AMSI SUMMER RESEARCH SCHOLARSHIPS 2023–24

SET YOUR SIGHTS ON RESEARCH THIS SUMMER

Stochastic Optimal Control and Robust Filtering Under Rough Paths

Jonathan Anthony Mavroforas

Supervised by Professor Anthony Dooley

University of Technology, Sydney

AMSI AUSTRALIAN MATHEMATICAL SCIENCES INSTITUTE

Abstract

Stochastic optimal control and robust filtering is considered from the path-wise perspective. This is done using the theory of rough paths, and a gentle introduction is given in the first section of this report. Path-wise optimal control results are developed using this machinery and applications are then given to robust stochastic filtering, where the filtering problem is transformed into a path-wise one. This path-wise filtering problem is further transformed into an optimal control problem where the rough path theory is applied.

Contents

1 Introduction

In this report, stochastic optimal control and robust filtering is studied from the path-wise perspective. This is done most naturally under the theory of rough paths since, for example, the sample trajectories of Brownian motion are not smooth; that is, Brownian motion is not differentiable. The Brownian sample paths are "rough" in a precise sense, which will be made clear. Naively applying path-wise optimal control theory to the stochastic case produces degeneracy issues that must be rectified, and this is done by introducing "cost functions". Lastly, "robust" stochastic filtering can be transformed into a path-wise optimal control problem, which lends itself to the optimal control theory constructed in this paper. A "robust" point estimate can then be obtained by solving a differential equation and an optimization problem.

1.1 A Note on Proofs

Proofs of the statements contained in this paper are relegated to the appendices, unless stated otherwise.

1.2 Statement of Authorship

The path-wise optimal control and filtering theorems in this paper were discovered by A. L. Allan and S. N. Cohen [2], with additional *lemmas* and *propositions* introduced by J. A. Mavroforas. Moreover, the work was reviewed by A. H. Dooley.

2 Rough Path Preliminaries

Some general results about rough path theory will be developed in this section. For a comprehensive overview of the subject, the reader is referred to [5].

2.1 General Integration Theory

In the following, J will denote the closed interval $[0, T]$ and $|\cdot|$ will denote the Euclidean norm on $\mathbb{R}^n$. We begin with some standard definitions.

Definition 2.1. Suppose that $X\colon J \to \mathbb{R}^n$, $t \mapsto X_t$ and $0 \leq s \leq t \leq T$.

$$\Delta_{[s,t]} \coloneqq \{(s,t) \in J^2 \colon 0 \leq s \leq t \leq T\} \tag{2.1}$$

$$X_{s,t} \coloneqq X_t - X_s \tag{2.2}$$

$$||X||_{p,J} \coloneqq \left[\sup_{\mathcal{D}} \sum |X_{t_i,t_{i+1}}|^p\right]^{\frac{1}{p}} \tag{2.3}$$

where $\mathcal{D} = (t_i)_{i=0}^n$ denotes a partition of $J = [0, T]$. To be explicit, the supremum in (2.3) is being taken over all partitions $\mathcal{D}$ of $[a, b]$.

Let us now denote the space of all continuous paths $X\colon J \to \mathbb{R}^n$ with finite p-variation (2.3) by $\mathcal{V}^{p\text{-var}}(J,\mathbb{R}^n)$. Furthermore, let $\mathcal{V}^{0,p\text{-var}}(J,\mathbb{R}^n)$ denote its closure with respect to the p-variation seminorm $||\cdot||_{p,J}$. Finally, define for $p \in [2,3)$ the set $\mathscr{C}^p(J,\mathbb{R}^n)$ consisting of all $\frac{1}{p}$-Hölder rough paths $\boldsymbol{\zeta} = (\zeta, \zeta^{(2)})$; that is, the set of all $\boldsymbol{\zeta}$ satisfying:

$$\zeta\colon J \to \mathbb{R}^n \tag{2.4}$$

$$\zeta^{(2)}\colon \Delta_{[s,t]} \to \mathbb{R}^n \otimes \mathbb{R}^n,\ (s,t) \mapsto \zeta^{(2)}_{s,t} \tag{2.5}$$

$$\zeta^{(2)}_{s,t} = \zeta^{(2)}_{s,r} + \zeta^{(2)}_{r,t} + \zeta_{s,r} \otimes \zeta_{r,t} \tag{2.6}$$

$$\begin{aligned} |||\boldsymbol{\zeta}|||_{\frac{1}{p}-\text{Höl}} &\coloneqq ||\zeta||_{\frac{1}{p}-\text{Höl}} + ||\zeta^{(2)}||_{\frac{2}{p}-\text{Höl}} \\ &< +\infty \end{aligned} \tag{2.7}$$

where

$$||\zeta||_{\frac{1}{p}-\text{Höl}} \coloneqq \sup_{s \neq t \in J} \frac{|\zeta_{s,t}|}{|t-s|^{\frac{1}{p}}} \tag{2.8}$$

and

$$||\zeta^{(2)}||_{\frac{2}{p}-\text{Höl}} \coloneqq \sup_{s \neq t \in J} \frac{|\zeta^{(2)}_{s,t}|}{|t-s|^{\frac{2}{p}}}. \tag{2.9}$$

The tensor product in (2.5) and (2.6) is the *Cartesian tensor product*. We also assume that (2.6) holds for all r such that $0 \leq s \leq r \leq t \leq T$. Equation (2.6) is called *Chen's relation* and $\zeta^{(2)}$ is called the *lift* (or *lift*) of ζ. Moreover, we sometimes call $\boldsymbol{\zeta} = (\zeta, \zeta^{(2)})$ the *lift* of ζ. The tensor product in (2.5) and (2.6) is the *Cartesian tensor product*. We also assume that (2.6) holds for all r such that $0 \leq s \leq r \leq t \leq T$. Equation (2.6) is called *Chen's relation* and $\zeta^{(2)}$ is called the *lift* of ζ.

The variation of a rough path $\boldsymbol{\zeta} = (\zeta, \zeta^{(2)})$ can now be defined.

Definition 2.2. Suppose that $\boldsymbol{\zeta} = (\zeta, \zeta^{(2)}) \in \mathscr{C}^p(J,\mathbb{R}^n)$. We define the variation of $\zeta^{(2)}$ and then the variation of $\boldsymbol{\zeta}$:

$$||\zeta^{(2)}||_{\frac{p}{2},J} \coloneqq \left[\sup_{\mathcal{D}} \sum |\zeta^{(2)}_{t_i,t_{i+1}}|^{\frac{p}{2}}\right]^{\frac{2}{p}} \tag{2.10}$$

$$|||\boldsymbol{\zeta}|||_{p,J} \coloneqq ||\zeta||_{p,J} + ||\zeta^{(2)}||_{\frac{p}{2},J} \tag{2.11}$$

where the supremum in (2.10) is taken over all partitions $\mathcal{D}$ of J.

An immediate consequence of **Definition 2.2** is that rough paths have finite p-variation for $p \in [2,3)$:

Proposition 2.1. *If* $\boldsymbol{\zeta} = (\zeta, \zeta^{(2)}) \in \mathscr{C}^p(J,\mathbb{R}^n)$*, then* $|||\boldsymbol{\zeta}|||_{p,J} < +\infty$.

We turn $\mathscr{C}^p(J,\mathbb{R}^n)$ into a metric space by:

Definition 2.3 (Rough Path Metrics)**.** Suppose that $\boldsymbol{\zeta} = (\zeta, \zeta^{(2)})$, $\boldsymbol{\eta} = (\eta, \eta^{(2)})$ are two rough paths. The $\frac{1}{p}$-Hölder and p-variation metrics over the interval J are defined to be

$$\varrho_{\frac{1}{p}\text{-Höl},J}(\boldsymbol{\zeta},\boldsymbol{\eta}) \coloneqq ||\zeta - \eta||_{\frac{1}{p}\text{-Höl},J} + \left|\left|\zeta^{(2)} - \eta^{(2)}\right|\right|_{\frac{2}{p}\text{-Höl},J} \tag{2.12}$$

$$\varrho_{p,J}(\boldsymbol{\zeta},\boldsymbol{\eta}) \coloneqq ||\zeta - \eta||_{p,J} + \left|\left|\zeta^{(2)} - \eta^{(2)}\right|\right|_{\frac{p}{2},J} \tag{2.13}$$

Remark 2.1 (Canonical Lift). Smooth paths ζ_t can be *lifted* in a canonical way:

$$\zeta^{(2)}_{s,t} := \int_s^t \zeta_{s,r} \otimes d\zeta_r$$

"Control" in the next definition is not to be confused with its meaning in optimal control theory. Our usage of the word will be clear from the context.

Definition 2.4 (Controlled Rough Paths [6]). Let $\boldsymbol{\zeta} \in \mathscr{C}^p(J, \mathbb{R}^d)$ be a rough path. The space of *controlled rough paths* (with respect to $\boldsymbol{\zeta}$) $\mathscr{D}^p_\zeta(J, \mathbb{R}^m)$ is the set of all

$$(X, X') \in \mathcal{V}^{p\text{-var}}(J, \mathbb{R}^m) \times \mathcal{V}^{p\text{-var}}(J, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^m))$$

such that the remainder term

$$R^X_{s,t} := X_{s,t} - X'_s \zeta_{s,t}$$

has finite $\frac{p}{2}$-variation. X' is called the *Gubinelli derivative* with respect to $\boldsymbol{\zeta}$.

Remark 2.2 (Norm on $\mathscr{D}^p$). $\mathscr{D}^p_\zeta(J, \mathbb{R}^m)$ equipped with the norm

$$|(X, X')| := |X_0| + |X'_0| + ||X'||_{p,J} + ||R^X||_{\frac{p}{2},J}$$

turns it into a *Banach space*.

The existence and uniqueness of rough integrals is given by the next result. The reader is referred to [6] for a proof as it is beyond the scope of this paper.

Theorem 2.2 (Rough Integration). *Recall that* $J = [0, T]$. *Suppose* $p \in [2, 3)$, $\boldsymbol{\zeta} = (\zeta, \zeta^{(2)}) \in \mathscr{C}^p(J, \mathbb{R}^d)$ *and* $(X, X') \in \mathscr{D}^p_\zeta(J, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^m))$. *Then*

$$\int_0^T X_r d\boldsymbol{\zeta}_r := \lim_{|\mathcal{D}| \to 0} \sum_{\mathcal{D}} \left[X_{t_i} \zeta_{t_i, t_{i+1}} + X'_{t_i} \zeta^{(2)}_{t_i, t_{i+1}} \right] \tag{2.14}$$

exists and is unique, and is called the rough integral *of* X *against* $\boldsymbol{\zeta}$. *Furthermore,*

$$\left| \int_s^t X_r d\zeta_r - X_s \zeta_{s,t} - X'_s \zeta^{(2)}_{s,t} \right| \tag{2.15}$$

$$\leq C \left(||R^X||_{\frac{p}{2},[s,t]} ||\zeta||_{p,[s,t]} + ||X'||_{p,[s,t]} ||\zeta^{(2)}||_{\frac{p}{2},[s,t]} \right)$$

where $0 \leq s \leq t \leq T$ *and* C *depends only on* p.

2.2 Integration Theory for Optimal Control

In the *stochastic* optimal control setting, one typically encounters differential equations of the form

$$dX_s = b(X_s, \gamma_s) dt + \sigma(X_s, \gamma_s) dB_s$$

where γ_t is a control process and B_t is a standard Brownian motion. Naively applying path-wise optimal control in the rough differential equation (RDE) setting, however, leads to degeneracy issues, as observed by Diehl, Friz and Gassiat [7] and Allan and Cohen [2]. These issues will be discussed and rectified in the next section, but first we aim to make sense of RDEs taking the form

$$dX_s = b(X_s, \gamma_s)ds + \lambda(X_s, \gamma_s)d\boldsymbol{\zeta}_s$$

which take us out of the standard setting.

Definition 2.5 (Rough Differential Equations)**.** Suppose that $p \in [2, 3)$, $\boldsymbol{\zeta} \in \mathscr{C}^p(J, \mathbb{R}^d)$ and $\gamma \in \mathcal{V}^{\frac{p}{2}\text{-var}}(J, \mathbb{R}^k)$. We will consider rough differential equations (RDEs) of the form

$$dX_s = b(X_s, \gamma_s)ds + \lambda(X_s, \gamma_s)d\boldsymbol{\zeta}_s \tag{2.16}$$

with $X_0 = x_0$, where the integration $\lambda(X_s, \gamma_s)d\boldsymbol{\zeta}_s$ is interpreted in the sense of *theorem 2.2*.

Remark 2.3 (Gubinelli Derivative). Suppose that $(X, X') \in \mathcal{D}^p$ and $\lambda \in C_b^2$. Then $\lambda(X, \gamma)$ has the Gubinelli derivative $\lambda(X, \gamma)' = \partial_x \lambda(X, \gamma)X'$, where $\partial_x \lambda$ is the Fréchet derivative [8] in the first variable defined by

$$\lim_{|h|\to 0} \frac{|\lambda(x + h, \gamma) - \lambda(x, \gamma) - Ah|}{|h|} = 0$$

for some linear transformation A.

Mirroring the proof of *proposition 2.1*, we have

Proposition 2.3. *α-Hölder continuous paths have finite $\frac{1}{\alpha}$-variation, where $\alpha \in (0, 1)$.*

The next lemma is standard and is stated without proof.

Lemma 2.4. *If $1 \le p \le q < +\infty$, then $\mathcal{V}^{1\text{-var}} \subseteq \mathcal{V}^{p\text{-var}} \subseteq \mathcal{V}^{q\text{-var}}$.*

Now we show that $t \mapsto ||X||_{1,t}$ has finite p-variation for $1 \le p < +\infty$.

Lemma 2.5. *If $X \in \mathcal{V}^{1\text{-var}}(J, \mathbb{R}^n)$, then $t \mapsto ||X||_{1,[0,t]}$ has finite p-variation, for $1 \le p < +\infty$.*

A proof of Jensen's inequality can be found in [10]

Lemma 2.6 (Jensen's Inequality)**.** *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is convex and $X : \Omega \to \mathbb{R}^n$ is a random vector. Then $f(\mathbb{E}X) \le \mathbb{E}f(X)$.*

Lemma 2.7. *If $p \ge 1$ and $x_1, \ldots, x_n$ are non-negative numbers, then*

$$(x_1 + \cdots + x_n)^p \le n^p(x_1^p + \cdots + x_n^p).$$

The next two lemmas will be used shortly.

Lemma 2.8. *Suppose that $X \in \mathcal{V}^{p\text{-var}}$. Then $||X||_{p,J} \le n \left[\sum_{\mathcal{D}} ||X||^p_{p,[t_i,t_{i+1}]}\right]$ for any partition $\mathcal{D} = \{t_0 < \cdots < t_n\}$ of J.*

Lemma 2.9. *Suppose that $d_1, \ldots, d_n \in \mathbb{N}_+$ and $x_i \in \mathbb{R}^{d_i}$. Then*

$$|(x_1, \ldots, x_n)| \le |x_1| + \cdots + |x_n|.$$

To simplify the notation, $|X_{s,t}| \lesssim |t-s|$ will mean $|X_{s,t}| \leq C|t-s|$ for some $C > 0$. Furthermore, the next proposition contains new results about RDEs.

Proposition 2.10 (Regularity). *Suppose that* $p \in [2,3)$, $\boldsymbol{\zeta} \in \mathscr{C}^p$, $M \in \mathbb{R}_+$, $|||\boldsymbol{\zeta}|||_{\frac{1}{p}\text{-Höl},J} \leq M$, $b \in Lip_b$, $\psi, \lambda \in C_b^2$, $\gamma \in \mathcal{V}^{\frac{p}{2}\text{-var}}$ *and* X *satisfies the RDE (2.16) with Gubinelli derivative* $X' = \lambda(X,\gamma)$. *Then we have the following regularity results*

1. $||\psi(X,\gamma)'||_{p,J} \lesssim ||X||_{p,J} + ||\gamma||_{\frac{p}{2},J}$
2. $||R^\psi||_{\frac{p}{2},J} \lesssim ||X||_{p,J}^2 + ||R^X||_{\frac{p}{2},J} + ||\gamma||_{\frac{p}{2},J}$
3. $||X||_{p,J} \lesssim 1 + ||\gamma||_{\frac{p}{2},J}^{1+p}$
4. $||R^X||_{\frac{p}{2},J} \lesssim 1 + ||\gamma||_{\frac{p}{2},J}^{2+p}$.

We refer the reader to [2] for proofs of the next two theorems. They are long but not difficult, and rely on standard arguments and the results above.

Theorem 2.11 (Existence & Uniqueness). *Suppose that* $b \in Lip_b$, $\lambda \in C_b^3$ *and* $\boldsymbol{\zeta} \in \mathscr{C}^p$. *If* $\gamma \in \mathcal{V}^{\frac{p}{2}\text{-var}}$ *and* x *is fixed, then there exists a unique solution* $(X, X') \in \mathscr{D}_\zeta^p$ *to the RDE*

$$dX_t = b(X_t, \gamma_s)dt + \lambda(X_t, \gamma_t)d\boldsymbol{\zeta}_t \tag{2.17}$$

with $X_0 = x$.

Theorem 2.12. *Suppose that* $b \in Lip_b$, $\lambda \in C_b^3$ *and the two rough paths* $\boldsymbol{\zeta}, \boldsymbol{\eta} \in \mathscr{C}^p$ *satisfy* $|||\boldsymbol{\zeta}|||_{\frac{1}{p}\text{-Höl}}, |||\boldsymbol{\eta}|||_{\frac{1}{p}\text{-Höl}} \leq M$, *where* $0 < M$. *If* $\gamma, \vartheta \in \mathcal{V}^{\frac{p}{2}\text{-var}}$ *and* $(X, X') = (X, \lambda(X,\gamma)) \in \mathscr{D}_\zeta^p$, $(Y, Y') = (Y, \lambda(Y, \vartheta)) \in \mathscr{D}_\eta^p$, *then*

$$||X' - Y'||_{p,J} \lesssim |x-y| + ||\gamma - \eta||_{\infty,J} + ||\gamma - \eta||_{p,J} + \varrho_{p,J}(\boldsymbol{\zeta}, \boldsymbol{\eta}) \tag{2.18}$$

and if $\psi \in C_b^3$,

$$\begin{aligned} \left|\left|\int_0^\cdot \psi(X_s, \gamma_s)d\boldsymbol{\zeta}_s - \int_0^\cdot \psi(Y_s, \vartheta_s)d\boldsymbol{\eta}_s\right|\right|_{p,J} &\lesssim |x-y| + ||\gamma - \eta||_{\infty,J} + ||\gamma - \eta||_{p,J} \\ &+ \varrho_{p,J}(\boldsymbol{\zeta}, \boldsymbol{\eta}) \end{aligned} \tag{2.19}$$

3 Optimal Control Under Rough Paths

Here we will develop some results regarding path-wise optimal control. To motivate the introduction of regularizing costs, suppose that ζ_t is a path with infinite 1-variation. Then the optimal control problem

$$v(t,x) = \sup_\gamma \mathbb{E} X_T^{t,x,\gamma}$$

governed by the dynamics $dX_s^{t,x,\gamma} = \gamma_s(\sigma dB_s + d\zeta_s)$ with $X_t^{t,x,\gamma} = x$, where the supremum is taken over all controls $-\varepsilon \leq \gamma_s \leq \varepsilon$, has the solution $v(t,x) = +\infty$ for all controls γ [2]. This is due to us being able to control the noise term $d\zeta_s$; we able to take advantage of the fine structure of the path ζ_t since we have infinitely precise observations.

The interpretation of the optimal control problem above is that of a trader maximizing his/her terminal wealth $X_T^{t,x,\gamma}$, trading the asset over the time $[t,T]$, where γ_s denotes the shares held at time s.

3.1 The Set-up

We now define the space of all geometric rough paths. $\mathscr{C}_g^{0,p} \subset \mathscr{C}^p$ will denote the $\varrho_{\frac{1}{p}\text{-Höl}}$ closure of canonical lifts of smooth paths. This is well-defined by the Stone-Weierstrass theorem.

Fix a geometric rough path $\boldsymbol{\zeta} \in \mathscr{C}_g^{0,p}(J, \mathbb{R}^d)$. In this section, we will consider the optimal control problem

$$v(t,x) \coloneqq \inf_{\gamma \in \mathcal{V}^{p/2\text{-var}}} J(t,x,\gamma) \tag{3.1}$$

where

$$J(t,x,\gamma) = \int_t^T f(X_s^{t,x,\gamma}, \gamma_s) ds + \int_t^T \psi(X_s^{t,x,\gamma}, \gamma_s) d\boldsymbol{\zeta}_s + g(X_T^{t,x,\gamma}) \tag{3.2}$$

and $X_s^{t,x,\gamma}$ satisfies the RDE (2.16) subject to $X_t^{t,x,\gamma} = x$. We call $v(t,x)$, $J(t,x,\gamma)$ the *value function* and *cost functional*, respectively. It is understood that $f \colon \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}$, $\psi \colon \mathbb{R}^m \times \mathbb{R}^k \to \mathcal{L}(\mathbb{R}^d, \mathbb{R})$ and $g \colon \mathbb{R}^m \to \mathbb{R}$.

Lemma 3.1. *Suppose the hypotheses of* theorem 2.11. *Then*

$$\left| \int_t^T \psi(X_s^{t,x,\gamma}, \gamma_s) d\boldsymbol{\zeta}_s \right| \lesssim 1 + ||\gamma||_{\frac{p}{2},[t,T]}^{2(1+p)}. \tag{3.3}$$

The lemma above allows us to place an upper bound on the magnitude of $\int_t^T \psi(X_s^{t,x,\gamma}, \gamma_s) d\boldsymbol{\zeta}_s$ in terms of the control $\gamma \in \mathcal{V}^{\frac{p}{2}\text{-var}}$. This upper bound is not meant to be sharp.

Definition 3.1 (Regularizing Cost)**.** Suppose that $\mathcal{S} \subseteq \mathcal{V}^{\frac{p}{2}\text{-var}}(J, \mathbb{R}^k)$ is a Banach space. A *regularizing cost* on $\mathcal{S}$ is a function

$$\beta \colon \Delta_J \times \mathcal{V}^{\frac{p}{2}\text{-var}}(J, \mathbb{R}^k) \to [0, +\infty] \tag{3.4}$$

such that

1. $\gamma \mapsto \beta_{r,t}(\gamma)$ is continuous
2. $\beta_{r,t}(\gamma) = +\infty$ on $\Delta_J \times \mathcal{V}^{\frac{p}{2}\text{-var}}(J, \mathbb{R}^k) \backslash \mathcal{S}$
3. $\frac{\beta_{r,t}(\gamma)}{||\gamma||_{\frac{p}{2},[r,t]}^{2(1+p)}} \to +\infty$ as $||\gamma||_{\frac{p}{2},[r,t]} \to +\infty$.

In practice, the regularizing cost depends on the phenomena being modelled. We shall see an application of this in the section on *robust stochastic filtering* where the regularizing cost takes the form of a negative log-likelihood function.

Let $\mathcal{V}^{0,p\text{-var}}$ denote the $||\cdot||_{p,}$-closure of smooth paths in $\mathcal{V}^{p\text{-var}}$. The value function is restated as

$$V(t,x) \coloneqq \inf_{\gamma \in \mathcal{V}^{0,\frac{p}{2}\text{-var}}} \left\{ J(t,x,\gamma) + \beta_{t,T}(\gamma) \right\}. \tag{3.5}$$

Now we can show that V is bounded under suitable conditions.

Proposition 3.2. *If f and g in (3.2) are bounded below, then $V(t,x)$ is bounded below.*

3.2 Dynamic Programming Principle

The dynamic programming principle is a fundamental result in optimal control. It enables us to solve problems by breaking them down into smaller sub-problems. Since regularizing costs need not be additive i.e. $\beta_{s,u}+\beta_{u,t} = \beta_{s,t}$ need not hold, it must be shown that there exists a suitable subset of additive costs so that the dynamic programming principle can be retained.

Lemma 3.3. *Suppose that $\mathcal{S} \subseteq \mathcal{V}^{0,\frac{p}{2}\text{-}var}$ contains all the smooth functions from J to $\mathbb{R}^k$. Then*

$$\begin{aligned} V(t,x) &= \inf_{\gamma \in \mathcal{V}^{0,\frac{p}{2}\text{-}var}} \{J(t,x,\gamma) + \beta_{t,T}(\gamma)\} \\ &= \inf_{\gamma \in \mathcal{S}} \{J(t,x,\gamma) + \beta_{t,T}(\gamma)\}. \end{aligned} \tag{3.6}$$

Proof. Fix $\gamma \in \mathcal{V}^{0,p\text{-var}}$ and choose a sequence $(\gamma^n)_{n=1}^{\infty}$ of paths in $\mathcal{S}$ such that $||\gamma^n - \gamma||_\infty \to 0$ as $n \to +\infty$. Then $||\gamma^n - \gamma||_{\frac{p}{2},J} \to 0$ as $n \to +\infty$, so (3.6) holds by *theorem 2.12*. □

Remark 3.1. If $u \in W^{1,q}$ and $\gamma_s^{t,a,u} := a + \int_t^s u_y dy$, where $W^{1,q}$ is a Sobolev space, then

$$\beta_{s,r}(\gamma^{t,a,u}) := \varepsilon \int_s^r |u_y|^q dy$$

is clearly additive by the properties of integration. We will see that regularizing costs taking this form are adequate for our purposes.

Definition 3.2 (Value Function)**.** Suppose that $u_s \in \mathcal{W}^{1,q}$, where $s \in J$. Let $d\gamma_s^{t,a,u} = u_s ds$, $\gamma_t^{t,a,u} = a$. The value function is now defined to be

$$v(t,x,a) := \inf_{u \in L^q} \left\{ J(t,x,\gamma^{t,a,u}) + \varepsilon \int_t^T |u_s|^q ds \right\} \tag{3.7}$$

Note that the regularizing cost in the definition above enables us to retain the dynamic programming principle, the proof of which is analogous to the standard case [12].

Theorem 3.4 (Dynamic Programming Principle)**.** *Let $X_s^{t,x,a,u} := X_s^{t,x,\gamma^{t,a,u}}$. If $1 \le q < +\infty$ and $r \in [t,T]$, then*

$$\begin{aligned} v(t,x,a) = \inf_{u \in L^q} \Big\{ & v(r, X_r^{t,x,a,u}, \gamma_r^{t,a,u}) + \int_t^r f(X_s^{t,x,a,u}, \gamma_s^{t,a,u}) ds \\ & + \int_t^r \psi(X_s^{t,x,a,u}, \gamma_s^{t,a,u}) d\boldsymbol{\zeta}_s + \varepsilon \int_t^r |u_s|^q ds \Big\} \end{aligned} \tag{3.8}$$

3.3 Generalized Control Problem

We may generalize the value function given by (3.7). Note that we can absorb the regularizing cost into the integral involving the function f. Thus, fixing a Banach space $(U, ||\cdot||_U)$, we may reformulate the optimal control problem as

$$\begin{aligned} J(t,x,a,u) := & \int_t^T f(X_s^{t,x,a,u}, \gamma_s^{t,a,u}) ds \\ & + \int_t^T \psi(X_s^{t,x,a,u}, \gamma_s^{t,a,u}) d\boldsymbol{\zeta}_s + g(X_T^{t,x,a,u}, \gamma_T^{t,a,u}) \end{aligned} \tag{3.9}$$

where

$$dX_s^{t,x,a,u} = b(X_s^{t,x,a,u}, \gamma_s^{t,a,u})ds + \lambda(X_s^{t,x,a,u}, \gamma_s^{t,a,u})d\boldsymbol{\zeta}_s \tag{3.10}$$

$$d\gamma_s^{t,a,u} = h(\gamma_s^{t,a,u}, u_s)ds \tag{3.11}$$

$$u\colon J \to U \tag{3.12}$$

with the value function

$$v(t,x,a) := \inf_{u \in L^\infty} J(t,x,a,u). \tag{3.13}$$

The following assumptions (*appendix (C.1)*), lemma and corollary are required for the main results in the next section.

Lemma 3.5. *Assume (C.1). Then for some* $0 < C < +\infty$*,*

$$\left| \int_t^T \psi(X_s^{t,x,a,u}, \gamma_s^{t,a,u})d\boldsymbol{\zeta}_s \right| \le C + \frac{1}{2}\int_t^T f(X_s^{t,x,a,u}, \gamma_s^{t,a,u}, u_s)ds. \tag{3.14}$$

Corollary 3.5.1. *Suppose that* $K \subseteq \mathbb{R}^m \times \mathbb{R}^k$ *is compact. Then there exists an* $M > 0$ *such that when* $(t,x,a) \in J \times K$*, the controls* $u \in U$ *may be restricted to the ones satisfying* $||\gamma^{t,a,u}||_{\frac{p}{2},J} \le M$*.*

3.4 Rough HJB Equation

We now derive the rough Hamilton-Jacobi-Bellman (HJB) equation. Fix a geometric rough path $\boldsymbol{\zeta} = (\zeta, \zeta^{(2)}) \in \mathscr{D}^p$ and a sequence $(\eta_t^n)_{n=1}^\infty$ of smooth paths such that $\eta_t^n \to \zeta_t$ as $n \to +\infty$. We lift η_t^n into rough path space by defining

$$(\eta^n)_{s,t}^{(2)} = \int_s^t \eta_{s,u}^n \otimes \eta_u^n$$

and set

$$dX_s^{t,x,a,u,n} = b(X_s^{t,x,a,u,n}, \gamma_s^{t,a,u})ds + \lambda(X_s^{t,x,a,u,n}, \gamma_s^{t,a,u})d\boldsymbol{\eta}_s^n \tag{3.15}$$

so letting $n \to +\infty$ yields (3.10) by *theorem 2.11*, where $\boldsymbol{\eta}^n = (\eta^n, (\eta^n)^{(2)})$ and

$$\left(X^{t,x,a,u,n}, (X^{t,x,a,u,n})'\right) = \left(X^{t,x,a,u,n}, \lambda(X^{t,x,a,u,n}, \gamma^{t,a,u})\right). \tag{3.16}$$

Furthermore, the integral in (3.15) is calculated in the Riemann-Stieltjes sense.

The rough HJB equation derived by firstly solving (3.15) using the results in [4] and then taking the limit as $n \to +\infty$. The above is now stated as a result.

Theorem 3.6 (Rough HJB Equation)**.** *Suppose (3.9) - (3.13) at the beginning of* §3.3*. Then*

$$-dv - b \cdot \nabla_x v dt - \inf_{u \in U}\{h \cdot \nabla_a v + f\}dt - (\lambda \cdot \nabla_x v + \psi)\, d\boldsymbol{\zeta} = 0 \tag{3.17}$$

subject to

$$v(T,x,a) = g(x,a) \tag{3.18}$$

where ∇_y *denotes the Laplacian with respect to the variable* y*.*

Before showing that v in *theorem 3.17* is the unique viscosity solution, we make the following definition.

Definition 3.3. Suppose that $v^{\boldsymbol{\eta}^n} \to v^{\boldsymbol{\zeta}}$ as $n \to +\infty$ in the sense of *theorem 3.6*. Then $v^{\boldsymbol{\zeta}}$ solves (3.17) if $v^{\boldsymbol{\eta}^n} \to v^{\boldsymbol{\zeta}}$ locally uniformly such that $\boldsymbol{\eta}^n \to \boldsymbol{\zeta}$ with respect to the $\frac{1}{p}$-Hölder rough path metric $\varrho_{\frac{1}{p}-\text{Höl}}$.

Theorem 3.7. *Suppose* assumption C.1. *Then the value function (3.13) solves (3.17), (3.18) in the sense of* definition 3.3. *Furthermore, for each fixed (t, x, a), the map $\boldsymbol{\zeta} \mapsto v^{\boldsymbol{\zeta}}(t, x, a)$ is uniformly continuous with respect to the rough path metrics $\varrho_{\frac{1}{p}-Höl,J}$ and $\varrho_{p,J}$, where $\boldsymbol{\zeta} \in \mathscr{C}_g^{0,p}(J, \mathbb{R}^d)$.*

4 Robust Filtering

Stochastic filtering enables us to make inferences about the state of a "signal process" that is not directly observed; more rigorously, we make an inference about the "signal process" $dS_t = \alpha_t S_t dt + \sigma_t dB_t^1$ through our observations $dY_t = c_t S_t dt + dB_t^2$. These inferences, however, require knowledge of the parameters α_t, σ_t occurring in the signal which causes difficulties in practice. The filtering problem has been well studied though [3] and various methods exist for handling inferences and parameter estimates. Our main focus will be on "robust" filtering, where an estimate of the current state of the signal is made through the observation process and is achieved by penalizing bad parameter estimates in an "intelligent" way. Moreover, as we will see, path-wise filtering problems can be transformed into path-wise optimal control problems, enabling us to use the results in section §3 to find a solution.

4.1 Kalman-Bucy Filter

We will make use of the Kalman-Bucy filter and it takes the form

$$dS_t = \alpha_t S_t dt + \sigma_t dB_t^1 \tag{4.1}$$

$$dY_t = c_t S_t dt + dB_t^2 \tag{4.2}$$

where (4.1) is the *signal* process and (4.2) is the *observation* process. We assume S_t is $\mathbb{R}^m$-valued, Y_t is $\mathbb{R}^d$-valued, $Y_0 = 0$, $S_0 \sim N(\mu_0, \Sigma_0)$, $\alpha\colon J \to \mathbb{R}^{m \times m}$, $\sigma\colon J \to \mathbb{R}^{m \times l}$ and $c\colon J \to \mathbb{R}^{d \times m}$. Furthermore, we assume that the quadratic covariation of the Brownian motions B_t^1, B_t^2 satisfy

$$d\langle B^1, B^2 \rangle_t = \rho_t dt$$

where $\rho_t \in \mathbb{R}^{l \times d}$ and

$$I - \rho_t \rho_t^\top$$

is positive semi-definite for all t.

Now, let $\mathcal{Y}_t$ denote the completed filtration generated by the observation process Y_t; that is, $\mathcal{Y}_t$ is the completion of the sigma algebra generated by $(Y_s)_{s \le t}$. Lastly, the prediction

$$q_t = \mathbb{E}\left[S_t | \mathcal{Y}_t\right] \tag{4.3}$$

satisfies the stochastic differential equation

$$dq_t = \alpha_t q_t dt + \left(R_t c_t^\top + \sigma_t \rho_t\right)(dY_t - c_t q_t dt) \tag{4.4}$$

and $R_t = \mathbb{E}\left[(S_t - q_t)(S_t - q_t)^\top | \mathcal{Y}_t\right]$ satisfies the Riccati equation

$$\frac{dR_t}{dt} = \sigma_t \sigma_t^\top + \alpha_t R_t + R_t \alpha_t^\top - \left(R_t c_t^\top + \sigma_t \rho_t\right)\left(c_t R_t + \rho_t^\top \sigma_t^\top\right). \tag{4.5}$$

4.2 Robust Filtering

We now develop the machinery for our robust filtering problem, starting with controls and convex expectations. Let $\gamma_t \coloneqq (\alpha, \sigma, c, \rho)_t \in \Gamma$ denote our controls, where

$$\Gamma \coloneqq \mathbb{R}^{m\times m} \times \mathbb{R}^{m\times l} \times \mathbb{R}^{d\times m} \times \Upsilon$$

and

$$\Upsilon \coloneqq \left\{\rho_t \in \mathbb{R}^{l\times d} \colon I - \rho_t \rho_t^\top \text{ is positive definite}\right\}.$$

If $\varphi \colon \mathbb{R}^m \to \mathbb{R}$ is a bounded Borel function, then we define the *convex expectation*

$$\mathcal{E}\left(\varphi(S_t)|\mathcal{Y}_t\right) \coloneqq \operatorname*{ess\,sup}_{(\gamma,\mu_0,\Sigma_0)} \left\{\mathbb{E}^{\gamma,\mu_0,\Sigma_0}\left[\varphi(S_t)|\mathcal{Y}_t\right] - \left(\frac{1}{k_1}\beta(\gamma,\mu_0,\Sigma_0|\mathcal{Y}_t)\right)^{k_2}\right\} \tag{4.6}$$

for $k_1 > 0$ and $k_2 \geq 1$.

The admissible controls and penalty function (regularizing cost) are defined as:

Definition 4.1 (Admissible Controls)**.** $\mathcal{A}$ will denote the space of all *admissible controls* $\gamma_t \in \Gamma$ such that $\gamma \colon J \to \Gamma$ is absolutely continuous with bounded derivative.

Definition 4.2 (Penalty Function)**.** The penalty function β will take the form of a negative log-likelihood function; that is,

$$\beta_t(\gamma,\mu_0,\Sigma_0|\mathcal{Y}_t) = -\ln\left(\pi_t(\gamma,\mu_0,\Sigma_0)L_t(\gamma,\mu_0,\Sigma_0|\mathcal{Y}_t)\right) \tag{4.7}$$

where π is the prior density and L is the likelihood function.

Remark 4.1 (Robust Point Estimate & Confidence Interval)*.* The definitions above permit the construction of a *robust point estimate* and a confidence interval, given by

$$\operatorname*{arg\,min}_{\xi\in\mathbb{R}} \mathcal{E}\left((\varphi(S_t) - \xi)^2 |\mathcal{Y}_t\right) \tag{4.8}$$

and

$$\left[-\mathcal{E}\left(-\varphi(S_t)|\mathcal{Y}_t\right), \mathcal{E}\left(\varphi(S_t)|\mathcal{Y}_t\right)\right], \tag{4.9}$$

respectively.

To make some progress, we need the following assumption:

Assumption 4.1. We will assume that the log prior density takes the form

$$-\ln(\pi_t(\gamma, \mu_0, \Sigma_0)) = \int_0^t z(q_s, R_s, \gamma_s)ds + g(\mu_0, \Sigma_0) \tag{4.10}$$

Now, the log-likelihood function $L_t(\cdot)$ can be represented as a Radon-Nikodym derivative

$$L_t(\gamma, \mu_0, \Sigma_0|\mathcal{Y}_t) = \left(\frac{d\mathbb{P}^{\gamma,\mu_0,\Sigma_0}}{d\mathbb{P}^{\gamma^*,\mu_0^*,\Sigma_0^*}}\right)_{\mathcal{Y}_t} \tag{4.11}$$

where $\gamma^*, \mu_0^*, \Sigma_0^*$ are fixed reference parameters. It turns out that an explicit expression for the likelihood function (4.11) can be found.

The innovation process [3] is given by

$$dV_s = dY_s - c_s q_s ds \tag{4.12}$$

and is a $\mathcal{Y}_t$-adapted Brownian motion under $\mathbb{P}^{\gamma,\mu_0,\Sigma_0}$ and the conditional mean and expectation process, q^*, V^*, respectively, satisfy

$$dV_s = dV_s^* - (c_s q_s - c_s^* q_s^*)\, ds. \tag{4.13}$$

By Girsanov's theorem [9], it follows that

$$L_t(\gamma, \mu_0, \Sigma_0|\mathcal{Y}_t) = \exp\left(\int_0^t (c_s q_s - c_s^* q_s^*) \cdot dV_s^* - \frac{1}{2}\int_0^t |c_s q_s - c_s^* q_s^*|^2 ds\right) \tag{4.14}$$

and substituting $dV_s^* = dY_s - c_s^* q_s^* ds$ into the above implies

$$\begin{aligned} -\ln L_t(\gamma, \mu_0, \Sigma_0|\mathcal{Y}_t) &= -\int_0^t (c_s q_s - c_s^* q_s^*) \cdot dY_s + \frac{1}{2}\int_0^t \left(|c_s q_s|^2 - |c_s^* q_s^*|^2\right) ds \\ &= -\int_0^t c_s q_s \cdot dY_s + \frac{1}{2}\int_0^t |c_s q_s|^2 ds + constant \end{aligned} \tag{4.15}$$

since the reference parameters are assumed to be fixed. In the following, we will omit this constant and assume our penalty function is correct up to an additive constant. In numerical computations, the constant may be chosen to ensure the penalty always takes the value 0 at its minimum [2].

We will now transform the Itô integral occurring in (4.15) into a Stratonovich integral in anticipation of the path-wise optimal control problem. Hence

$$-\int_0^t c_s q_s \cdot dY_s = -\int_0^t c_s q_s \circ dY_s + \frac{1}{2}\langle cq, Y\rangle_t. \tag{4.16}$$

It can also be shown [2] that

$$\langle cq, Y\rangle_t = \int_0^t \operatorname{trace}\left(c_s(R_s c_s^\top + \sigma_s \rho_s)\right) ds. \tag{4.17}$$

This implies

$$-\ln L_t(\gamma, \mu_0, \Sigma_0|\mathcal{Y}_t) = -\int_0^t c_s q_s \circ dY_s + \frac{1}{2}\int_0^t \left(|c_s q_s|^2 + \operatorname{trace}\left(c_s(R_s c_s^\top + \sigma_s \rho_s)\right)\right) ds. \tag{4.18}$$

Now, set

$$w(q, R, \gamma) = z(q, R, \gamma) + \frac{1}{2}\left(|cq|^2 + \operatorname{trace}\left(c(Rc^\top + \sigma\rho)\right)\right)$$

and

$$\psi(q,\gamma) = -cq$$

so that

$$\mathcal{E}\left(\varphi(S_t)|\mathcal{Y}_t\right) = \operatorname*{ess\,sup}_{\gamma,\mu_0,\Sigma_0}\left\{\mathbb{E}\left[\varphi(S_t)|\mathcal{Y}_t\right] - \left(\frac{1}{k_1}\left(\int_0^t w(q_s,R_s,\gamma_s)ds + \int_0^t \psi(q_s,\gamma_s)\circ dY_s + g(\mu_0,\Sigma_0)\right)\right)^{k_2}\right\} \tag{4.19}$$

4.3 Lifting Into Rough Path Space

In practice, filtering is performed with respect to a fixed observation path. Thus, one might want to fix a path $\zeta_t = Y_t(\omega)$ and solve the filtering problem with respect to it. Doing so requires ζ_t to be lifted into rough path space, however. Define the *lift* of ζ_t by

$$\zeta^{(2)}_{s,t} = \int_s^t Y_{s,r}(\omega)\otimes \circ dY_r(\omega) \tag{4.20}$$

so that $\boldsymbol{\zeta} = (\zeta,\zeta^{(2)}) \in \mathscr{C}^{0,p}_g$ for $p\in(2,3)$. Then the prediction q_t satisfies the rough differential equation

$$dq_t = \alpha_t q_t dt + \left(R_t c_t^\top + \sigma_t\rho_t\right)\left(d\zeta_t - c_t q_t dt\right) \tag{4.21}$$

and

$$\begin{aligned} q_{s,t} &= \int_s^t \left(R_r c_r^\top + \sigma_r\rho_r\right) d\zeta_r + O(|t-s|) \\ &= \left(R_r c_r^\top + \sigma_r\rho_r\right)\zeta_{s,t} + O(|t-s|) \end{aligned} \tag{4.22}$$

by *theorem 2.2*. Hence the Gubinelli derivative of $\psi(q,\gamma) = -cq$ satisfies $\psi(q,\gamma)' = -c\left(R_r c_r^\top + \sigma_r\rho_r\right)$. Furthermore,

$$\int_0^\cdot \psi(q_s,\gamma_s)d\boldsymbol{\zeta}_s$$

exists as a rough integral and coincides with the Stratonovich integral.

4.4 The Optimal Control Problem

Now we are ready to transform the filtering problem into a path-wise optimal control problem. Let

$$k_t(\mu,\Sigma) := \inf\left\{\int_0^t w(q_s,R_s,\gamma_s)ds + \int_0^t \psi(q_s,\gamma_s)d\boldsymbol{\zeta}_s + g(q_0,R_0)\right\} \tag{4.23}$$

where the infimum is taken over all γ, q_0, R_0 such that $(q_t,R_t) = (\mu,\Sigma)$. Additionally, we set $g(\mu_0,\Sigma_0) = +\infty$ for $(\mu_0,\Sigma_0)\notin \mathbb{R}^m\times\mathcal{S}^m_+$, where $\mathcal{S}^m_+$ is the set of all $m\times m$ symmetric, positive-definite matrices over $\mathbb{R}$. This allows us to rewrite the convex expectation (4.19) as

Lemma 4.1. *Let $\phi(\cdot|\mu,\Sigma)$ denote the probability density function of a $N(\mu,\Sigma)$ distribution. If $\boldsymbol{\zeta} = (\zeta,\zeta^{(2)})$ is defined as above, then for any bounded measurable function φ we have*

$$\mathcal{E}\left(\varphi(S_t)|\mathcal{Y}_t\right) = \sup\left\{\int_{\mathbb{R}^m}\varphi(x)d\phi(x|\mu,\Sigma) - \left(\frac{1}{k_1}k_t(\mu,\Sigma)\right)^{k_2}\right\} \tag{4.24}$$

where the supremum is taken over all $(\mu,\Sigma)\in\mathbb{R}^m\times\mathcal{S}^m_+$.

The proof of *lemma 4.1* can be found in [1]. Thus, for $q^{t,\mu,\Sigma}, R^{t,\mu,\Sigma}$ that satisfy $(q_t^{t,\mu,\Sigma}, R_t^{t,\mu,\Sigma}) = (\mu, \Sigma)$, we have the optimal control problem

$$k_t(\mu, \Sigma) = \inf_{\gamma} \left\{ \int_0^t w(q_s^{t,\mu,\Sigma}, R_s^{t,\mu,\Sigma}, \gamma_s) ds + \int_0^t \psi(q_s^{t,\mu,\Sigma}, \gamma_s) d\boldsymbol{\zeta}_s + g(q_0^{t,\mu,\Sigma}, R_0^{t,\mu,\Sigma}) \right\}. \tag{4.25}$$

Note that the optimal control problem in (4.25) is lacking a regularizing cost. Again, we must introduce a regularizing cost to prevent degeneracy. Consider the dynamics

$$d\gamma_s^{t,a,u} = h(\gamma_s^{t,a,u}, u_s) ds$$

where

$$h \colon \Gamma \times U \to U$$

$$u \colon J \to U \text{ is bounded}$$

and $U \coloneqq \mathbb{R}^{m\times m} \times \mathbb{R}^{m\times l} \times \mathbb{R}^{d\times m} \times \mathbb{R}^{l\times d}$. The new terminal condition is

$$(q_t^{t,\mu,\Sigma,a,u}, R_t^{t,\mu,\Sigma,a,u}, \gamma_t^{t,a,u}) = (\mu, \Sigma, a). \tag{4.26}$$

We may allow w and g to depend on γ_0 without affecting the proof of *lemma 4.1* [2]. Thus,

$$\tilde{k}_t(\mu, \Sigma) \coloneqq \inf_{a\in\Gamma} v(t, \mu, \Sigma, a) \tag{4.27}$$

where

$$\begin{aligned} v(t, \mu, \Sigma, a) \coloneqq \inf_{u \text{ bounded}} \Bigg\{ & \int_0^t w(q_s^{t,\mu,\Sigma,a,u}, R_s^{t,\mu,\Sigma,a,u}, \gamma_s^{t,a,u}, u_s) ds \\ & + \int_0^t \psi(q_s^{t,\mu,\Sigma,a,u}, \gamma_s^{t,a,u}) d\boldsymbol{\zeta}_s + g(q_0^{t,\mu,\Sigma,a,u}, R_0^{t,\mu,\Sigma,a,u}, \gamma_0^{t,a,u}) \Bigg\} \end{aligned}$$

is our new value function.

4.5 The Associated HJB Equation

We are now in a position derive a rough HJB equation for the filtering problem. To simplify the notation, we write the following:

$$dq_s^{t,\mu,\Sigma,a,u} = b_\mu(q_s^{t,\mu,\Sigma,a,u}, R_s^{t,\Sigma,a,u}, \gamma_s^{t,a,u}) ds + \lambda(R_s^{t,\Sigma,a,u}, \gamma_s^{t,a,u}) d\zeta_s, \;\; q_t^{t,\mu,\Sigma,a,u} = \mu$$

$$dR_s^{t,\Sigma,a,u} = b_\Sigma(R_s^{t,\Sigma,a,u}, \gamma_s^{t,a,u}) ds, \;\; R_t^{t,\Sigma,a,u} = \Sigma$$

$$d\gamma_s^{t,a,u} = h(\gamma_s^{t,a,u}, u_s) ds, \;\; \gamma_t^{t,a,u} = a$$

where

$$\gamma = (\alpha, \sigma, c, \rho)$$

$$b_\mu(q, R, \gamma) \coloneqq \alpha q - \left(Rc^\top + \sigma\rho\right) cq$$

$$b_\Sigma(R, \gamma) \coloneqq \sigma\sigma^\top \alpha R + R\alpha^\top - \left(Rc^\top + \sigma\rho\right)\left(cR + \rho^\top\sigma^\top\right)$$

$$\lambda(R, \gamma) \coloneqq Rc^\top + \sigma\rho.$$

Remark 4.2 (Backward Control Problem). The dynamics above define a "backward" control problem; that is, the equations above satisfy a terminal condition at time t and a cost is prescribed to the initial values q_0, R_0, γ_0 by the function g occurring in $v(t, \mu, \Sigma, a)$ defined in the previous section. Thus, some work needs to be done to transform the problem into one where the results of §3 can be applied.

Let us define the following:

$$||A|| := \text{trace}(A^\top A)$$

$$||\gamma|| := \max\{||\alpha||, ||\sigma||, ||c||, ||\rho||\}$$

and if $A \in \mathcal{S}_+^m$, let $\lambda_{min}(A), \lambda_{max}(A)$ denote the minimum and maximum eigenvalue of A, respectively. We will also require some assumptions (*appendix D.1*).

The assumption $\{h(\gamma, u) : u \in U\} = U$ is used to guarantee the existence of a control u such that the state trajectories remain inside their respective domains, irrespective of the terminal condition (t, μ, Σ, a), ensuring that the value function v is finite [2].

The next two results are analogous to *lemma 3.5* and *corollary 3.5.1*. The purpose of these is to derive a result similar to *theorem 3.7*, which will be the main statement of this section.

Lemma 4.2. *Suppose* assumption D.1. *Then for any terminal condition* (t, μ, Σ, a) *and control* u *we have*

$$\left|\int_0^t \psi(q_s, \gamma_s) d\boldsymbol{\zeta}_s\right| \leq C + \frac{1}{2}\left(\int_0^t w(q_s, R_s, \gamma_s) ds + g(q_0, R_0, \gamma_0)\right).$$

The proof is long and relies on the simplifying notation in the beginning of this section, as well as *assumption D.1*. The idea is to use *theorem 2.2* and to bound estimates involving $|\psi(q, \gamma)|, ||\psi(q, \gamma)'||, \left|\left|R^{\psi(q,\gamma)}\right|\right|_{\frac{p}{2},[0,t]}$ and $||\psi(q, \gamma)'||_{p,[0,t]}$ in a special way. Thus, it is omitted and the reader is referred to [2].

Corollary 4.2.2. *Suppose that* $K \subseteq \mathbb{R}^m \times \mathcal{S}_+^m \times \Gamma$ *is compact. Then one may restrict to controls* u *such that the norms* $||q||_\infty, ||R||_\infty, ||\gamma||_\infty, ||R||_{1,[0,t]}, ||\gamma||_{1,[0,t]}$ *are bounded by* $0 < M < +\infty$ *when* $(t, \mu, \Sigma, a) \in [0, T] \times K$.

To derive the HJB equation, we proceed as we did in §3.4; we approximate $\boldsymbol{\zeta} = (\zeta, \zeta^{(2)})$ with smooth functions $\boldsymbol{\eta}^n = (\eta^n, (\eta^n)^{(2)})$ by the Stone-Weierstrass theorem, solve the problem with respect to $\boldsymbol{\eta}^n = (\eta^n, (\eta^n)^{(2)})$ using classical methods and then take the rough HJB equation as the limiting case.

Before stating the HJB equation, let $A : B$ denote the inner product between two elements A, B of the same inner product space. If A, B are matrices, define their inner product $A : B := \text{trace}(A^\top B)$.

Theorem 4.3 (The HJB Equation)**.** *Suppose* assumption D.1. *Then the value function* v *satisfies*

$$dv + (b_\mu \cdot \nabla_\mu v + b_\Sigma : \nabla_\Sigma v)\, dt + \sup_{u \in U} \{h : \nabla_a v - w\}\, dt + (\lambda \cdot \nabla_\mu v - \psi)\, d\boldsymbol{\zeta} = 0 \tag{4.28}$$

subject to

$$v(0, \mu, \Sigma, a) = g(\mu, \Sigma, a). \tag{4.29}$$

To guarantee a unique solution to (4.28), one should restrict to solutions $\tilde{v}(t, \mu, \Sigma, a)$ that approach $\pm\infty$ as $|\mu| + ||\Sigma|| + ||a|| \to +\infty$, $\lambda_{min}(\Sigma) \to 0$ and when ρ is random, as $\lambda_{max}(\rho^\top \rho) \to 1$. Denote this space of value functions by $\mathcal{H}$. The reasoning, which is beyond the scope of this paper, is given by [1].

Two additional results are required before proving the main result:

Theorem 4.4 (Young Integral)**.** *Suppose that V, W are Banach spaces and $1 \leq p, q \leq +\infty$ satisfy $\frac{1}{p} + \frac{1}{q} > 1$. If $X \in \mathcal{V}^{p\text{-}var}(J, V)$ and $Y \in \mathcal{V}^{q\text{-}var}(J, \boldsymbol{L}(V, W))$, then for each $t \in J$,*

$$\int_0^t Y_s dX_s = \lim_{|\mathcal{D}| \to 0} \sum_{\mathcal{D}} Y_{t_i} (X_{t_{i+1}} - X_{t_i}) \tag{4.30}$$

and

$$\left|\left| \int_0^{\cdot} (Y_s - Y_0)\, dX_s \right|\right|_{p,J} \lesssim ||Y||_{q,J}\, ||X||_{p,J}\,. \tag{4.31}$$

A proof of this can be found in [11].

Lemma 4.5 (Grönwall's Inequality)**.** *Suppose that $f(t) \geq 0$ and $f(t) \leq C + A \int_0^t f(s) ds$ for some $A, C \in \mathbb{R}_+$ holds for all $t \in J$. Then $f(t) \lesssim e^{At}$ for all $t \in J$.*

We close this section with the main theorem:

Theorem 4.6. *The value function v in* theorem 4.3 *solves the HJB in the sense of* definition 3.3*. Also, the map $\boldsymbol{\zeta} \mapsto v^{\boldsymbol{\zeta}}(t, \mu, \Sigma, a)$, $\boldsymbol{\zeta} \in \mathscr{C}_g^{0,p}$ is locally uniformly continuous with respect to the rough path metrics $\varrho_{\frac{1}{p}\text{-}H\ddot{o}l,J}(\cdot, \cdot), \varrho_{p,J}(\cdot, \cdot)$, locally uniformly in (t, μ, Σ, a).*

5 Research Questions

We have seen how path-wise optimal control theory under rough paths is degenerate when no modifications are made to penalize the variation of the path $\boldsymbol{\zeta}$. We have also seen how this can be rectified by introducing a regularizing cost. By restricting to a suitable class of regularizing costs, we were able to retain the *dynamic programming principle*, which permitted the derivation of the rough HJB equation. Lastly, we showed that solutions to the HJB equation are unique in a certain sense (*theorem 3.7*). Naturally, one might ask whether there exists a "rough" version of the *verification theorem*, i.e. if one has a function w and a control γ^* which satisfies the HJB equation, then w is the unique value function and γ^* is the optimal control, suggesting future research.

We have also seen how robust stochastic filtering can be treated from the path-wise optimal control perspective. Another natural question, as remarked by [2], is about the convergence properties of the convex expectation $\mathcal{E}\left(\varphi(S_t)|\mathcal{Y}_t\right)$ to the actual expectation $\mathbb{E}\left[\varphi(S_t)|\mathcal{Y}_t\right]$. Furthermore, the performance of the path-wise robust filter in practice is also of interest, which also suggests an area of future research.

Appendices

A Rough Path Motivation

Example A.1 (Rough path motivation). Consider a differentiable function

$$f\colon \mathbb{R}^n \to \mathbb{R}$$

and a continuous path

$$X\colon J \to \mathbb{R}^n.$$

Taylor's theorem implies for $s \leq r$

$$f(X_r) = f(X_s) + \nabla f(X_s) \cdot (X_r - X_s) + o(|X_r - X_s|).$$

If we ignore the $o(|X_r - X_s|)$ term and assume X is "regular enough", then

$$\int_s^t f(X_r)dX_r \approx f(X_s)(X_t - X_s) + \nabla f(X_s)\int_s^t (X_r - X_s) \otimes dX_r$$

where $u \otimes v \coloneqq uv^\top$ is the *Cartesian tensor* in $\mathbb{R}^n$, $u, v \in \mathbb{R}^n$. Let us temporarily define

$$X^{(2)}_{s,t} = \int_s^t (X_r - X_s) \otimes dX_r$$

to be the "*lift*" of the path X. This additional information allows us to obtain a better estimate of the integral $\int_s^t f(X_r)dX_r$. In fact, the notion of the "lift" of a rough path is necessary for our integration theory to hold, giving rise to the existence and uniqueness of *rough integrals* and *rough differential equations*.

Also, a tedious calculation shows for $s \leq u \leq t$

$$X^{(2)}_{s,t} = X^{(2)}_{s,u} + X^{(2)}_{u,t} + X_{s,u} \otimes X_{u,t}.$$

The identity above is known as *Chen's relation*.

B Rough Path Theory Proofs

Proof of proposition 2.1. Note that

$$\begin{aligned}
\sum_{\mathcal{D}} |\zeta_{t_i,t_{i+1}}|^p &= \sum_{\mathcal{D}} \frac{|\zeta_{t_i,t_{i+1}}|^p}{t_{i+1} - t_i}(t_{i+1} - t_i) \\
&\leq \sum_{\mathcal{D}} ||\zeta||^p_{\frac{1}{p}-\text{Höl}}(t_{i+1} - t_i) \\
&= ||\zeta||^p_{\frac{1}{p}-\text{Höl}} \sum_{\mathcal{D}}(t_{i+1} - t_i) \\
&= ||\zeta||^p_{\frac{1}{p}-\text{Höl}} \cdot T \\
&< +\infty.
\end{aligned}$$

Taking the supremum over all $\mathcal{D}$ shows that

$$||\zeta||_{p,J}^p \leq ||\zeta||_{\frac{1}{p}-\text{Höl}}^p \cdot T$$

is finite, hence

$$||\zeta||_{p,J} \leq ||\zeta||_{\frac{1}{p}-\text{Höl}} \cdot T^{\frac{1}{p}}$$

is also finite. An analagous argument applied to $\zeta^{(2)}$ shows that

$$||\zeta^{(2)}||_{\frac{p}{2},J} < +\infty.$$

Thus, in light of the above and (2.11), $|||\boldsymbol{\zeta}|||_{p,J} < +\infty$. □

Proof of proposition 2.3. By α-Hölder continuous, we mean paths $X\colon J \to \mathbb{R}^n$ such that

$$\sup_{s\neq t\in J} \frac{|X_{s,t}|}{|t-s|^\alpha} < +\infty.$$

Applying the same method of proof as in *proposition 2.1* gives the result. □

Proof of lemma 2.5. Define

$$f\colon t \mapsto ||X||_{1,[0,t]}\,.$$

It is not hard to show that f is monotonically increasing on J. With this in mind, it follows that for any partition $\mathcal{D} = \{0 = t_0 < \cdots < t_n = T\}$

$$\begin{aligned}\sum_{\mathcal{D}} |f(t_{i+1}) - f(t_i)| &= \sum_{\mathcal{D}} f(t_{i+1}) - f(t_i) \\ &= f(T) - f(0) \\ &< +\infty\end{aligned}$$

so f has finite 1-variation, hence finite p-variation by *lemma 2.4*. □

Proof of lemma 2.7. Consider the probability measure on $\Omega = \{x_1, \ldots, x_n\}$ defined by

$$P\{x_i\} = \frac{1}{n}.$$

Since the map $t \mapsto t^p$ is convex (for $t \geq 0$), we have

$$\begin{aligned}\left(\frac{x_1 + \cdots + x_n}{n}\right)^p &= \frac{1}{n^p}(x_1 + \cdots + x_n)^p \\ &\leq \frac{1}{n}(x_1^p + \cdots + x_n^p).\end{aligned}$$

by *Jensen's inequality 2.6*. Multiplying through the above by $n^p > 1$ yields

$$\begin{aligned}(x_1 + \cdots + x_n)^p &\leq n^{p-1}(x_1^p + \cdots + x_n^p) \\ &< n^p(x_1^p + \cdots + x_n^p).\end{aligned}$$

□

Proof of lemma 2.8. Fix a partition $\mathcal{D} = \{t_0 < \cdots < t_n\}$ of J and let $\mathcal{D}_k = \{s_0^k < \cdots < s_{n_k}^k\}$ be an partition of $[t_k, t_{k+1}]$ for $k = 1, \ldots, n$. *Lemma 2.7* above implies

$$\begin{aligned}\sum_{\mathcal{D}} |X_{t_{i+1}} - X_{t_i}|^p &\leq \sum_{k=0}^{n-1} \left[\sum_{i=0}^{n_k-1} |X_{s_{i+1}^k} - X_{s_i^k}| \right]^p \\ &\leq n^p \sum_{k=0}^{n-1} \sum_{i=0}^{n_k-1} |X_{s_{i+1}^k} - X_{s_i^k}|^p \\ &\leq n^p \sum_{k=0}^{n-1} ||X||_{p,[t_i,t_{i+1}]}^p\end{aligned}$$

proving the result. □

Proof of lemma 2.9. The proof proceeds by applying the triangle inequality to

$$(x_1, \ldots, x_n) = (x_1, 0) + (0, x_2, 0) + \cdots + (0, x_n).$$

□

Proof of proposition 2.10. The regularity results above hold for any sub-interval $[s,t]$ of $J = [0,T]$ so we will restrict ourselves to $[s,t]$. Recall *lemmas 2.4, 2.5, 2.7, 2.8* and *2.9.* Since $\psi(X,\gamma)' = \partial_x \psi(X,\gamma) X' = \partial_x \psi(X,\gamma)\lambda(X,\gamma)$ and $\psi, \lambda \in C_b^2$, it follows that that $\partial_x\psi, \lambda$ are Lipschitz continuous due to their bounded derivatives. To simplify the notation further, let $\Delta_s^t X \coloneqq X_{s,t}$. Then

$$\begin{aligned}|\Delta_s^t \psi(X,\gamma)'| &\lesssim |(X_{s,t}, \gamma_{s,t})| \\ &\lesssim |X_{s,t}| + |\gamma_{s,t}| \\ &\lesssim ||X||_{p,[s,t]} + ||\gamma||_{\frac{p}{2},[s,t]}\end{aligned}$$

so $||\psi||_{p,[s,t]} \lesssim ||X||_{p,[s,t]} + ||\gamma||_{\frac{p}{2},[s,t]}$, proving (1).

To prove (2) we expand R^ψ using Taylor's theorem, i.e.

$$\begin{aligned}R_{s,t}^\psi &= \Delta_s^t \psi(X,\gamma) - \psi(X_s,\gamma_s)'(X_{s,t},\gamma_{s,t}) \\ &= \frac{1}{2}\partial_x^2 \psi(X_s + hX_{s,t}, \gamma_s)(X_{s,t},\gamma_{s,t})^{\otimes 2}\end{aligned} \tag{B.1}$$

for some $h \in [0,1]$. Before proceeding, note that $p \mapsto ||X||_{p,[s,t]}$ is non-increasing for any path X. Hence

$$\begin{aligned}|R_{s,t}^\psi| &\lesssim |(X_{s,t},\gamma_{s,t})|^2 \\ &\lesssim ||X||_{p,[s,t]}^2 + ||\gamma||_{\frac{p}{2},[s,t]} \\ &\lesssim ||X||_{p,[s,t]}^2 + \left|\left|R^X\right|\right|_{\frac{p}{2},[s,t]} + ||\gamma||_{\frac{p}{2},[s,t]}\end{aligned}$$

by (B.1).

Now we prove (3). By *theorem 2.2* and since $b \in Lip_b$, we have

$$\begin{aligned}
|R^X_{s,t}| &= |X_{s,t} - X'_s \zeta_{s,t}| \\
&= \left| \int_s^t b(X_u, \gamma_u) du + \int_s^t \lambda(X_u, \gamma_u) d\boldsymbol{\zeta}_u - \lambda(X_s, \gamma_s)\zeta_{s,t} \right| \\
&\leq \left| \int_s^t \lambda(X_u, \gamma_u) d\boldsymbol{\zeta}_u - \lambda(X_s, \gamma_s)\zeta_{s,t} - \lambda(X_s, \gamma_s)' \zeta^{(2)}_{s,t} \right| \\
&+ \left| \int_s^t b(X_u, \gamma_u) du \right| + \left| \lambda(X_s, \gamma_s)' \zeta^{(2)}_{s,t} \right| \\
&\lesssim ||R^\lambda||_{\frac{p}{2},[s,t]} ||\zeta||_{p,[s,t]} + ||\lambda(X,\gamma)'||_{\frac{p}{2},[s,t]} \left|\left|\zeta^{(2)}\right|\right|_{\frac{p}{2},[s,t]} \\
&+ |t-s| + \left|\left|\zeta^{(2)}\right|\right|_{\frac{p}{2},[s,t]}.
\end{aligned}$$

In light of (1), (2) and the above, it follows that

$$\begin{aligned}
||R^X||_{\frac{p}{2},[s,t]} &\lesssim \left(||X||^2_{p,[s,t]} + ||R^X||_{\frac{p}{2},[s,t]} + ||\gamma||_{\frac{p}{2},[s,t]} \right) ||\zeta||_{p,[s,t]} \\
&+ \left(||X||_{p,[s,t]} + ||\gamma||_{\frac{p}{2},[s,t]} \right) \left|\left|\zeta^{(2)}\right|\right|_{\frac{p}{2},[s,t]} \\
&+ |t-s| + \left|\left|\zeta^{(2)}\right|\right|_{\frac{p}{2},[s,t]} \\
&\lesssim \left(||X||^2_{p,[s,t]} + ||R^X||_{\frac{p}{2},[s,t]} + ||\gamma||_{\frac{p}{2},[s,t]} \right) ||\zeta||_{p,[s,t]} \\
&+ \left(1 + ||X||^2_{p,[s,t]} + ||\gamma||_{\frac{p}{2},[s,t]} \right) \left|\left|\zeta^{(2)}\right|\right|_{\frac{p}{2},[s,t]} \\
&+ |t-s| + \left|\left|\zeta^{(2)}\right|\right|_{\frac{p}{2},[s,t]}.
\end{aligned} \tag{B.2}$$

Looking at the proof of *proposition 2.1*, we see that $||\zeta||_{p,[s,t]} \leq M|t-s|^{\frac{1}{p}}$, and similarly for $\zeta^{(2)}$. Also, we saw in the proof of (2) above that

$$|R^X_{s,t}| \lesssim ||X||^2_{p,J} + ||\gamma||_{\frac{p}{2},J}$$

so we may drop $||R^X||_{\frac{p}{2},[s,t]}$ from the right-hand side in (B.2). Supposing without loss of generality that $||\zeta||_{p,[s,t]}, \left|\left|\zeta^{(2)}\right|\right|_{\frac{p}{2},[s,t]} \leq \frac{1}{2}$ whenever $|t-s| < \delta$ (we can do this and extend to $J = [0,T]$ via *lemma 2.8*), we have

$$\begin{aligned}
||R^X||_{\frac{p}{2},[s,t]} &\lesssim \left(||X||^2_{p,[s,t]} + ||\gamma||_{\frac{p}{2},[s,t]} \right) ||\zeta||_{p,[s,t]} \\
&+ \left(1 + ||X||^2_{p,[s,t]} + ||\gamma||_{\frac{p}{2},[s,t]} \right) \left|\left|\zeta^{(2)}\right|\right|_{\frac{p}{2},[s,t]} \\
&+ |t-s| + \left|\left|\zeta^{(2)}\right|\right|_{\frac{p}{2},[s,t]}
\end{aligned} \tag{B.3}$$

and by definition of $||R^X||_{\frac{p}{2},[s,t]}$

$$||X||_{p,[s,t]} \lesssim ||\zeta||_{p,[s,t]} + ||R^X||_{\frac{p}{2},[s,t]}$$

hence

$$\begin{aligned}||X||_{p,[s,t]} &\lesssim ||\zeta||_{p,[s,t]} + \left|\left|R^X\right|\right|_{\frac{p}{2},[s,t]} \\ &\lesssim ||\zeta||_{p,[s,t]} + \left(||X||^2_{p,[s,t]} + ||\gamma||_{\frac{p}{2},[s,t]}\right)||\zeta||_{p,[s,t]} \\ &+ \left(1 + ||X||^2_{p,[s,t]} + ||\gamma||_{\frac{p}{2},[s,t]}\right)\left|\left|\zeta^{(2)}\right|\right|_{\frac{p}{2},[s,t]} \\ &+ |t-s| + \left|\left|\zeta^{(2)}\right|\right|_{\frac{p}{2},[s,t]}.\end{aligned}$$

Expanding the right-hand side and noting that $||X||^2_{p,[s,t]}\,||\zeta||_{p,[s,t]} \leq ||X||^2_{p,[s,t]}$ since $||\zeta||_{p,[s,t]} < \frac{1}{2}$, we get

$$||X||_{p,[s,t]} \lesssim \left(1 + ||\gamma||_{\frac{p}{2},[s,t]}\right)\left(||\zeta||_{p,[s,t]} + \left|\left|\zeta^{(2)}\right|\right|_{\frac{p}{2},[s,t]} + |t-s|\right) + ||X||^2_{p,[s,t]}.$$

Let $I \subseteq [s,t]$ denote a sub-interval such that $||X||_{p,I} < \frac{1}{2}$ and set r equal to the length of I. Then

$$\begin{aligned}||X||_{p,I} &\lesssim \left(1 + ||\gamma||_{\frac{p}{2},I}\right)\left(||\zeta||_{p,I} + \left|\left|\zeta^{(2)}\right|\right|_{\frac{p}{2},I} + |I|\right) + ||X||^2_{p,I} \\ &\lesssim 2\left(1 + ||\gamma||_{\frac{p}{2},I}\right)\left(||\zeta||_{p,I} + \left|\left|\zeta^{(2)}\right|\right|_{\frac{p}{2},I} + |I|\right) \\ &\lesssim \left(1 + ||\gamma||_{\frac{p}{2},I}\right)\left(M(\delta^*)^{\frac{1}{p}} + M(\delta^*)^{\frac{2}{p}} + \delta^*\right) \\ &\lesssim 1 + ||\gamma||_{\frac{p}{2},I}.\end{aligned}$$

where $|I| = r$. Now we extend to $J = [0,T]$. Set $\delta^* = \min\{\delta, r\}$ and choose a partition of J as in *lemma 2.8* such that the mesh size that is lesser than δ^*. Then

$$\left(1 + ||\gamma||_{\frac{p}{2},I}\right)^p \leq 2^p\left(1 + ||\gamma||^p_{\frac{p}{2},I}\right)$$

by *lemma 2.7*, so

$$\begin{aligned}||X||_{p,J} &\lesssim \sum_{\mathcal{D}}\left(1 + ||\gamma||^p_{\frac{p}{2},[t_i,t_{i+1}]}\right) \\ &\lesssim 1 + ||\gamma||^p_{\frac{p}{2},J}\end{aligned}$$

but $1 + ||\gamma||^p_{\frac{p}{2},J} \lesssim 1 + ||\gamma||^{1+p}_{\frac{p}{2},J}$, proving (3).

To prove (4) we use (B.3) and approach the situation in an analogous fashion, arriving at the inequality

$$\begin{aligned}\left|\left|R^X\right|\right|_{\frac{p}{2},J} &\lesssim 1 + ||\gamma||^{p/2}_{\frac{p}{2},J} \\ &\lesssim 1 + ||\gamma||^{2+p}_{\frac{p}{2},J}.\end{aligned}$$

□

C Optimal Control Proofs

Proof of lemma 3.1. By *theorem 2.2* and *proposition 2.10*, we have

$$\begin{aligned}\left|\int_t^T \psi(X_s^{t,x,\gamma},\gamma_s)d\boldsymbol{\zeta}_s\right| &\lesssim ||R^\psi||_{\frac{p}{2},[t,T]}\,||\zeta||_{p,[t,T]} + ||\psi'||_{p,[t,T]}\left|\left|\zeta^{(2)}\right|\right|_{\frac{p}{2},[t,T]} \\ &+ |\psi(x,\gamma_t)\zeta_{t,T}| + |\psi(x,\gamma_t)'\zeta^{(2)}_{t,T}| \\ &\lesssim \left(||X^{t,x,\gamma}||^2_{p,[t,T]} + ||R^X||_{\frac{p}{2},[t,T]} + ||\gamma||_{\frac{p}{2},[t,T]}\right)||\zeta||_{p,[t,T]} \\ &+ \left(||X^{t,x}||_{p,[t,T]} + ||\gamma||_{\frac{p}{2},[t,T]}\right)\left|\left|\zeta^{(2)}\right|\right|_{\frac{p}{2},[t,T]} \\ &+ |\psi(x,\gamma_t)\zeta_{t,T}| + |\psi(x,\gamma_t)'\zeta^{(2)}_{t,T}|.\end{aligned}$$

Now, $|\psi(x,\gamma_t)\zeta_{t,T}| + |\psi(x,\gamma_t)'\zeta^{(2)}_{t,T}|$ is a constant and

$$||R^X||_{\frac{p}{2},[t,T]} \lesssim 1 + ||\gamma||^{2+p}_{\frac{p}{2},[t,T]} \tag{C.1}$$

$$||X||_{p,[t,T]} \lesssim 1 + ||\gamma||^{1+p}_{\frac{p}{2},[t,T]} \tag{C.2}$$

so *lemma 2.7* applied to the square of the right-hand side of (C.2) gives (3.3). □

Proof of proposition 3.2. By *propositions 2.10* and *3.1* we have

$$\begin{aligned}\left|\int_t^T \psi d\boldsymbol{\zeta}_s\right| &\lesssim 1 + ||\gamma||^{2(1+p)}_{\frac{p}{2},[t,T]} \\ &\lesssim 1 + \frac{\beta_{t,T}(\gamma)}{2}\end{aligned}$$

so

$$J(t,x,\gamma) + \beta_{t,T}(\gamma) \geq \int_t^T f ds + g(X_T^{t,x,\gamma}) + \frac{\beta_{t,T}(\gamma)}{2} - C$$

for some $C > 0$, proving the result. □

Assumption C.1. We will assume the following:

1. *$b \in Lip_b$ and $\lambda, \psi \in C_b^3$*
2. *$f(x,a,u)$ and $g(x,a)$ are continuous, bounded below, Lipschitz continuous in (x,a) and f is uniformly continuous in u*
3. *$h(a,u)$ is continuous, Lipschitz continuous in a, uniformly continuous in u, is bounded in a, locally uniformly in u, and for some $\delta \geq 1$*

$$\sup_{a\in\mathbb{R}^k}\frac{|h(a,u)|}{||u||_U^\delta} \to 0$$

as $||u||_U \to +\infty$

4. *for the same δ above,*

$$\inf_{x\in\mathbb{R}^m, a\in\mathbb{R}^k}\frac{|f(x,a,u)|}{||u||_U^{2\delta(1+p)}} \to +\infty$$

as $||u||_U \to +\infty$

Proof of lemma 3.5. Note again that $p \mapsto ||\gamma||_{p,J}$ is non-increasing for $1 \leq p < +\infty$. By *lemma 3.1* and Hölder's inequality we have

$$\begin{aligned}\left|\int_t^T \psi(X_s^{t,x,a,u}, \gamma_s^{t,a,u})d\boldsymbol{\zeta}_s\right| &\leq C\left(1 + ||\gamma^{t,a,u}||^{2(1+p)}_{\frac{p}{2},[t,T]}\right) \\ &\leq C\left(1 + ||\gamma^{t,a,u}||^{2(1+p)}_{1,[t,T]}\right) \\ &= C\left(1 + \left[\int_t^T |h(\gamma_s^{t,a,u}, u_s)|ds\right]^{2(1+p)}\right) \\ &\leq C\left(1 + T^{\frac{2(1+p)}{p'}} \int_t^T |h(\gamma_s^{t,a,u}, u_s)|^{2(1+p)}ds\right)\end{aligned}$$

since $T - t \leq T$, where p' is the Hölder conjugate of $2(1+p)$. The result now follows by *assumption C.1*. □

Proof of corollary 3.5.1. By *lemma 3.5* we have

$$J(t, x, a, u) \geq \frac{1}{2}\int_t^T f(X_s^{t,x,a,u}, \gamma_s^{t,a,u}, u_s)ds - \tilde{C}$$

for some $\tilde{C} > 0$. If we fix $u^* \in U$, then we may ignore all controls u that satisfy

$$\frac{1}{2}\int_t^T f(X_s^{t,x,a,u}, \gamma_s^{t,a,u}, u_s)ds - \tilde{C} \geq \sup_{(t^*,x^*,a^*)} J(t^*, x^*, a^*, u^*).$$

Thus, the proof of *lemma 3.5* also gives an upper bound on $||\gamma^{t,a,u}||_{\frac{p}{2},J}$. The result holds by *assumption C.1* and since u^* is arbitrary. □

Proof of theorem 3.7. Suppose that $\boldsymbol{\eta} \in \mathscr{C}_g^{0,p}(J, \mathbb{R}^d)$ is another geometric rough path such that $|||\boldsymbol{\zeta}|||_{\frac{1}{p}-\text{Höl},J}, |||\boldsymbol{\eta}|||_{\frac{1}{p}-\text{Höl},J} \leq M$ so that the conditions of (2.12) are satisfied given the paths $X_s^{\boldsymbol{\eta}} := X_s^{t,x,a,u,\boldsymbol{\eta}}$, $X_s^{\boldsymbol{\zeta}} := X_s^{t,x,a,u,\boldsymbol{\zeta}}$, i.e. they satisfy the RDE driven by $\boldsymbol{\eta}, \boldsymbol{\zeta}$. Now, we may restrict to controls γ satisfying $||\gamma||_{\frac{p}{2},J} \leq L$ for some $L > 0$ by *corollary 3.5.1*. Hence,

$$||X^{\boldsymbol{\zeta}} - X^{\boldsymbol{\eta}}||_{\infty,J} \lesssim \varrho_{p,J}(\boldsymbol{\zeta}, \boldsymbol{\eta})$$

and

$$\left|\left|\int_0^\cdot \psi(X_s^{\boldsymbol{\zeta}}, \gamma_s^{t,a,u})d\boldsymbol{\zeta}_s - \int_0^\cdot \psi(X_s^{\boldsymbol{\eta}}, \gamma_s^{t,a,u})d\boldsymbol{\eta}_s\right|\right|_{p,J} \lesssim \varrho_{p,J}(\boldsymbol{\zeta}, \boldsymbol{\eta})$$

by *proposition 2.12*. Lastly, if we let U^L denote the set of all u such that $d\gamma_s^{t,a,u} = u_s ds$ and $||\gamma||_{\frac{p}{2},J} \leq L$, then

$$\begin{aligned}
|v^{\boldsymbol{\zeta}}(t,x,a) - v^{\boldsymbol{\eta}}(t,x,a)| \leq \sup_{u \in U^L} \bigg| & \int_t^T f(X_s^{t,x,a,u,\boldsymbol{\zeta}}, \gamma_s^{t,a,u}, u_s) ds \\
& - \int_t^T f(X_s^{t,x,a,u,\boldsymbol{\eta}}, \gamma_s^{t,a,u}, u_s) ds \\
& + \int_t^T \psi(X_s^{t,x,a,u,\boldsymbol{\zeta}}, \gamma_s^{t,a,u}) d\boldsymbol{\zeta}_s \\
& - \int_t^T \psi(X_s^{t,x,a,u,\boldsymbol{\eta}}, \gamma_s^{t,a,u}) d\boldsymbol{\eta}_s \\
& + g(X_T^{t,x,a,u,\boldsymbol{\zeta}}, \gamma_T^{t,a,u}) - g(X_T^{t,x,a,u,\boldsymbol{\eta}}, \gamma_T^{t,a,u}) \bigg| \\
& \lesssim \sup_{u \in U^L} \left(\int_t^T |X_s^{t,x,a,u,\boldsymbol{\zeta}} - X_s^{t,x,a,u,\boldsymbol{\eta}}| + \varrho_{p,J}(\boldsymbol{\zeta}, \boldsymbol{\eta}) \right. \\
& \left. + |X_T^{t,x,a,u,\boldsymbol{\zeta}} - X_T^{t,x,a,u,\boldsymbol{\eta}}| \right) \\
& \lesssim \varrho_{p,J}(\boldsymbol{\zeta}, \boldsymbol{\eta}) \\
& \lesssim \varrho_{\frac{1}{p}\text{-Höl},J}(\boldsymbol{\zeta}, \boldsymbol{\eta})
\end{aligned}$$

by the Lipschitz assumptions on f, g. Replacing $\boldsymbol{\eta}$ with $\boldsymbol{\eta}^n$, where

$$\lim_{n \to +\infty} \varrho_{\frac{1}{p}\text{-Höl},J}(\boldsymbol{\zeta}, \boldsymbol{\eta^n}) = 0$$

and $\boldsymbol{\eta}^n$ is smooth proves the result. □

D Robust Filtering Proofs

Assumption D.1.

- *$w(q,R,\gamma,u)$ and $g(q,R,\gamma)$ are continuous, bounded below and locally Lipschitz in (q,R,γ), uniformly in u*

- *$h(\gamma,u)$ is continuous, $\{h(\gamma,u) \colon u \in U\} = U$ for any $\gamma \in \Gamma$, Lipschitz in γ, uniformly in u, is bounded in γ, locally uniformly in u, and for some δ_1*

$$\sup_{\gamma \in \Gamma} \frac{||h(\gamma,u)||}{||u||^{\delta_1}} \to 0$$

  as $||u|| \to +\infty$

- *for some $\delta_2 > \delta_1$,*

$$\frac{|f(q,R,\gamma,u)|}{(1+|q|+||R||^2+||\gamma||^2)\,||u||^{\delta_2} + (1+|q|^2+||R||^2)\,(1+||\gamma||^4)} \to +\infty$$

  as $|q|+||R||+||\gamma||+||u|| \to +\infty$

- *g satisfies*
$$\frac{|g(q,R,\gamma)|}{|q|^2 + (1+||R||)(1+||\gamma||^2)} \to +\infty$$
as $|q| + ||R|| + ||\gamma|| \to +\infty$ *and*
$$\inf_{(q,\gamma)\in\mathbb{R}^m\times\Gamma} |g(q,R,\gamma)| \to +\infty$$
as $\lambda_{min}(R) \to 0$
- $\inf_{q,R,\alpha,\sigma,c} |g(q,R,\gamma)| \to +\infty$ *as* $\lambda_{max}(\rho\rho^\top) \to 1$
- $||h(\gamma,u)|| \le (1-\lambda_{max}(\rho\rho^\top))||u||$ *for all* $(\gamma,u) \in \Gamma \times U$.

Proof of corollary 4.2.2. We may obtain an upper bound for $||\gamma||_{1,[0,t]}$ by an argument analogous to *corollary 3.5.* Also, the path R_t lies in a bounded set by the ODE for R_t and *assumption D.1*, so $||R||_\infty < +\infty$, hence $||R||_{1,[0,t]} < +\infty$. Lastly, $||q||_\infty < +\infty$ by observing the differential equation for q. □

Proof of lemma 4.5. Let
$$g(t) := \int_0^t f(s)ds$$
so that
$$g'(t) \le C + Ag(t).$$
Solving the ODE
$$g' - Ag - C = 0$$
implies
$$g(t) = k\mathrm{e}^{At} - \frac{C}{A}$$
for some k. Lastly, setting $g(0) = C$ and solving for k implies $k = C\frac{A}{A-1} < C$, hence
$$\begin{aligned} g(t) &\le C\mathrm{e}^{At} - \frac{C}{A} \\ &\le C\mathrm{e}^{At}. \end{aligned}$$
Differentiating the above and noting that $g'(t) \le C + Ag(t)$ proves the result. □

Proof of theorem 4.6. Fix another rough path $\boldsymbol{\eta} \in \mathscr{C}_g^{0,p}$ such that (without loss of generality) $\varrho_{\frac{1}{p}\text{-Höl},J}(\boldsymbol{\zeta},\boldsymbol{\eta}) \le 1$ and
$$|||\boldsymbol{\zeta}|||_{\frac{1}{p}\text{-Höl},J}, |||\boldsymbol{\eta}|||_{\frac{1}{p}\text{-Höl},J} \le M^*,$$
where $M^* := \max\{|||\boldsymbol{\zeta}|||_{\frac{1}{p}\text{-Höl},J}, |||\boldsymbol{\eta}|||_{\frac{1}{p}\text{-Höl},J}\}$. Also, let $q^{\boldsymbol{\zeta}}, q^{\boldsymbol{\eta}}$ denote the prediction driven by the $\boldsymbol{\zeta},\boldsymbol{\eta}$, respectively, and similarly for the value functions $v^{\boldsymbol{\zeta}}, v^{\boldsymbol{\eta}}$. Now, let the bound M be defined by *corollary 4.2.2* such that it holds for $\boldsymbol{\zeta},\boldsymbol{\eta}$. Now, by (4.31), we have
$$\left|\int_s^t \left(R_r c_r^\top + \sigma_r \rho_r\right) d(\eta-\zeta)_r\right| \lesssim ||\eta-\zeta||_{p,J}$$
so that
$$|q_s^{\boldsymbol{\eta}} - q_s^{\boldsymbol{\zeta}}| \lesssim \int_s^t |q_r^{\boldsymbol{\eta}} - q_r^{\boldsymbol{\zeta}}|dr + ||\eta-\zeta||_{p,J}$$

implying

$$\left|\left|q^{\boldsymbol{\eta}} - q^{\boldsymbol{\zeta}}\right|\right|_{\infty,J} \lesssim ||\eta - \zeta||_{p,J}$$

by Grönwall's inequality. In light of *theorem 2.12* and the above,

$$\left|\left|\int_0^\cdot \psi(q_s^{\boldsymbol{\eta}}, \gamma_s) d\boldsymbol{\eta}_s - \int_0^\cdot \psi(q_s^{\boldsymbol{\zeta}}, \gamma_s) d\boldsymbol{\zeta}_s\right|\right|_{p,J} \lesssim \varrho_{p,J}(\boldsymbol{\eta}, \boldsymbol{\zeta}).$$

Thus, for any terminal condition $(t, \mu, \Sigma, a) \in J \times K$

$$\begin{aligned}
|v^{\boldsymbol{\eta}}(t, \mu, \Sigma, a) - v^{\boldsymbol{\zeta}}(t, \mu, \Sigma, a)| &\leq \sup_u \left| \int_0^t \left(w(q_s^{\boldsymbol{\eta}}, R_s, \gamma_s, u_s) - w(q_s^{\boldsymbol{\zeta}}, R_s, \gamma_s, u_s)\right) ds \right. \\
&\quad + \int_0^t \psi(q_s^{\boldsymbol{\eta}}, \gamma_s) d\boldsymbol{\eta}_s - \int_0^t \psi(q_s^{\boldsymbol{\zeta}}, \gamma_s) d\boldsymbol{\zeta}_s + g(q_0^{\boldsymbol{\eta}}, R_0, \gamma_0) - g(q_0^{\boldsymbol{\zeta}}, R_0, \gamma_0) \\
&\lesssim \sup_u \left(\int_0^t |q_s^{\boldsymbol{\eta}} - q_s^{\boldsymbol{\zeta}}| ds + \varrho_{p,J}(\boldsymbol{\eta}, \boldsymbol{\zeta}) + |q_0^{\boldsymbol{\eta}} - q_0^{\boldsymbol{\zeta}}| \right) \\
&\lesssim \varrho_{p,J}(\boldsymbol{\eta}, \boldsymbol{\zeta}) \\
&\lesssim \varrho_{\frac{1}{p}\text{-Höl},J}(\boldsymbol{\eta}, \boldsymbol{\zeta}).
\end{aligned}$$

where the supremum is taken over all u such that γ satisfies the assumptions at the beginning of the proof. The remainder of the proof proceeds as in *theorem 3.7*. □

References

[1] A. L. Allan and S. N. Cohen. "Parameter uncertainty in the Kalman-Bucy filter". In: *SIAM J. Control Optim.* (2019).

[2] A. L. Allan and S. N. Cohen. *Pathwise Stochastic Control with Applications to Robust Filtering*. 2019. arXiv: 1902.05434 [math.PR].

[3] A. Bain and D. Crisan. *Fundamentals of Stochastic Filtering*. Springer Science & Business Media, 2008.

[4] M. Bardi and F. Da Lio. "On the Bellman equation for some unbounded control problems". In: *Nonlinear Differ. Equ. Appl.* (1997).

[5] P. K. Friz and M. Hairer. *A Course on Rough Paths With an Introduction to Regularity Structures*. 2nd ed. Springer Switzerland, 2020.

[6] M. Gubinelli. "Controlling Rough Paths". In: *J. Funct. Anal.* (2004).

[7] P. K. Friz J. Diehl and P. Gassiat. "Stochastic Control with Rough Paths". In: *Appl. Math. Optim.* (2017).

[8] L. H. Loomis and S. Sternberg. *Advanced Calculus*. Revised Edition. World Scientific Publishing Company, 2014.

[9] B. Øksendal. *Stochastic Differential Equations: An Introduction with Applications*. 6th ed. Springer Science & Business Media, 2013.

[10] W. Rudin. *Real and Complex Analysis*. McGraw-Hill Education, 1987.

[11] M. Caruana T. J. Lyons and T. Lévy. *Differential Equations Driven by Rough Paths*. Springer, 2007.

[12] J. Yong and X. Y. Zhou. *Stochastic Controls, Hamiltonian Systems and HJB Equations*. Springer-Verlag, New York, 1999.