordingly.

Note that for a $C.W - Complex$, the homology $H_*(X^k, X^{k-1}) = H_*(X^k/X^{k-1}, pt) = \tilde{H}_*(X^k/X^{k-1})$ because X^k, X^{k-1} is a good pair. And $\tilde{H}_*(X^k/X^{k-1})$ is very easy to calculate as the space X^k/X^{k-1} is just the wedge of many spheres.

Note that given an equivariant map $C_n(X) \to C_{n-1}(X)$ induces a map $d_n : C_n(X)^H \to C_{n-1}(X)^H$ on fixed point sets which leads us to the following definition:

Definition 11. For each G/H we can apply the fixed point functor to $C_*(X)$ to obtain a complex:

$$... \to C_{n+1}(X)^H \to C_n(X)^H \to C_{n-1}(X)^H \to ... \to C_0(X)^H \to 0$$

taking homology we obtain **equivariant homology groups**

$$H_n(X)(G/H) = \frac{kerd_n}{imd_{n+1}}$$

Example 2. For our example of S^1 acted on by $G = C_2 \times C_2$ we have

$$C_0(X) = H_0(X^0) = \mathbb{Z}x \oplus \mathbb{Z}y \oplus \mathbb{Z}z \oplus \mathbb{Z}t$$

and

$$C_1(X) = \tilde{H}_1(S^1 \vee S^1 \vee S^1 \vee S^1) = \mathbb{Z}X \oplus \mathbb{Z}Y \oplus \mathbb{Z}Z \oplus \mathbb{Z}T$$

with map $d_1 : C_1(X) \to C_0(X)$ given by the following action on generators

$$X \to y - x$$

$$Y \to y - z$$

$$Z \to t - z$$

$$T \to t - x$$

Now d_1 has kernel $\mathbb{Z}(X - Y + Z - T)$ and image $\mathbb{Z}(y - x) \oplus \mathbb{Z}(y - z) \oplus \mathbb{Z}(t - z)$. Since the fixed point functor for subgroup 0 is identical to the same complex

$$0 \to \mathbb{Z}X \oplus \mathbb{Z}Y \oplus \mathbb{Z}Z \oplus \mathbb{Z}T \to \mathbb{Z}x \oplus \mathbb{Z}y \oplus \mathbb{Z}z \oplus \mathbb{Z}t \to 0$$

we have the following (equivariant) homology groups

$$\begin{aligned} H_0(S^1)(G) &= \frac{kerd_0}{imd_1} \\ &= \frac{ker0}{imd_1} \\ &= \frac{\mathbb{Z}x \oplus \mathbb{Z}y \oplus \mathbb{Z}z \oplus \mathbb{Z}t}{\mathbb{Z}(y - x) \oplus \mathbb{Z}(y - z) \oplus \mathbb{Z}(t - z)} \\ &= \frac{\mathbb{Z}(x) \oplus \mathbb{Z}(y - x) \oplus \mathbb{Z}(z - y) \oplus \mathbb{Z}(t - z)}{\mathbb{Z}(y - x) \oplus \mathbb{Z}(y - z) \oplus \mathbb{Z}(t - z)} \\ &= \mathbb{Z} \end{aligned}$$

and

$$\begin{aligned}H_1(S^1)(G) &= \frac{kerd_1}{im0}\\ &= kerd_1\\ &= \mathbb{Z}(X - Y + Z - T)\\ &= \mathbb{Z}\end{aligned}$$

Now let's consider the subgroup $\langle\sigma\rangle$ to find $H_*(S^1)(\langle\sigma\rangle)$. We take the chain complex

$$0 \to \mathbb{Z}X \oplus \mathbb{Z}Y \oplus \mathbb{Z}Z \oplus \mathbb{Z}T \to \mathbb{Z}x \oplus \mathbb{Z}y \oplus \mathbb{Z}z \oplus \mathbb{Z}t \to 0$$

and apply the fixed point functor on $G/\langle\sigma\rangle$ to obtain

$$0 \to (\mathbb{Z}X \oplus \mathbb{Z}Y \oplus \mathbb{Z}Z \oplus \mathbb{Z}T)^{\langle\sigma\rangle} \to (\mathbb{Z}x \oplus \mathbb{Z}y \oplus \mathbb{Z}z \oplus \mathbb{Z}t)^{\langle\sigma\rangle} \to 0$$

To compute this, notice that $\sigma \cdot (aX + bY + cZ + dT) = aT + bZ + cY + dX$ so $(\mathbb{Z}X \oplus \mathbb{Z}Y \oplus \mathbb{Z}Z \oplus \mathbb{Z}T)^{\langle\sigma\rangle}$ is generated by $X + T$ and $Y + Z$. Similarly it can be shown that $(\mathbb{Z}x \oplus \mathbb{Z}y \oplus \mathbb{Z}z \oplus \mathbb{Z}t)^{\langle\sigma\rangle}$ is generated by x, z and $y + t$ so our complex is given by

$$0 \to \mathbb{Z}(X + T) \oplus \mathbb{Z}(Y + Z) \to \mathbb{Z}x \oplus \mathbb{Z}z \oplus \mathbb{Z}(y + t) \to 0$$

. Now the restriction of d_1 onto fixed point sets gives maps $X + T \to t + y - 2x$ and $Y + Z \to t + y - 2z$ so the kernel is 0 and the image is $\mathbb{Z}(t + y - 2x) \oplus \mathbb{Z}(t + y - 2z)$ hence we have the following equivariant homology groups

$$\begin{aligned}H_0(S^1)(G/\langle\sigma\rangle) &= \frac{kerd_0}{imd_1}\\ &= \frac{\mathbb{Z}x \oplus \mathbb{Z}z \oplus \mathbb{Z}(y + t)}{\mathbb{Z}(t + y - 2x) \oplus \mathbb{Z}(t + y - 2z)}\\ &= \frac{\mathbb{Z}x \oplus \mathbb{Z}z \oplus \mathbb{Z}(y + t - 2z)}{\mathbb{Z}(2z - 2x) \oplus \mathbb{Z}(t + y - 2z)}\\ &= \frac{\mathbb{Z}(x) \oplus \mathbb{Z}(z - x)}{\mathbb{Z}(2(z - x))}\\ &= \mathbb{Z} \oplus \mathbb{Z}/2\end{aligned}$$

and

$$\begin{aligned}H_1(S^1)(G/\langle\sigma\rangle) &= kerd_1\\ &= 0\end{aligned}$$

By the symmetry between action σ and ϵ we can say that $H_n(S^1)(G/\langle\sigma\rangle) = H_n(S^1)(G/\langle\epsilon\rangle)$ for all n.

Now looking at the $\langle\sigma \otimes \epsilon\rangle$ invariant chain groups we get complex

$$0 \to \mathbb{Z}(Z + X) \oplus \mathbb{Z}(Y + T) \to \mathbb{Z}(x + z) \oplus \mathbb{Z}(y + t) \to 0$$

with constant middle map with image $\mathbb{Z}(y+t-(x+z))$ and kernel $\mathbb{Z}(X+Z-Y-T)$ so we obtain the following equivariant homology groups

$$H^0(S^1)(G/\langle\sigma\otimes\epsilon\rangle) = \frac{(C^0(X))^{\langle\sigma\otimes\epsilon\rangle}}{imd_1} = \frac{\mathbb{Z}(x+z)\oplus\mathbb{Z}(y+t)}{\mathbb{Z}(y+t-x-z)} = \mathbb{Z}$$

and

$$H^1(S^1)(G/\langle\sigma\otimes\epsilon\rangle) = \mathbb{Z}(X+Z-Y-T) = \mathbb{Z}.$$

Now for subgroup G we have that the fixed 0 chains are generated by $x+z$ and $y+t$ and the fixed 1 chains are generated by $X+Y+Z+T$ so the complex is given by

$$0 \to \mathbb{Z}(X+Y+Z+T) \to \mathbb{Z}(x+z)\oplus\mathbb{Z}(y+t) \to 0$$

with map $X+Y+Z+T \to 2y+2t-2x-2t$ so the kernel is 0 and the image is $\mathbb{Z}(2y+2t-2x-2z)$ so the equivariant homology groups are

$$H_0(S^1)(G/G) = \frac{\mathbb{Z}(x+z)\oplus\mathbb{Z}(y+t)}{\mathbb{Z}(2y+2t-2x-2z)} = \mathbb{Z}\oplus\mathbb{Z}/2$$

and

$$H_1(S_1)(G/G) = 0.$$

5 Equivariant Cohomology

Now that we understand the notion of equivariant homology lets define the dual notion **Equivariant Cohomology**. Recall that for a $G-C.W$ complex we have complex $(C_*(X), d_*)$. Dualise this to obtain complex $(C^*(X), d^*)$:

$$... \leftarrow C^{n+1}(X) \leftarrow C^n(X) \leftarrow C^{n-1}(X) \leftarrow ... \leftarrow C^0(X) \leftarrow 0$$

where $C^k(X) = Hom(C_k(X), \mathbb{Z})$ and for some $f \in C^k(X)$ we have $d^{k+1}(f) = f \circ d_k$.

Definition 12. Applying the fixed point functor to $C^*(X)$ we obtain complex

$$... \leftarrow (C^{n+1}(X))^H \leftarrow (C^n(X))^H \leftarrow (C^{n-1}(X))^H \leftarrow ... \leftarrow (C^0(X))^H \leftarrow 0$$

and define **equivariant cohomology groups**

$$H^n(X)(G/H) = \frac{ker d^{n+1}}{imd^n}$$

Now to aid with computation, suppose that for some $G-C.W$ complex X we have $C_k(X) = \bigoplus \mathbb{Z}v_i$, then from a relativley well known isomorphism we have

$$C^k(X) = Hom(\bigoplus \mathbb{Z}v_i, \mathbb{Z}) = \Pi Hom(\mathbb{Z}v_i, \mathbb{Z})$$

And denote v_i^* to be the map $v_i \to 1$ and $v_j \to 1$ for $i \neq j$. Then we can write

$$C^k(X) = \Pi \mathbb{Z} v_i^*$$

.

Example 3. For the example of $G = C_2 \times C_2$ acting on $X = S^1$ we have complex $(C_*(X), d_*)$

$$0 \to \mathbb{Z}X \oplus \mathbb{Z}Y \oplus \mathbb{Z}Z \oplus \mathbb{Z}T \to \mathbb{Z}x \oplus \mathbb{Z}y \oplus \mathbb{Z}z \oplus \mathbb{Z}t \to 0$$

with map $d_1 : X \to y - x, Y \to y - z, Z \to t - z, T \to t - x$. So when we dualise we have complex $(C^*(X), d^*)$

$$0 \leftarrow \mathbb{Z}X^* \oplus \mathbb{Z}Y^* \oplus \mathbb{Z}Z^* \oplus \mathbb{Z}T^* \leftarrow \mathbb{Z}x^* \oplus \mathbb{Z}y^* \oplus \mathbb{Z}z^* \oplus \mathbb{Z}t^* \leftarrow 0$$

Now let's compute the map d^1. It suffices to show what d^1 does on x^*, y^*, z^* and t^*. First let's compute the map $d^1(x^*)$.

$$\begin{aligned} d^1(x^*)(X) &= (x^* \circ d_1)(X) \\ &= x^*(y - x) \\ &= -1 \end{aligned}$$

$$\begin{aligned} d^1(x^*)(Y) &= (x^* \circ d_1)(Y) \\ &= x^*(y - z) \\ &= 0 \end{aligned}$$

$$\begin{aligned} d^1(x^*)(Z) &= (x^* \circ d_1)(Z) \\ &= x^*(t - z) \\ &= 0 \end{aligned}$$

$$\begin{aligned} d^1(x^*)(T) &= (x^* \circ d_1)(T) \\ &= x^*(t - x) \\ &= -1 \end{aligned}$$

Hence we have that $d^1(x^*) = -X^* - T^*$. A very similar computation which will be omitted shows that

$$d^1(y^*) = X^* + Y^*, d^1(z^*) = -Y^* - Z^*, d^1(t^*) = Z^* + T^*$$

Hence we can see that the kernel is $\mathbb{Z}(x^*+y^*+z^*+t^*)$ and the image is $\mathbb{Z}(-X^*-T^*)\oplus\mathbb{Z}(-Y^*-Z^*)\oplus\mathbb{Z}(X^*+Y^*)$. Now can can compute the cohomology as follows

$$H^0(S^1)(G) = kerd_1 = \mathbb{Z}$$

and

$$H^1(S^1)(G) = \frac{ker0}{imd^1} = \frac{\mathbb{Z}X^* \oplus \mathbb{Z}Y^* \oplus \mathbb{Z}Z^* \oplus \mathbb{Z}T^*}{\mathbb{Z}(-X^* - T^*) \oplus \mathbb{Z}(-Y^* - Z^*) \oplus \mathbb{Z}(X^* + Y^*)} = \mathbb{Z}$$

Now to compute the equivariant cohomology groups $H^*(X)(G/H)$ for other subgroups H we will use the following result.

Proposition 13. *For $v \in C_k(X)$, $g \cdot v^* = (g^{-1} \cdot v)^*$.*

Proof. Note that for some $x \in C_k(X)$ we have

$$(g \cdot v^*)(x) = v^*(g \cdot x)$$

which is 1 if $x = g^{-1} \cdot v$ and 0 otherwise which means $g \cdot v^* = (g^{-1} \cdot v)^*$. □

So it is easy to see that

$$\underline{C^0}(X)(G/\langle\sigma\rangle) = (C^0(X))^{\langle\sigma\rangle} = \mathbb{Z}x^* \oplus \mathbb{Z}z^* \oplus \mathbb{Z}(y^* + t^*)$$

and

$$\underline{C^1}(X)(G/\langle\sigma\rangle) = (C^1(X))^{\langle\sigma\rangle} = \mathbb{Z}(X^* + T^*) \oplus \mathbb{Z}(Y^* + Z^*)$$

and the boundary map $d^1 : (C^0(X))^{\langle\sigma\rangle} \to (C^1(X))^{\langle\sigma\rangle}$ given by

$$x^* \to -(X^* + T^*), z^* \to -(Y^* + Z^*), y^* + t^* \to X^* + T^* + Y^* + Z^*$$

So the kernel is $\mathbb{Z}(x^*+y^*+z^*+t^*)$ and image is $\mathbb{Z}(X^*+T^*)\oplus\mathbb{Z}(Y^*+Z^*)$ Hence we have equivariant cohomology groups

$$H^0(X)(G/\langle\sigma\rangle) = kerd^1 = \mathbb{Z}$$

and

$$H^1(X)(G/\langle\sigma\rangle) = \frac{(C^1(X))^{\langle\sigma\rangle}}{imd^1} = \frac{\mathbb{Z}(X^* + T^*) \oplus \mathbb{Z}(Y^* + Z^*)}{\mathbb{Z}(X^* + T^*) \oplus \mathbb{Z}(Y^* + Z^*)} = 0$$

Similar to Homology notice that $H^*(X)(G/\langle\sigma\rangle) = H^*(X)(G/\langle\epsilon\rangle)$ as both subgroups induce isomorphic actions.

Now looking at the $\sigma \otimes \epsilon$ invariant chains we get cochain complex

$$0 \leftarrow \mathbb{Z}(X^* + Z^*) \oplus \mathbb{Z}(Y^* + T^*) \leftarrow \mathbb{Z}(x^* + z^*) \oplus \mathbb{Z}(y^* + t^*) \leftarrow 0$$

with middle map $x^* + z^* \to -(X^* + Y^* + Z^* + T^*)$ and $y^* + t^* \to X^* + Y^* + Z^* + T^*$ so the kernel is $\mathbb{Z}(x^* + y^* + z^* + t^*)$ and image $\mathbb{Z}(X^* + Y^* + T^* + Z^*)$ so we have equivariant cohomology groups

$$H^0(X)(G/\langle\sigma \otimes \epsilon\rangle) = \mathbb{Z}$$

and

$$H^1(X)(G/\langle\sigma\otimes\epsilon\rangle) = \frac{\mathbb{Z}(X^*+Z^*)\otimes\mathbb{Z}(Y^*+T^*)}{\mathbb{Z}(X^*+Y^*+Z^*+T^*)} = \mathbb{Z}$$

Finally we consider G invariant complexes. Note that the G orbit of X^* is all of $C^1(X)$ so $(C^1(X))^G = \mathbb{Z}(X^*+Y^*+Z^*+T^*)$ and for the 0 cells we have orbits on generators $\{x^*, z^*\}, \{y^*, t^*\}$ so we have cochain complex $((C^*(X))^G, d^*)$

$$0 \leftarrow \mathbb{Z}(X^*+Y^*+Z^*+T^*) \leftarrow \mathbb{Z}(x^*+z^*)\oplus\mathbb{Z}(y^*+t^*) \leftarrow 0$$

with map $x^*+z^* \to -(X^*+Y^*+T^*+Z^*)$ and $y^*+t^* \to X^*+Y^*+Z^*+T^*$ so the kernel is $\mathbb{Z}(y^*+y^*-x^*-z^*)$ and image is $\mathbb{Z}(X^*+Y^*+Z^*+T^*)$ hence the equivariant cohomology groups are

$$H^0(S^1)(G/G) = kerd^1 = \mathbb{Z}$$

and

$$H^1(S^1)(G/G) = \frac{\mathbb{Z}(X^*+Y^*+Z^*+T^*)}{\mathbb{Z}(X^*+Y^*+Z^*+T^*)} = 0$$

6 Representation Spheres

Definition 14. For a (finite dimensional) orthogonal representation V of a finite group G (or equivalently any $\mathbb{R}[G]$-module) define S^V as its **one point compactification**.

Define $S(V) := \{v \in V : |v| = 1\}$ and $D(V) := \{v \in V : |v| \leq 1\}$, then we have $S^V = D(V)/S(V)$ with cofiber sequence $S(V) \to D(V) \to S^V$.

Proposition 15. *Let $G = C_{p^l}$ where p is prime and l is a positive integer. Let G^i denote the subgroup of G with index p^i and define $G_i := G/G^i$. Let V be a non trivial represnetation of G. Then S^V has $G-C.W$ complex decomposition with*

- *A single $|V^G|-cell$*
- *For each m with $|V^{G^{i-1}}| < m \leq |V^{G^i}|$ a $m-cell$ $G_i \times D^m$*

Lemma 16. *For $G = C_{p^l}$ as above, a G action on a G-space X induces an action of G_i on X^{G^i}.*

Proof. Let X be a G space. Define action on X^{G^i} by $G_i = G/G^i$ by $gG^i \cdot x = g \cdot x$. This is well defined as if $gG^i = g'G^i$ then there exist $g_i \in G^i$ such that $g = g_i g'$ hence

$$(gG^i)\cdot x = g\cdot x = g^i g' \cdot x = g'x = (g'G^i)\cdot x$$

The rest of the axioms are easy to see. □

Now we will prove **proposition 13**

Proof. Note that we have fixed point sets

$$S^{V^G} \subset S^{V^{G^1}} \subset S^{V^{G^2}} \subset ... \subset S^{V^{G^{l-1}}} \subset S^{V^{G^l}} = S^V$$

Now take $G/G \times D^0$ be a single point and attach a $|V^G|$ cell $G/G \times D^{|V^G|}$ via constant attaching map $G/G \times S^{|V^G|-1} \to X^0$ to get S^{V^G}. Note that if $|V^G| = 0$, the $0-$cell will be attached via $S^{-1} \to X^0$ which by convention is the map $\emptyset \to S^0$ which gives two points ($0 \in V$ and the point at infinity).

Now, given a $G - C.W$ structure on $S^{V^{G^{i-1}}}$ we inductively put a $G - C.W$ structure on $S^{V^{G^i}}$ where the extension is non trivial. First note that we need cells of up to dimension $|V^{G^i}|$. Suppose that in $S^{V^{G^{i-1}}}$ we already have cells of dimension $|V^{G^{i-1}}|$. Then we need to attach cells of the form $G/G^k \times D^m$ where $|V^{G^{i-1}}| < m \leq |V^{G^i}|$. We want $k = i$ as we need elements fixed by G^i. Thus we have

$$S^{V^{G^i}} = S^{V^{G^i}} \cup (\bigcup_{|V^{G^{i-1}}| < m \leq |V^{G^i}|} G_i \times D^m)$$

and explicitly we have cell decomposition

$$S^V = X^0 \cup (\bigcup_{i=1}^{l} \bigcup_{|V^{G^{i-1}}| < m \leq |V^{G^i}|} G_i \times D^m)$$

with one cell for each dimension. For a fixed i we attach $G_i \times D^{m+1}$ to $G^i \times D^m$ in the obvious way.

To attach $G_i \times D^m$ to $G_{i-1} \times D^{m-1}$ we can simply attach the $G_{i-1} \hookrightarrow G_i$ subgroup in the obvious way and the rest of the attaching should be clear from equivariance. $\square$

Example 4. Take $G = C_{2^2}$. Let $V = \mathbb{R}^2$ with representation $\rho : G \to GL(V)$

$$0 \to id_{\mathbb{R}^2}$$

$$1 \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$2 \to \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$3 \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Then we have fixed point sets

$$S^{V^G} \subseteq S^{V^{G^1}} \subseteq S^V$$

where $S^{V^G} = S^{V^{G^1}} = \{0, \infty\}$; the south and north poles of $S^V = S^2$. And since $V^G = 0$ we attach a single 0-cell to a point. Then since $0 = |S^{V^{G^1}}| < 1, 2 \leq |S^V| = 2$ we attach cells $G \times D^1$ and $G \times D^2$. This gives

G-C.W complex $S^V = G/G \times D_0^0 \cup G/G \times D_1^0 \cup \times G \times D^1 \cup G \times D^2$.
The attaching map $\varphi_1 : G \times S^0 \to X^0 = G/G \times D^0 \cup G/G \times D^0$ is given by

$$(g^i, 0) \to G/G \times D_0^0$$

$$(g^i, 1) \to G/G \times D_1^0$$

for all i. Now we have $X^1 = \{(0, y, z) \in \mathbb{R}^3 : y^2 + z^2 = 1\} \cup \{(x, 0, z) \in \mathbb{R}^3 : x^2 + z^2 = 1\}$. To construct the attaching map $\varphi_2 : G \times S^1 \to X^1$, we attach $e \times S^1$ to X^1 by

$$(e, e^{it}) \to \begin{cases} \begin{pmatrix} sint \\ 0 \\ -cost \end{pmatrix} & t \in [0, \pi] \\ \begin{pmatrix} 0 \\ sint \\ -cost \end{pmatrix} & t \in (\pi, 2\pi) \end{cases}$$

which attaches it to $e \times D^1$ and $g \times D^1$, the rest follows from equivariance. The picture is given in figure 2.

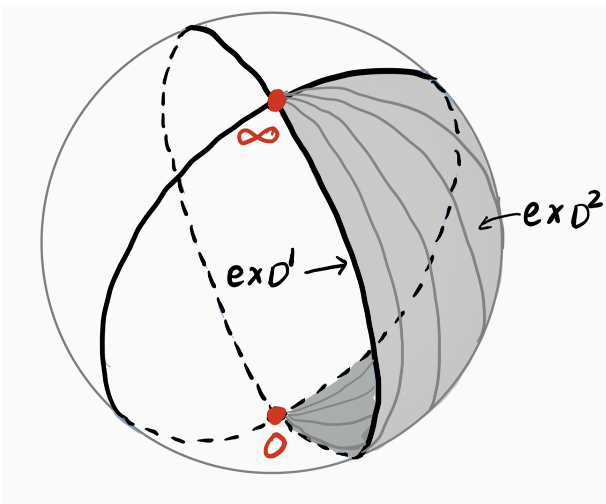


Figure 2: S^V

7 Discussion and Conclusion

In conclusion, we have developed the theories of homology and cohomology in an equivariant setting where we upgraded our topological spaces to be equip with symmetries of a finite group. This was made computationally possible by considering such spaces as $G-C.W$ complexes and taking a slight modification to the typical cellular homology and cohomology used in basic algebraic topology. Another advantage to considering group actions is that given a topological space X and group G one cannot always equip X with a non trivial G action hence even the construction of a G space reveals alot about the underlying (non-equivariant) space.

This subject is quite dense and in my research I have struggled to find many concrete examples and computations without using quite advanced techniques, hence I tried to give examples here which can be easily followed by one with minimal background in the subject. These examples illustrate the underlying geometric notions of the subject The major result of this paper is the constructive proof that the representation sphere for any non trivial representation of C_{p^l} has a $G-C.W$ complex decomposition with a single cell in each dimension.

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