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The Spectra of Toeplitz operator on the Hardy space

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Abstract

Toeplitz operators are an important class of bounded linear operators on the Hardy space, and to discuss the existence and uniqueness of solutions to Toeplitz equations, one is motivated to describe the spectrum of Toeplitz operators. For Toeplitz operators with specific properties, such as being analytic, and associated with continuous symbols, the spectrum is completely understood, while an area of continuing research is the spectral structure of more general Toeplitz operators.

In this report, we will start by summarising existing results for well-understood Toeplitz operators and their spectra, and provide concrete examples to verify the theorems. We will then discuss methods of finding the spectra of general Toeplitz operators, including investigating the connection of the numerical range of a Toeplitz operator to its spectrum, and using numerical method with computer programs.

Statement of Authorship

I thank Associate Professor Pinhas Grossman for proofreading this report and his guidance throughout the project. No new results are provided in this report, definitions and theories are summarised from literature. Martínez-Avendaño's (2007) book *An Introduction to Operators on the Hardy-Hilbert Space* gives an excellent treatment to existing results, and is central to Section 2-3 in this report. Associate Professor Pinhas Grossman proposed the ideas for the various methods to investigate the spectra of a general Toeplitz operator in Section 4; the MATLAB code used in Section 4.2 is developed by Enxi Lin.

Contents

1 Introduction

Given a linear operator $A : \mathcal{H} \to \mathcal{H}'$ between Hilbert spaces, one is often interested in the solution to linear equations $A(h) = h'$ for $h \in \mathcal{H}$ and $h' \in \mathcal{H}'$; when $\mathcal{H}$ and $\mathcal{H}'$ and are finite dimensional, the solution is completely understood by employing the theories of eigenvalues and eigenvectors to the matrix that represents A. However, many of these results fail in infinite-dimensional spaces, where operators may have no eigenvalues. One is therefore motivated to define the spectrum of a (bounded) linear operator, to generalise the notion for the eigenvalues to infinite-dimensional spaces.

One class of operator on Hilbert spaces that has attracted particular interest is the Toeplitz operators, which are the compressions of the multiplication operators on L^2 to the Hardy space $\mathcal{H}^2$. The study of this class of operators originated with Otto Toeplitz, a German mathematician, around 1900s, and there has been ongoing research dedicated to it as increasing applications of Toeplitz operator to various fields arise, such as noncommutative geometry and singular integral equations in pure mathematics, and signal processing and time-series analysis in engineering.

Extensive work has been done to compute spectra of Toeplitz operators, In particular, the spectra are well-understood for Toeplitz operators with specific properties, such as being analytic, and associated with continuous symbols (Arveson 2002). An area of continuing research is the spectral structure of more general Toeplitz operators, which is also the interest for our research project.

In this report, we will summarise existing results on well-understood Toeplitz operators and their spectra, and provide concrete examples to verify the theorems. We will then discuss methods of (partially) computing the spectra of general Toeplitz operators, including investigating the connection of the numerical range of a Toeplitz operator to its spectrum, and using computational method with computer programs.

2 Preliminary definitions

2.1 Hardy space

We will begin by defining the function spaces we will be working on.

The L^2 space, a special case in of the L^p spaces, is central to many problems in Analysis. In this project, we will specifically look at the L^2 spaces on the circle, which is naturally associated with Fourier series. Consider the measure space of S^1 with the standard Lebesgue measure. A function $f : S^1 \to \mathbb{C}$ is in $L^2(S^1)$ if it is measurable and square-integrable, i.e., $\int_0^{2\pi} |f(e^{i\theta})|^2 d\theta < \infty$. We can then define the L^2-norm of f:

$$||f||_{L^2} = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta \right)^{\frac{1}{2}}$$

which is normalised so that the measure of the entire circle is 1.

A caveat to note is that there is a technical issue with this definition: $||f||_{L^2} = 0$ does not imply $f = 0$, but $f = 0$ a.e., making $||\cdot||_{L^2}$ only a semi-norm on the set of all square-integrable functions. Therefore, the precise definition of L^2 requires the equivalence relation: f and g are equivalent if $f = g$ a.e.

Definition 1. $L^2 = L^2(S^1)$ consists of all (equivalence classes of) square-integrable functions on S^1 with respect to Lebesgue measure. The norm of an element in L^2 is defined as the L^2-norm of a representative function in the equivalence class.

However, in practice, there is little risk of regarding elements in L^2 as functions, rather than equivalence classes of functions.

Some elementary properties of L^2 are useful to note. Firstly, when equipped with the inner product: $\langle f, g\rangle_{L^2} = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}d\theta$, L^2 is a Hilbert space (Conway 1990). Moreover, the set $\{b_n : b_n(e^{i\theta}) = e^{in\theta}\}_{n\in\mathbb{Z}}$ forms an **orthonormal basis** for L^2 (Conway 1990), this means every $f \in L^2$ has Fourier series of form

$$\sum_{n=-\infty}^{\infty} \langle f, e^{in\theta}\rangle e^{in\theta} = \sum_{n=-\infty}^{\infty} \hat{f}_n e^{in\theta},$$

which converges to f in L^2. A key result is that sequence of Fourier coefficients $\hat{f} = \{\hat{f}_n\}_{n\in\mathbb{Z}}$ are square-summable, i.e., $\sum_{-\infty}^{\infty} |f_n|^2 < \infty$. This allows us to identify elements in L^2 with $\ell^2(\mathbb{Z})$ via the Fourier transform.

Theorem 2. *(Riesz-Fischer) If $f \in L^2$, then the Fourier transform*

$$\hat{f}_n = \int_0^{2\pi} f(e^{i\theta})e^{-in\theta}d\theta, \text{ for } n \in \mathbb{Z}$$

provides a Hilbert space isomorphism between L^2 and $l^2(\mathbb{Z})$.

In particular, this isomorphism is norm-preserving:

$$||f||_{L^2} = ||\hat{f}||_{\ell^2} = \left(\sum_{-\infty}^{\infty} |f_n|^2\right)^{\frac{1}{2}}$$

Next, we shall examine the space on which the Toeplitz operators acts: the **Hardy space**, which is the (closed) subspace of L^2 with vanishing negative Fourier coefficients.

Definition 3. The Hardy space (on S^1), denoted $\tilde{\mathcal{H}}^2$, is defined as

$$\tilde{\mathcal{H}}^2 = \{f \in L^2 : \langle f, b_n\rangle = 0\ \forall n < 0\} = \left\{ f \in L^2 : f(e^{i\theta}) = \sum_{n=0}^{\infty} f_n e^{in\theta} \right\}$$

Clearly, the set $\{b_n : b_n(e^{i\theta}) = e^{in\theta}\}_{n\in\mathbb{N}}$ forms an orthonormal basis for $\tilde{\mathcal{H}}^2$, and we can similarly identify elements in $\tilde{\mathcal{H}}^2$ with those in $\ell^2(\mathbb{N})$.

We may extend functions in $\tilde{\mathcal{H}^2}$ to the open unit disc $\mathbb{D}$, yielding a space of analytic functions with square-summable power series coefficients $\mathcal{H}^2(\mathbb{D}) = \mathcal{H}^2$:

$$\mathcal{H}^2 = \left\{ (f : \mathbb{D} \to \mathbb{C}) : f(z) = \sum_{n=0}^{\infty} f_n z^n \text{ and} \sum_{n=0}^{\infty} |f_n|^2 \leq \infty \right\}$$

with the inner product and norm

$$\langle \sum_{n=0}^{\infty} f_n z^n, \sum_{n=0}^{\infty} g_n z^n \rangle_{\mathcal{H}^2} = \sum_{n=0}^{\infty} f_n \overline{g_n} \quad \text{and} \quad || \sum_{n=0}^{\infty} f_n z^n ||_{\mathcal{H}^2} = \left(\sum_{n=0}^{\infty} |f_n|^2 \right)^{\frac{1}{2}}$$

respectively.

The extension is constructed as follows: for $\tilde{f} \in \tilde{\mathcal{H}^2}$, we define

$$\tilde{f}_r(z) := \tilde{f} * P_r(\theta) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t) \tilde{f}(e^{it}) dt, \ 0 < r < 1.$$

where $P_r(\theta)$ is the Poisson Kernel and $\tilde{f}_r = \tilde{f} * P_r \in \mathcal{H}^2(\mathbb{D})$ (Rossi 2011).

Theorem 4. *For every $\tilde{f} \in \tilde{\mathcal{H}}$, there exist a unique $f \in \mathcal{H}^2(\mathbb{D})$, such that*

$$||\tilde{f}_r - f||_{\mathcal{H}^2(\mathbb{D})} \to 0 \quad \textit{as} \quad r \to 1^- \ , \textit{ and } \ ||\tilde{f}||_{\tilde{\mathcal{H}^2}} = ||f||_{\mathcal{H}^2(\mathbb{D})}$$

Further, if $\tilde{f} = \sum_{n=0}^{\infty} f_n e^{in\theta}$, then the corresponding $f \in \mathcal{H}^2(\mathbb{D})$ is $f = \sum_{n=0}^{\infty} f_n z^n$

Proof of this theorem may be found in Rossi (2011, p.461-464).

Next, we shall define another important class of functions.

Definition 5. The space $L^\infty = L^\infty(S^1)$ consists of **essentially bounded** measurable functions, that is,

$$L^\infty = \{\phi : \mu\left(\{e^{i\theta} : |\phi(e^{i\theta})| > M\}\right) = 0 \text{ for some } M > 0\}$$

where μ is the standard Lebesgues measure on S^1.

Similar to the technicality with defining L^2, L^∞ is more precisely defined in terms of equivalence classes of functions modulo sets of measure zero.

Further, the norm on a (representative) function $\phi \in L^\infty$ is the **essential norm**

$$||\phi||_{L^\infty} = \inf \left\{ M : \mu\left(\{e^{i\theta} : |\phi(e^{i\theta)|} > M\} \right) = 0 \right\}.$$

Moreover, it is clear that $L^\infty \subseteq L^2$.

We may similarly defined $\tilde{\mathcal{H}}^\infty$ as the subset of L^∞ with vanishing negative Fourier coefficients: $\tilde{\mathcal{H}}^\infty = \tilde{\mathcal{H}}^2 \cap L^\infty$, and $\mathcal{H}^\infty$ as the analytic extension of $\tilde{\mathcal{H}}^\infty$ to the unit disc:

Definition 6. The space $\mathcal{H}^\infty$ consists of analytic and bounded functions on the open unit disc; and the norm of $\phi \in \mathcal{H}^\infty$ is $||f||_{\mathcal{H}^\infty} = \sup\{|f(z)| : z \in \mathbb{D}\}$.

Clearly, $\mathcal{H}^\infty \subseteq \mathcal{H}^2$.

2.2 Toeplitz operator

In this section, we will define and discuss the basic properties of Toeplitz operators.

The modern formulation begins with the multiplication operator:

Definition 7. For $\phi \in L^\infty$. The **multiplication operator** by ϕ, denotad M_ϕ, is defined by

$$M_\phi : L^2 \to L^2 : M_\phi f = \phi f.$$

An interesting property of the multiplication operator is its matrix representation:

Theorem 8. *Let $\phi \in L^\infty$ with Fourier series*

$$\sum_{-\infty}^{\infty} \phi_n e^{in\theta}.$$

Then the matrix of M_ϕ with respect to the orthonormal basis $\{e^{in\theta}\}_{n\in\mathbb{Z}}$ of L^2 is

$$M_\phi = \begin{pmatrix} \ddots & & & & & & \\ & \phi_0 & \phi_{-1} & \phi_{-2} & & & \\ & \phi_1 & \phi_0 & \phi_{-1} & \phi_{-2} & & \\ & \phi_2 & \phi_1 & \boxed{\phi_0} & \phi_{-1} & \phi_{-2} & \\ & & \phi_2 & \phi_1 & \phi_0 & \phi_{-1} & \\ & & & \phi_2 & \phi_1 & \phi_0 & \\ & & & & & & \ddots \end{pmatrix}$$

where the boxed entry represents the (0,0) position

Matrices of this form are an example of Toeplitz matrix:

Definition 9. A (finite, singly infinite, or doubly infinite) matrix is called a **Toeplitz matrix** if its entries are constant along each diagonal, i.e., the matrix $(a_{m,n})$ is Toeplitz if $a_{m_1,n_1} = a_{m_2,n_2}$ whenever $m_1 - n_1 = m_2 - n_2$.

Returning to the definition of a multiplication operator, a natural question is to ask: for such a multiplication operator to be well-defined, is ϕ necessarily a L^∞ function? It is easy to see L^2 is not closed under this operation (consider $\phi(z) = f(z) = z^{-1/4}$). In fact, the multiplication of an L^2 function by a L^p, $p \geq 1$ function is not necessarily in L^2. A theorem on the boundedness criterion elaborates on this note:

Theorem 10. *(Brown and Halmos 1963) Consider a Toeplitz matrix corresponding to the sequence $(a_n)_{n\in\mathbb{Z}}$, which represent an operator, A, on ℓ^2, then A is bounded if and only if there exists a function $\phi \in L^\infty(S^1)$ such that $a_n = \hat{\phi}_n$, that is, $A = M_\phi$ if we let it acts on $L^2(S^1)$.*

Next, We shall see that Toeplitz operator is the compression of a multiplication operator on L^2 to the Hardy space $\tilde{\mathcal{H}}^2$:

Definition 11. For $\phi \in L^\infty$, the **Toeplitz operator** with **symbol** ϕ is the operator T_ϕ defined by

$$T_\phi : \tilde{\mathcal{H}}^2 \to \tilde{\mathcal{H}}^2 : T_\phi f = P\phi f,$$

where P is the **orthogonal projection** of L^2 onto $\tilde{\mathcal{H}}^2$

Regarding the matrix representation of the Toeplitz operator, we have the following corollary of Theorem 10:

Corollary 12. *An operator on $\tilde{\mathcal{H}}^2$ is a Toeplitz operator with symbol $\phi \in L^\infty$ if and only if its matrix with respect to the basis $\{e^{in\theta}\}_{n=0}^\infty$ of $\tilde{\mathcal{H}}^2$ is*

$$T_\phi = \begin{pmatrix} \phi_0 & \phi_{-1} & \phi_{-2} & \phi_{-3} & \ddots \\ \phi_1 & \phi_0 & \phi_{-1} & \phi_{-2} & \ddots \\ \phi_2 & \phi_1 & \phi_0 & \phi_{-1} & \ddots \\ \phi_3 & \phi_2 & \phi_1 & \phi_0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

where ϕ_k is the kth Fourier coefficient of ϕ.

Let's now consider an example.

Example 1 - Shift operators

On $\ell^2 = \ell^2(\mathbb{N})$, we define the unilateral right shift operator U by

$$U : \ell^2 \to \ell^2 : U(a_0, a_1, a_2, ...,) = (0, a_0, a_1, a_2, ...)$$

The adjoint of U is the unilateral left shift:

$$U^* : \ell^2 \to \ell^2 : U(a_0, a_1, a_2, a_3, ...,) = (a_1, a_2, a_3, ...)$$

These well-known shift operators are Toeplitz operators:

Lemma.The operator $T_{e^{i\theta}}$ is unitarily equivalent to U on ℓ^2, and the operator $T_{e^{-i\theta}}$ is unitarily equivalent to U^*.

This lemma is proven by considering the isomorphism between $\tilde{\mathcal{H}}^2$ and ℓ^2:

$$h : \tilde{\mathcal{H}}^2 \to \ell^2 : \sum_{n=0}^{\infty} a_n e^{in\theta} \to (a_n)_{n=0}^\infty.$$

Then $T_{e^{i\theta}} = hUh^*$ and $T_{e^{-i\theta}} = hU^*h^*$.

Intuitively, $T_{e^{-i\theta}}$ shifts the Fourier coefficient of the function to the right by raising the power of each term by 1, which is equivalent to the right shift operator on ℓ^2.

Here are some basic properties of the Toeplitz operators.

Lemma 13. *If T_ϕ is a Toeplitz operator*

(a) $||T_\phi|| = ||\phi||_\infty$

(b) $T_\phi^* = T_{\overline{\phi}}$

Proof. Proof of these results can be found in (Martínez-Avendaño and Rosenthal 2007). □

2.3 Spectrum

In applications, it is usually important to solve Toeplitz equations $T_\phi f = g$, and to discuss the existence and uniqueness of solutions. On finite-dimentional spaces, problems like this boil down to characterising its eigenvalues. However Toeplitz operators on the infinite-dimensional space may have no eigenvalues. It is therefore necessary to define the spectrum as a more generalised notion of the eignevalues.

Definition 14. For a bounded linear operator on a Hilbert space, $A \in \mathcal{B}(\mathcal{H})$, the **resolvent** of A, denoted $\rho(A)$ is the set of all complex numbers λ such that $A - \lambda$ is one-to-one and onto; the **spectrum** of A, denoted $\sigma(A)$, is the set of all complex numbers λ such that $A - \lambda$ is not invertible, i.e., $\sigma(A) = \mathbb{C} \setminus \rho(A)$

There are various subsets of the spectrum, for example:

- λ is an **eigenvalue** of A if $Af = \lambda f$ for some non-zero f (i.e., $A - \lambda$ is not one-to-one). The set of all eigenvalues of A is called the **point spectrum** of A, denoted $\Pi_0(A)$.
- The **approximate point spectrum** is the set $\Pi(A)$ of complex numbers λ such that there exist a sequence $\{f_n\}$ of unit vectors satisfying $\{||(A - \lambda)f_n|| \to 0\}$ as $n \to \infty$.

Below are some basic properties of the spectrum.

Lemma 15. .

(a) *(Spectral radius formula).* $r(A) := \sup\{|\lambda| : \lambda \in \sigma(A)\} = \lim_{n\to\infty} ||A^n||^{\frac{1}{n}}$. *In particular,* $r(A) \le ||A||$.

(b) $\sigma(A^*) = \{\overline{\lambda} : \lambda \in \sigma(A)\}$

(c) $\Pi_0(A) \subseteq \Pi(A) \subseteq \sigma(A)$

Proof. Proof of these results can be found in (Conway 1990). □

3 Spectral theorem for Toeplitz operators

In this section, we will discuss the spectra of 'well-behaved' Toeplitz operators, specifically those with $\tilde{\mathcal{H}}^\infty$ or continuous symbols.

3.1 Analytic symbols

Definition 16. A Toeplitz operator T_ϕ is called an **analytic Toeplitz operator** if $\phi \in \tilde{\mathcal{H}}^\infty$ (and hence will have analytic extension to $\mathbb{D}$).

The spectrum of analytic Toeplitz operator is characterised as follow:

Theorem 17. *For an analytic Toeplitz operator $T_{\tilde{\phi}}$, $\tilde{\phi} \in \tilde{\mathcal{H}}^\infty$, the spectrum is $\sigma(T_\phi) = \overline{\phi(\mathbb{D})}$ where ϕ is the analytic extension of $\tilde{\phi}$ onto $\mathbb{D}$.*

Proof. Recall $\tilde{\mathcal{H}}^2(S^1)$ is isomorphic to $\mathcal{H}^2(\mathbb{D})$, so the spectrum will be preserved if we regard T_ϕ as acting on $\mathcal{H}^2(\mathbb{D})$ with symbol $\phi \in \mathcal{H}^\infty(\mathbb{D})$. Also note that $T_\phi f = P\phi f = \phi f$, since $\phi \in \mathcal{H}^\infty \subseteq \mathcal{H}^2$.
We will firstly show that $\sigma(T_\phi) \subseteq \phi(\mathbb{D})$. Suppose $\lambda \notin \phi(\mathbb{D})$, there exist $\epsilon > 0$ such that $|\phi(z) - \lambda| \geq \epsilon$ for all $z \in \mathbb{D}$, then $\frac{1}{\phi - \lambda}$ is defined on $\mathbb{D}$ and $\frac{1}{\phi-\lambda} \in \mathcal{H}^\infty$, since $|\frac{1}{\phi-\lambda}(z)| \leq \frac{1}{\epsilon}$. Hence, $T_{\frac{1}{\phi-\lambda}}$ is well-defined and $(T_\phi - \lambda)^{-1} = T_{\frac{1}{\phi-\lambda}}$, and so $\lambda \notin \sigma(T_\phi)$.
Next, we will show $\phi(\mathbb{D}) \subseteq \sigma(T_\phi)$. Consider $\lambda = \phi(z)$, for some $z \in \mathbb{D}$; for all $g = (T_\phi - \lambda)f$, $g(w) = (\phi(w) - \lambda)f(w) = 0$. Hence $T_\phi - \lambda$ is not surjective and $\lambda \in \sigma(T_\phi)$. □

Example 1 - Shift operators (continued)
The right shift operator $T_{e^{i\theta}}$ is analytic, we can apply the theorem to find its spectrum: $\sigma(T_{e^{i\theta}}) = \overline{\mathbb{D}}$, since the images of $\mathbb{D}$ under the identity map $e^{i\theta}$ is $\mathbb{D}$. We will prove this from definition in the next subsection.
On the other hand, the left shift operator $\sigma(T_{e^{-i\theta}})$ is not analytic, but we have another spectral theorem for Toeplitz operator with continuous symbols.

3.2 Continuous symbol

To examine spectra of Toeplitz operators with continuous symbols, it requires the concept of a winding number.

Definition 18. Let γ be a continuous closed curve on S^1 ($\gamma : [0,1] \to \mathbb{C}$), and let $a \in \mathbb{C} \setminus \operatorname{ran}(\gamma)$. We define the **winding number** of γ about a:

$$\operatorname{Wnd}_a \gamma = \frac{\theta_a(1) - \theta_a(0)}{2\pi}$$

where $\theta_a : [0,1] \to \mathbb{R}$ is defined such that for

$$f_a : [0,1] \to S^1 : f_a(u) = \frac{\gamma(u) - a}{|\gamma(u) - a|},$$

$f_a(u) = e^{\theta(u)}$.

Informally, it is just the number of times that the curve winds around a point.
Then, we have the following theorem that completely describes the spectrum of a Toeplitz operator with a continuous symbol.

Theorem 19. *Let ϕ be a continuous function on S^1, then*

$$\sigma(T_\phi) = ran\ \phi \cup \{a \in \mathbb{C} : a \notin ran\ \phi \text{ and } Wnd_a \phi \neq 0\}$$

Proof. Proof of this theorem can be found in (Martínez-Avendaño and Rosenthal 2007). □

We shall see a simple application to clarify the theorem.

Example 2

Consider T_ϕ, where $\phi(z) = -2iz^3 - 3z^2 + 2iz^{-1} - z^{-2} + z^{-3}$ which is clearly continuous. Then the spectrum of T_ϕ is given by $\phi(S^1)$ together with the region where the winding number is non-zero (Figure [1]).

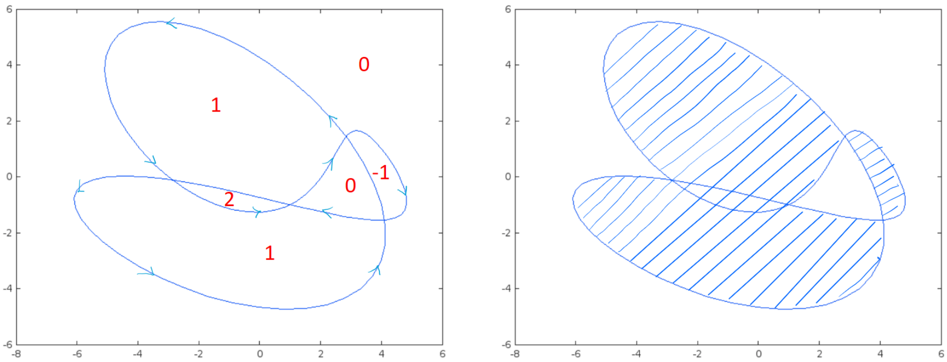

Figure 1: left: image of ϕ and winding number for each region; right: the spectrum of T_ϕ

.

Example 1 - Shift operators (continued)

Recall the shift operators, the theorem says $\sigma(T_{e^{i\theta}}) = \overline{\mathbb{D}} = \sigma(T_{e^{-i\theta}})$, since the images of S^1 under the identity map $e^{i\theta}$ and the conjugate map $e^{-i\theta}$ are both S^1, and the set of points with non-zero winding number is $\mathbb{D}$ for both curves (Figure [2]); $S^1 \cup \mathbb{D} = \overline{\mathbb{D}}$.

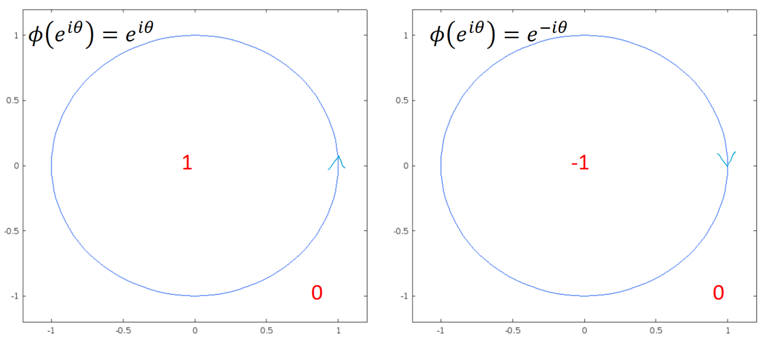

Figure 2: Range of the symbols for $T_{e^{i\theta}}$ and $T_{e^{-i\theta}}$

We will verify this from the definition of the spectrum.

Proof. We will firstly consider $T_{e^{-i\theta}}$. We have that $||T_{e^{-i\theta}}|| = ||e^{-i\theta}||_\infty = 1$ (Lemma 13a), and the spectral radius formula (Lemma 15a) implies that $|\lambda| \leq ||T_{e^{-i\theta}}|| = 1$, $\forall \lambda \in \sigma(T_{e^{-i\theta}})$, hence $\sigma(T_{e^{-i\theta}}) \subseteq \overline{\mathbb{D}}$.
For the reverse inclusion, consider first $\lambda \in \mathbb{D}$, clearly $f = \sum_{n=0}^{\infty} \lambda^n e^{in\theta} \in L^2$. Then,

$$T_{e^{-i\theta}} f = PM_{e^{-i\theta}} f = P(\sum_{n=-1}^{\infty} \lambda^n e^{i(n-1)\theta}) = \sum_{n=0}^{\infty} \lambda^{n+1} e^{in\theta} = \lambda f,$$

Hence $\lambda \in \Pi_0(T_{e^{-i\theta}}) \subseteq \sigma(T_{e^{-i\theta}})$.
Now, we will show that $S^1 \in \Pi(T_{e^{-i\theta}}) \subseteq \sigma(T_{e^{-i\theta}})$: consider $\lambda = e^{i\alpha} \in S^1$, we can construct a sequence of unit vectors $\{f_n\}_{n\in\mathbb{N}}$ in $\tilde{\mathcal{H}}^2$ such that $||(T_{e^{-i\theta}} - \lambda)f_n|| \to 0$: each f_n has Fourier coefficient

$$\hat{f_n} = \left(\frac{1}{\sqrt{n}} e^{i\alpha}, \frac{1}{\sqrt{n}} e^{2i\alpha}, ..., \frac{1}{\sqrt{n}} e^{ni\alpha}, 0, 0, 0, ...\right).$$

Then

$$||(T_{e^{-i\theta}} - e^{i\alpha})f_n||^2 = \sum_{j=0}^{\infty} |f_n^{j+1} - e^{i\alpha} f_n^j|^2 = \frac{1}{n} \to 0$$

Finally, for $T_{e^{i\theta}}$, by Lemma 13b)

$$\sigma(T_{e^{i\theta}}) = \sigma(T^*_{e^{-i\theta}}) = \{\overline{\lambda} : \lambda \in \sigma(e^{-i\theta})\} = \overline{\mathbb{D}}$$

□

We also computed the spectrum of a Toeplitz operator which bears significant in applications. In signal processing or control theory, a Toeplitz matrix A represents a time-invariant map, since $A_{i-1,j} = A_{i,j+1}$; an analytic Toeplitz operator (with lower-triangular matrix) represents a causal system (the output depends on past and current inputs but not future inputs), Toeplitz operator with an upper-triangular matrix represents an anti-causal system (output depends only on future values).

Example 2 - Rational transfer function
In signal processing, a transfer function provides description of the input-output relation for a linear time invariant dynamical system system, which can be regarded as the transfer function acting as a symbol for a Toeplitz operator.
For example, Consider the rational transfer function, $\phi(z) = \frac{1}{1-kz}$, where $k \in \mathbb{C} \setminus \{0, 1\}$, $\phi \in L^\infty(S^1)$.

For $0 < |k| < 1$, ϕ is analytic, and ϕ has Fourier series $\sum_{n=0}^{\infty} k^n e^{in\theta}$, that is, the inverse Fourier transform is $\hat{\phi} = (..., 0, 0, \boxed{1}, k, k^2, k^3, ...)$

Hence, the Toeplitz operator T_ϕ has matrix representation:

$$T_\phi = \begin{pmatrix} 1 & 0 & 0 & 0 & \ddots \\ k & 1 & 0 & 0 & \ddots \\ k^2 & k & 1 & 0 & \ddots \\ k^3 & k^2 & k & 1 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

In fact, the matrix representation of all analytic Toeplitz operator is lower triangular (Martínez-Avendaño and Rosenthal 2007).

ϕ maps the unit disc to the disc centres at $\frac{1}{1-|k|^2}$ with radius $\frac{|k|}{1-|k|^2}$. Hence $\sigma(T_\phi) = \overline{\phi(\mathbb{D})} = \overline{\mathbb{D}\left(\frac{1}{1-|k|^2}, \frac{|k|}{1-|k|^2}\right)}$.

Proof. See the proof for theorem [17]. □

On the other hand, for $|k| > 1$, $\phi \notin H^\infty$:

$$\phi(e^{i\theta}) = \frac{\frac{1}{k}e^{-i\theta}}{\frac{1}{k}e^{-i\theta} - 1} = (-\frac{1}{k}e^{-i\theta})\sum_{n=0}^{\infty}\frac{1}{k^n}e^{-in\theta} = -\sum_{n=1}^{\infty}\frac{1}{k^n}e^{-in\theta} = \sum_{n=-\infty}^{-1} -k^n e^{in\theta}$$

The Toeplitz operator T_ϕ has matrix representation:

$$T_\phi = \begin{pmatrix} 0 & -\frac{1}{k} & -\frac{1}{k^2} & -\frac{1}{k^3} & \ddots \\ 0 & 0 & -\frac{1}{k} & -\frac{1}{k^2} & \ddots \\ 0 & 0 & 0 & -\frac{1}{k} & \ddots \\ 0 & 0 & 0 & 0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

ϕ maps the unit circle to the circle centred at $\frac{1}{|k|^2-1}$ with radius $\frac{|k|}{|k|^2-1}$.

Nevertheless, ϕ is continuous and Theorem [19] implies that $\sigma(T_\phi) = \overline{\mathbb{D}\left(\frac{1}{|k|^2-1}, \frac{|k|}{|k|^2-1}\right)}$. We shall also verify this result.

Proof. Note that if we restrict ϕ to $\phi : \mathbb{C} \setminus \{\frac{1}{k}\} \to \mathbb{C} \setminus \{0\}$, it is bijective.

We will firstly show that $\phi(S^1) \subseteq \sigma(T_\phi)$. For $\lambda \in \phi(S^1)$, we have $\phi(w) = \lambda$ for some $w \in S^1$. But for all $g = (T_\phi - \lambda)f$, $g(w) = (\phi(w) - \lambda)f(w) = 0$. Hence $T_\phi - \lambda$ is not surjective and $\lambda \in \sigma(T_\phi)$.

Now, we will prove that $\mathbb{D}\left(\frac{1}{|k|^2-1}, \frac{|k|}{|k|^2-1}\right) \subseteq \sigma(T_\phi)$. Firstly, $0 \notin \sigma(T_\phi)$, since $T_\phi f$ does not depend on f_0, so T_f is not injective. For $\lambda \in \mathbb{D}\left(\frac{1}{|k|^2-1}, \frac{|k|}{|k|^2-1}\right) \setminus \{0\}$, we can also show that $T_\phi - \lambda$ is not injective. Let

$\lambda = \phi(w)$ for some $|w| > 1$ and $(T_\phi - \frac{1}{1-kw})f = 0$ for some $f = \sum_{n=0}^{\infty} f_n e^{in\theta} \in L^2$:

$$\begin{pmatrix} \frac{1}{k}f_1 + \frac{1}{k^2}f_2 + \frac{1}{k^3}f_3 + ... \\ \frac{1}{k}f_2 + \frac{1}{k^2}f_3 + ... \\ \frac{1}{k}f_3 + ... \\ \vdots \end{pmatrix} + \frac{1}{1-kw}\begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \end{pmatrix} = 0$$

Solving simultaneously yields $f_{n+1} = \frac{1}{w}f_n$ for all $n \geq 0$. Note that $|\frac{1}{w}| \leq 1$ and so $(f_0, \frac{1}{w}f_0, (\frac{1}{w})^2 f_0, ...) \in \ell^2$ for all f_0 which means $\ker(T_\phi - \lambda) \neq \{0\}$.

Finally, we will prove that $\sigma(T_\phi) \subseteq \overline{\mathbb{D}\left(\frac{1}{|k|^2-1}, \frac{|k|}{|k|^2-1}\right)}$. Consider $\lambda \in \mathbb{C} \setminus \mathbb{D}\left(\frac{1}{|k|^2-1}, \frac{|k|}{|k|^2-1}\right)$, then $\lambda = \phi(w)$ for some $|w| < 1$. By similar argument as in the previous paragraph, with $(f_0, \frac{1}{w}f_0, (\frac{1}{w})^2 f_0, ...) \in \ell^2$ iff $f_0 = 0$ in this case as $|\frac{k}{kw}| > 1$. Hence $\ker(T_\phi - \lambda) = \{0\}$. To show surjectivity, it suffices to show that for every orthonormal basis e_n of $\mathcal{H}^2$, there exists a $f \in \mathcal{H}^2$, such that $(T_\phi - \lambda)f = e_n$. This is clear: just pick $f(e^{i\theta}) = -\frac{1}{k}(1-kw)^2 e^{i(n-1)\theta} + (1-kw)e^{in\theta}$ for each $n \in \mathbb{N}$.

□

4 Spectrum of General Toeplitz operator

The spectrum of a general Toeplitz operator (with discontinuous symbol) is usually hard to compute. One can instead look at the numerical range of Toeplitz operators to gain insight into its spectrum.

Definition 20. The numerical range of an operator $A \in \mathcal{B}(\mathcal{H})$, denoted $W(A)$ is the subset of the complex plane

$$W(A) = \{\langle Af, f\rangle : f \in \mathcal{H}, ||f|| = 1\}.$$

Note that the nuemrical range is a convex subset of the complex plane (Toeplitz-Hausdorff Theorem).
For any arbitrary operator, the spectrum is contained in the closure of the numerical range:

Theorem 21. *For every operator $A \in \mathcal{B}(\mathcal{H})$, $\sigma(A) \subseteq \overline{W(A))}$.*

For a Toeplitz operator, there is more we can say about that connection between its spectrum and numerical range.

4.1 Numerical range of Toeplitz operator

We will start with a definition.

Definition 22. For a set X, the **convex hull** of X, denoted $\mathrm{Conv}(X)$, is the smallest convex set containing X; the closure of the convex hull, $\overline{\mathrm{Conv}(X)}$, is called the closed convex hull.

For a (general) Toeplitz operators, we have a complete description of the numerical range in terms of its spectrum:

Theorem 23. *For $\phi \in L^\infty$*

$$\overline{W(T_\phi)} = \overline{Conv(\sigma(T_\phi))}$$

In particular, $W(T_\phi) = Int(Conv(\sigma(T_\phi))$ ("Int" denotes "interior")

Proof. Proof of this theorem can be found in (Klein 1972). □

Example 1 - Shift operators (continued)
Theorem [23] implies that $W(T_{e^{i\theta}}) = \mathbb{D} = W(T_{e^{-i\theta}})$. We will also verify this from the definition.

Proof. We will firstly consider $T_{e^{-i\theta}}$. Let $f \in L^\infty$ with $||f||_\infty = 1$.

$$|\langle T_{e^{-i\theta}} f, f\rangle| \leq ||T_{e^{-i\theta}} f||_\infty ||f||_\infty = ||T_{e^{-i\theta}} f||_\infty \leq ||T_{e^{-i\theta}}||_{\mathcal{H}^2} = 1.$$

Hence $W(T_{e^{-i\theta}}) \subseteq \overline{\mathbb{D}}$. Suppose now $|\langle T_{e^{-i\theta}} f, f\rangle| = 1$, this implies $T_{e^{-i\theta}} f = \lambda f$ for some $|\lambda| = 1$, i.e, λ is an eigenvalue. But

$$\sum_{n=0}^{\infty} f_n e^{i(n+1)\theta} = \sum_{n=0}^{\infty} \lambda f_n e^{in\theta} \implies |f_{n+1}| = |f_n| \ \forall n \in \mathbb{N}$$

contradicting $||f||_\infty = 1$. Hence $W(T_{e^{-i\theta}}) \subseteq \mathbb{D}$
For the reverse inclusion, let $\lambda \in \mathbb{D}$, from previous investigation of this example we know $\lambda \in \Pi_0(T_{e^{-i\theta}})$, that is, $T_{e^{-i\theta}} g = \lambda g$ for some $g \neq 0$. Let $f = \frac{g}{||g||_\infty}$, then $||f||_\infty = 1$ and $\langle T_{e^{-i\theta}} f, f\rangle = \lambda$, and $\lambda \in W(T_{e^{-i\theta}})$
Finally, for $T_{e^{i\theta}}$,

$$\langle T_{e^{i\theta}} f, f\rangle = \langle f, T^*_{e^{i\theta}} f\rangle = \overline{\langle T_{e^{-i\theta}} f, f\rangle}$$

Hence, $W(T_{e^{i\theta}})$ is the complex conjugate of $W(T_{e^{-i\theta}})$ which is also $\mathbb{D}$. □

There are two simple but useful results regarding the numerical range derived in the proof above:

Lemma 24. *For $A \in \mathcal{B}(\mathcal{H})$:*
a) $W(A^*) = \overline{W(A)}$
b) $\Pi_0(A) \subseteq W(A)$

4.2 Computational method

Although being able to provide information about the spectrum of Toeplitz operator with discontinuous symbol, the numerical range is still a challenge to compute analytically in general. In this project, we experimented using a computer program to simulate points in the numerical range: given a Toeplitz operator, we take truncation of size n of its (singly infinite) Toeplitz matrix, denote T_ϕ^{tr} and randomly generate unit vectors of size n, f^{rand};

then we compute and plot $\langle T_\phi^{tr} f^{rand}, f^{rand}\rangle$, yielding a subset of T_ϕ.

Figure [3] shows the points generated by this program verifying the numerical range for the right shift operator.

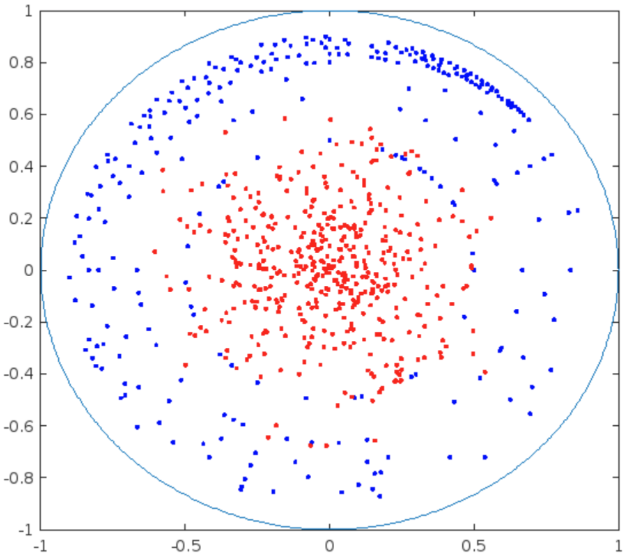

Figure 3: simulated points in the numerical range of $T_{e^{i\theta}}$.

We then use the same program on a Toeplitz operator with discontinuous symbol, whose spectrum and numerical is unknown.

Example 3 - discontinuous symbol

Consider T_ϕ, where

$$\phi(e^{i\theta}) = \begin{cases} e^{2i\theta} + 2i & \theta \in [0, \pi) \\ e^{2i\theta} - 2i & \theta \in [\pi, 2\pi) \end{cases}$$

The image of ϕ is shown in Figure [4a]. We calculated the Fourier coefficients of ϕ which defines the matrix of T_ϕ:

$$\hat{\phi_n} = \begin{cases} \frac{2}{n\pi}[1 - (-1)^n] & n \neq 0, 2 \\ 0 & n = 0 \\ 1 & n = 2 \end{cases}$$

The simulation is provided in Figure [4b].

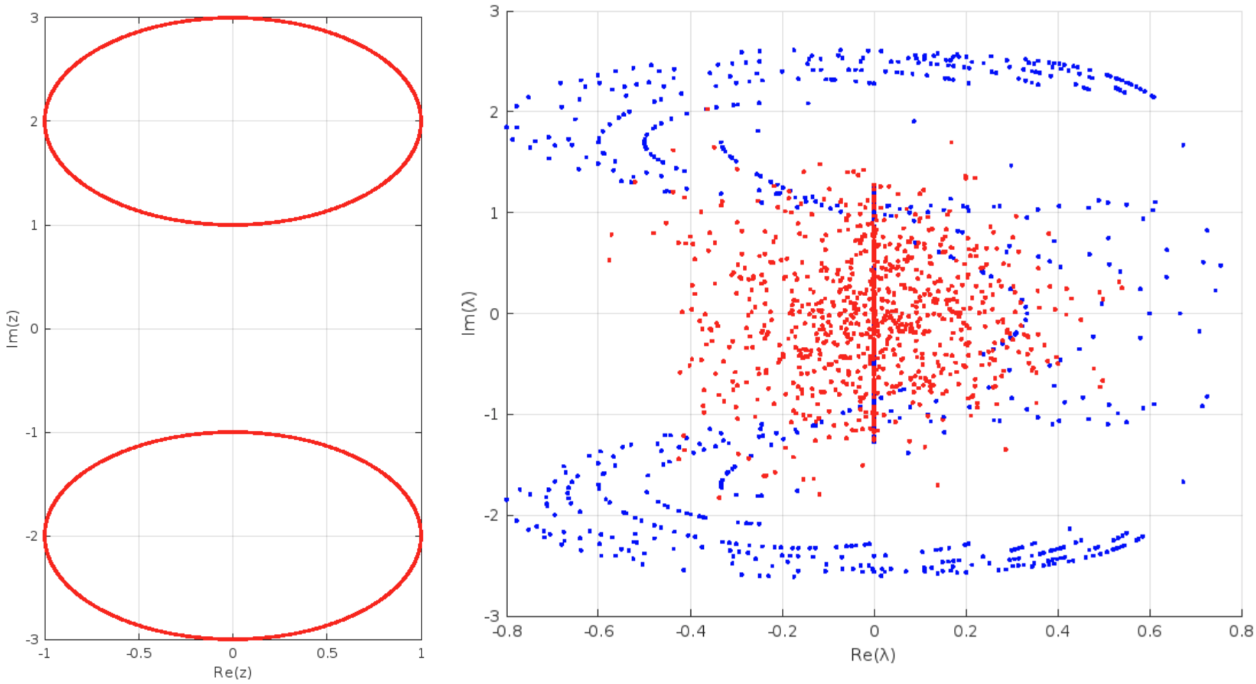

Figure 4: (left) Image of the symbol ϕ; (right) simulated points in the numerical range of a Toeplitz operator with discontinuous symbol ϕ.

An important caveat to note is that, one weakness of using the numerical range to study the spectrum is that it only detects the convex hull, and so we don't know whether regions between parts of the spectrum will be also in the spectrum. In this example, it appears that the region in-between the two disconnected circles (the red dots) is in the numerical range, however, we cannot determine if it is part of the spectrum, since excluding it will not change the convex hull of the two circles.

5 Conclusion and Future Direction

In this report we summarised existing results for well-understood Toeplitz operators and their spectra, and provide concrete examples to verify the theorems. We discussed methods of finding the spectra of general Toeplitz operators, including investigating the connection of the numerical range of a Toeplitz operator to its spectrum, and using numerical method with computer programs.

Future studies may continue to explore the spectrum of general Toeplitz operators, attempt to provide a complete description given some constraints, such as having piecewise continuous symbols, and investigate the applicability of this computational method to more complicated Toeplitz operators.

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Appendix - code

```
%define the symbol by its Fourier series
syms x
y=piecewise(x~=0 &x~=2, 2/(pi*x)*(1-(-1)^x), x==0, 0, x==2, 1)

nu=100;%no.points for each truncation size
figure
grid on
for j = 2:10 %truncation size
    %create truncated Toeplitz matrix:
    A=[];
    for n=0:j-1
         a=transpose(subs(y,x,-n:-n+j-1));
         A = [A,a];
    end

    %first set of points - using randomly generated unit vectors with
       uniformly distributed angles
    B = exp(1i*2*pi*rand(j,nu)).*randn(j,nu); %generate nu points with
       length j
    for i=1:nu
            g=B(:,i)/norm(B(:,i));
            l=dot(A*g,g);
            plot(real(l),imag(l),'.','Color', 'r')
            hold on
    end

    %second set of points - using unit vectors where arguments in
       consecutive coordinate is differ by a (random) constant
    %this is to resolve the issue for the first set of unit vector generated
        giving points with vanishing magnitude as truncation size increases
    v = ones(j,nu)/sqrt(j);
    for t = 1:nu
            r = randi([-100 100],1,1);
            u = diag(ones(1,j).*exp(1i*100*pi*[0:1/(r*(j-1)):1/r]))*v(:,t);
            s = dot(u,A*u);
```

```
            plot(real(s),imag(s),'.','Color','b')
            hold on
        end
        A=[];
    end

    xlabel('Re(   )')
    ylabel('Im(   )')
    hold off
```