AMSI SUMMER RESEARCH SCHOLARSHIPS 2023–24

SET YOUR SIGHTS ON RESEARCH THIS SUMMER

# $SO(n)$-invariant Einstein metrics

Joseph Kwong
Supervisor: Dr Ramiro Lafuente
The University of Queensland

AMSI AUSTRALIAN MATHEMATICAL SCIENCES INSTITUTE

**Abstract**

In this report, we show that if $g$ is a complete $SO(n)$-invariant Einstein metric on $M = \mathbb{R}^n$ or $M = \mathbb{S}^n$, then, up to scaling, $(M, g)$ is isometric to one of the three model spaces. The proof uses the Killing-Hopf Theorem and warped product metrics.

## Contents



# 1 Introduction

*Smooth manifolds* are generalisations of smooth curves and surfaces to higher dimensions. A *Riemannian metric* is a mathematical object defined on a smooth manifold which allows us to talk about distances, angles, and curvatures. A smooth manifold equipped with a Riemannian metric is called a *Riemannian manifold*. Arguably, the most famous examples of Riemannian manifolds are the *three model spaces: Euclidean space*, the *round sphere*, and *hyperbolic space*.

It turns out that every any smooth manifold admits infinitely many Riemannian metrics. Thus, geometers have been interested in the following question:

*Given a smooth manifold $M$, are there any "distinguished" Riemannian metrics on $M$?*

One interpretation of what it means for a metric to be "distinguished" is having *constant curvature*. Two important curvatures associated with any Riemannian metric are the *sectional curvature* and the *Ricci curvature*. A metric having constant sectional curvature implies that it also has constant Ricci curvature, but the converse is not true in general.

Metrics with constant sectional curvature are well-understood. Indeed, the *Killing-Hopf Theorem* tells us that, given some standard assumptions, the only Riemannian manifolds with constant sectional curvature are the three model spaces. In particular, not every smooth manifold admits a metric with constant sectional curvature.

A metric with constant Ricci curvature is called *Einstein*. More precisely, a Riemannian metric $g$ is *Einstein* if

$$\mathrm{Ric}_g = \lambda g, \tag{1}$$

where $\mathrm{Ric}_g$ is the Ricci curvature of $g$, and $\lambda \in \mathbb{R}$ is a constant. Equation (1) is called the *Einstein equation*. Finding Einstein metrics is difficult: the Einstein equation is a non-linear partial differential equation.

One way that geometers have been able to find Einstein metrics is by introducing *symmetry*. The hope is that symmetry simplifies the Einstein equation. Mathematically, introducing symmetry comes in two steps: first, we pick a Lie group $G$ acting on our smooth manifold $M$; next, we seek Einstein metrics on $M$ which are invariant under the action of $G$. These metrics are called *$G$-invariant*.

This approach has been very successful in finding new Einstein metrics. For example, many Einstein metrics on the smooth manifold $\mathbb{R}^n$ have been found via *Einstein solvmanifolds*, which are solvable Lie groups equipped with an Einstein metric which is invariant under left-multiplication (see [Lau09] for a survey of Einstein solvmanifolds). On the other hand, non-round Einstein metrics have been found on the $n$-sphere $\mathbb{S}^n$ by considering metrics invariant under particular group actions (for examples, see [Jen73] and [B9̈8]).

In this report, we consider the case when the group is the *special orthogonal group* $SO(n)$, and the smooth manifold is $\mathbb{R}^n$ or $\mathbb{S}^n$. Our main result is that this symmetry does not give us any new Einstein metrics:

**Theorem 1.1.** *Fix $n \geq 2$, and let $g$ be a complete $SO(n)$-invariant Einstein metric on $M = \mathbb{R}^n$ or $M = \mathbb{S}^n$. Then, up to scaling, $(M, g)$ is isometric to one of the three model spaces.*

We remark that Theorem 1.1 is well-known to experts. However, to the of best of the author's knowledge, a full proof is not written down anywhere in the literature.

This report is organised as follows. In Section 2, we discuss the background material relevant for the proof of Theorem 1.1. In Section 3.1, we show that any $SO(n)$-invariant metric on $\mathbb{R}^n$ or $\mathbb{S}^n$ can be written as a warped product metric on a dense open subset. In Section 3.2, we show that on the warped product $I \times_f \mathbb{S}^{n-1}$, the Einstein condition is equivalent to having constant sectional curvature. Finally, we prove Theorem 1.1 in Section 3.3.

**Statement of authorship.** All research in this report was done by the stated author. Any previously established results have been cited appropriately and clearly.

**Acknowledgements.** I would like to thank my supervisor Dr Ramiro Lafuente for his advice, encouragement, and guidance. I would also like to thank Adam Thompson for many helpful conversations.

# 2 Preliminaries

## 2.1 Einstein metrics

Let $M$ be a smooth manifold and $g$ a Riemannian metric on $M$. Let $\mathrm{Ric}_g$ denote the Ricci curvature of $g$. We say that $g$ is *Einstein* if $\mathrm{Ric}_g = \lambda g$ for some $\lambda \in \mathbb{R}$. In this case, we say that the Riemannian manifold $(M, g)$ is an *Einstein manifold*.

*Remark* 2.1. Einstein metrics are often called *metrics with constant Ricci curvature*. This is because $\mathrm{Ric}_g = \lambda g$ if and only if $\mathrm{Ric}_g(v, v) = \lambda$ for every unit vector $v \in UM$.

Let $\mathrm{Sec}_g$ denote the sectional curvature of $g$. Recall that we say $g$ has *constant sectional curvature* $c \in \mathbb{R}$ if $\mathrm{Sec}_g(\Pi_p) = c$ for every $p \in M$ and 2-dimensional subspace $\Pi_p \leq T_pM$. If $g$ has constant sectional curvature, then it is Einstein:

**Proposition 2.2** ([Lee18, Proposition 8.36])**.** *Let $M$ be a smooth manifold and $g$ a Riemannian metric on $M$. Then $g$ has constant sectional curvature $c \in \mathbb{R}$ if and only if*

$$\mathrm{Rm}_g = \frac{1}{2} c \, g \owedge g,$$

*where $\mathrm{Rm}_g$ denotes the $(0, 4)$-Riemann tensor of $g$ and $\owedge$ is the Kulkarni–Nomizu product. In this case, $g$ is Einstein with $\mathrm{Ric}_g = c(n-1)g$.*

The following three examples of Riemannian manifolds are known as the *three model spaces*. Each model space is a simply-connected complete Riemannian manifold of constant sectional curvature (and so is in particular Einstein). Fix $n \geq 2$.

**Example 2.3** (Euclidean space)**.** Consider $\mathbb{R}^n$ as a smooth manifold. Let $x^1, \dots, x^n : \mathbb{R}^n \to \mathbb{R}$ denote the standard coordinates of $\mathbb{R}^n$. The *Euclidean metric* $\overline{g}$ is the Riemannian metric on $\mathbb{R}^n$ defined by

$$\overline{g} := (dx^1)^2 + \cdots + (dx^n)^2.$$

If we identify each tangent space of $\mathbb{R}^n$ with $\mathbb{R}^n$ itself in the usual way, then the Euclidean metric at each point is just the standard inner product on $\mathbb{R}^n$. The Riemannian manifold $(\mathbb{R}^n, \overline{g})$ is called *Euclidean space*. The Euclidean metric has constant sectional curvature 0, and so is Einstein with $\mathrm{Ric}_{\overline{g}} = 0$.

**Example 2.4** (The round sphere)**.** Let $\mathbb{S}^n$ denote the $n$-dimensional unit sphere, which is the embedded submanifold of $\mathbb{R}^{n+1}$ consisting of all points in $\mathbb{R}^{n+1}$ with Euclidean norm 1. The *round metric* $\mathring{g}$ is the Riemannian metric on $\mathbb{S}^n$ given by $\mathring{g} := \iota^*\overline{g}$, where $\iota : \mathbb{S}^n \hookrightarrow \mathbb{R}^n$ denotes the inclusion map and $\overline{g}$ is the Euclidean metric on $\mathbb{R}^{n+1}$. In other words, the round metric on $\mathbb{S}^n$ is just the Euclidean metric on $\mathbb{R}^{n+1}$ restricted to the tangent spaces of $\mathbb{S}^n$. The Riemannian manifold $(\mathbb{S}^n, \mathring{g})$ is called the *round sphere*. The round metric has constant sectional curvature 1 and so is Einstein with $\mathrm{Ric}_{\mathring{g}} = (n-1)\mathring{g}$.

**Example 2.5** (Hyperbolic space)**.** Let $\mathbb{H}^n$ denote the *upper-half sheet of the two-sheeted hyperboloid*, which is the embedded submanifold of $\mathbb{R}^{n+1}$ defined by

$$\mathbb{H}^n := \left\{(x^1, \dots, x^{n+1}) \in \mathbb{R}^{n+1} \;\middle|\; (x^1)^2 + \cdots + (x^n)^2 - (x^{n+1})^2 = -1, \;\; x^{n+1} > 0\right\}.$$

The *hyperbolic metric* $\breve{g}$ is the Riemannian metric $\breve{g}$ on $\mathbb{H}^n$ given by $\breve{g} := \iota^*\overline{q}$, where $\iota : \mathbb{H}^n \hookrightarrow \mathbb{R}^{n+1}$ denotes the inclusion map and $\overline{q}$ denotes the *Minkowski metric* on $\mathbb{R}^{n+1}$, which is the Lorentzian (non-Riemannian) metric on $\mathbb{R}^{n+1}$ given by

$$\overline{q} := (dx^1)^2 + \cdots + (dx^n)^2 - (dx^{n+1})^2.$$

Here, $(x^i)$ denote the standard coordinates on $\mathbb{R}^{n+1}$. The hyperbolic metric has constant sectional curvature $-1$, and so is Einstein with $\mathrm{Ric}_{\breve{g}} = -(n-1)\breve{g}$.

*Remark* 2.6. As a smooth manifold, hyperbolic space is diffeomorphic to $\mathbb{R}^n$. Thus, the hyperbolic metric can be viewed as a Riemannian metric on $\mathbb{R}^n$.

The following shows why the three model spaces are so important:

**Proposition 2.7** (Killing-Hopf Theorem [Lee18, Theorem 12.4])**.** *Let $(M, g)$ be a simply-connected complete Riemannian manifold. Suppose $g$ has constant sectional curvature. Then, up to scaling, $(M, g)$ is isometric to one of the three model spaces.*

Are the model spaces the only simply-connected complete Einstein manifolds? The answer is *yes* when $\dim M \leq 3$:

**Proposition 2.8** ([Lee18, Corollary 8.28, Problem 8-14])**.** *Let $(M, g)$ be a Riemannian manifold with $\dim M \in \{2, 3\}$. If $g$ is Einstein, then $g$ has constant sectional curvature.*

However, the proposition above does not generalise to higher dimensions. Indeed, there exist simply-connected complete Einstein manifolds which are *not* model spaces, (for example, *complex projective space* equipped with the *Fubini-study metric*, or *complex hyperbolic space* [Bes87, 7.15, 7.17]).

## 2.2 Warped product metrics

Let $(B, g_B)$ and $(F, g_F)$ denote Riemannian manifolds, and let $f : B \to \mathbb{R}^+$ be a positive smooth function. The *warped product of $(B, g_B)$ and $(F, g_F)$ with warping function $f$*, denoted $B \times_f F$, is the Riemannian manifold defined in the following manner: as a smooth manifold, $B \times_f F$ is just the product manifold $B \times F$. The Riemannian metric $g$ on $B \times_f F$ is given by

$$g := (\pi_B)^* g_B + (f \circ \pi_B)^2 (\pi_F)^* g_F,$$

where $\pi_B : B \times F \to B$ and $\pi_F : B \times F \to F$ denote the usual projection maps.

We say that $(B, g_B)$ is the *base manifold* and $(F, g_F)$ is the *fibre manifold*. The metric $g$ is called the *warped product metric*. Often we abuse notation and write $g = g_B + f^2 g_F$.

We now show how the warped product metric evaluates tangent vectors: fix $(p,q) \in B \times_f F$, and identify $T_{(p,q)}(B \times F) \cong T_pB \oplus T_qF$ in the usual way. Then for $v = (v_B, v_F)$ and $w = (w_B, w_F)$ in $T_{(p,q)}(B \times F)$, we have

$$g_{(p,q)}(v,w) = g_B|_p(v_B, w_B) + f(p)^2 g_F|_q(v_F, w_F).$$

*Remark* 2.9. For the rest of the report, we only consider the special case when the base manifold is an interval $I$ equipped with the Euclidean metric $dt^2$, and the fibre manifold is the $(n-1)$-dimensional round sphere $(\mathbb{S}^{n-1}, \mathring{g})$. In other words, we only consider the warped products of the form $I \times_f \mathbb{S}^{n-1}$ with warped product metric $g = dt^2 + f^2\mathring{g}$.

The following three examples show that, on each of the three model spaces, an open dense subset can be viewed as a warped product $I \times_f \mathbb{S}^{n-1}$. The proofs of the following facts can be found in [Pet16, Section 1.4.4]. Fix $n \geq 2$.

**Example 2.10** (Euclidean space)**.** Consider when $I = \mathbb{R}^+$ and $f(t) = t$. Then the warped product $\mathbb{R}^+ \times_t \mathbb{S}^{n-1}$ is isometric to Euclidean space minus the origin. An explicit isometry $F : \mathbb{R}^+ \times_t \mathbb{S}^{n-1} \to \mathbb{R}^n \backslash \{0\}$ is given by

$$F : (t,s) \mapsto ts, \qquad F^{-1} : x \mapsto \left(\|x\|, \frac{x}{\|x\|}\right),$$

where $t \in \mathbb{R}^+$, $s \in \mathbb{S}^{n-1}$, $x \in \mathbb{R}^n \backslash \{0\}$, and $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$.

**Example 2.11** (The round sphere)**.** Consider when $I = (0,\pi)$ and $f(t) = \sin(t)$. Then the warped product $(0,\pi) \times_{\sin(t)} \mathbb{S}^{n-1}$ is isometric to the round sphere minus the north and south poles, which are given by $N := (0,\ldots,0,1)$ and $S := (0,\ldots,0,-1)$, respectively. An explicit isometry $F : (0,\pi) \times_{\sin(t)} \mathbb{S}^{n-1} \to \mathbb{S}^n \backslash \{N, S\}$ is given by

$$F : (t,s) \mapsto \Big(\sin(t)s, \cos(t)\Big), \qquad F^{-1} : (x, x^{n+1}) \mapsto \left(\arccos(x^{n+1}), \frac{x}{\|x\|}\right),$$

where $t \in \mathbb{R}^+$, $s \in \mathbb{S}^{n-1}$, $x \in \mathbb{R}^n$, and $x^{n+1} \in \mathbb{R}$.

**Example 2.12** (Hyperbolic space)**.** Consider when $I = \mathbb{R}^+$ and $f(t) = \sinh(t)$. Then the warped product $\mathbb{R}^+ \times_{\sinh(t)} \mathbb{S}^{n-1}$ is isometric to hyperbolic space minus the point $P := (0,\ldots,0,1)$. An explicit isometry $F : \mathbb{R}^+ \times_{\sinh(t)} \mathbb{S}^{n-1} \to \mathbb{H}^n \backslash \{P\}$ is given by

$$F : (t,s) \mapsto \Big(\sinh(t)s, \cosh(t)\Big), \qquad F^{-1} : (x, x^{n+1}) \mapsto \left(\operatorname{arccosh}(x^{n+1}), \frac{x}{\|x\|}\right),$$

where $t \in \mathbb{R}^+$, $s \in \mathbb{S}^{n-1}$, $x \in \mathbb{R}^n$, and $x^{n+1} \in \mathbb{R}$.

*Remark* 2.13 (Scaling a warped product metric)*.* Let $I$ be an open interval, let $f : I \to \mathbb{R}$ be a positive smooth function, and consider the warped product $I \times_f \mathbb{S}^{n-1}$ with warped product metric $g = dt^2 + f^2\mathring{g}$. Fix a real number $R > 0$. Then $(I \times \mathbb{S}^{n-1}, R^2 g)$ is isometric to the warped product $\widetilde{I} \times_{\widetilde{f}} \mathbb{S}^{n-1}$ with warped product metric $dt^2 + (\widetilde{f})^2\mathring{g}$, where $\widetilde{I} := RI$ and $\widetilde{f} : \widetilde{I} \to \mathbb{R}$ is given by $\widetilde{f}(t) := R\, f(t/R)$. An explicit isometry is given by

$$F : I \times \mathbb{S}^{n-1} \to \widetilde{I} \times \mathbb{S}^{n-1}, \qquad (t,s) \mapsto (Rt, s).$$

## 2.3 Smooth actions of $SO(n)$

A *Lie group* is an abstract group endowed with a smooth manifold structure such that the multiplication and inversion maps are smooth. The Lie group of interest in this report is the *special orthogonal group in dimension* $n$, which we denote by $SO(n)$. We define $SO(n)$ to be the Lie group of all $n$ by $n$ matrices with real entries $A$ which satisfy $AA^\top = I_n$ and $\det(A) = 1$. As a smooth manifold, $SO(n)$ is compact, connected, and has dimension $n(n-1)/2$.

For the rest of this subsection, let $G$ denote a Lie group and $M$ a smooth manifold. A *smooth (left) action of $G$ on $M$* is a group homomorphism

$$\theta : G \to \mathrm{Diff}(M), \qquad \alpha \mapsto \theta_\alpha$$

such that the map $G \times M \to M$ given by $(\alpha, p) \mapsto \theta_\alpha(p)$ is smooth. Here, $\mathrm{Diff}(M)$ denotes the diffeomorphism group of $M$. Often, we write $\alpha \cdot p := \theta_\alpha(p)$ if there is no ambiguity about the choice of action $\theta$.

Let $\theta : G \to \mathrm{Diff}(M)$ be a smooth action of $G$ on $M$. For each $p \in M$, the *orbit of $p$*, denoted by $G \cdot p$, is the set of all points in $M$ which can be written as $\alpha \cdot p$ for some $\alpha \in G$. The orbits partition $M$, and each orbit is an embedded submanifold of $M$. We say *$G$ acts transitively on $M$* if there is only one orbit.

Next, for each $p \in M$, the *isotropy subgroup at $p$* (or the *stabiliser of $p$*), denoted $G_p$, is the subgroup of $G$ consisting of all $\alpha \in G$ such that $\alpha \cdot p = p$. When $G$ is compact, each isotropy subgroup is an embedded Lie subgroup of $G$. The isotropy subgroup $G_p$ acts smoothly and *linearly* on $T_pM$ via $\alpha \cdot v := d\theta_\alpha(v)$. By *linearly*, we mean that $v \mapsto \alpha \cdot v$ is linear for all $\alpha \in G$. This action of $G_p$ on $T_pM$ is called the *isotropy representation of $\theta$ at $p$*.

The following four examples give smooth actions of $SO(n)$ on various smooth manifolds. Fix $n \geq 2$.

**Example 2.14** ($SO(n)$ acting on $\mathbb{R}^n$)**.** The Lie group $SO(n)$ acts smoothly on the smooth manifold $\mathbb{R}^n$ via $A \cdot x := Ax$. The orbits are the origin and the *Euclidean spheres* $\mathbb{S}^{n-1}(R)$ for $R > 0$. Here, $\mathbb{S}^{n-1}(R)$ denotes the embedded submanifold of $\mathbb{R}^n$ consisting of points with Euclidean norm $R > 0$. The isotropy subgroup at the origin is $SO(n)$, and the isotropy subgroup at any other point is (isomorphic to) $SO(n-1)$.

**Example 2.15** ($SO(n)$ acting on $\mathbb{S}^n$)**.** The Lie group $SO(n)$ acts smoothly on the smooth manifold $\mathbb{S}^n$ via $A \cdot (x, t) := (Ax, t)$, where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. The orbits are the north pole $N := (0, \dots, 0, 1)$, the south pole $S := (0, \dots, 0, -1)$, and $\mathbb{S}^{n-1}(\sqrt{1-t^2}) \times \{t\}$ for $t \in (-1, 1)$. The isotropy subgroups at the north and south poles are both $SO(n)$, and the isotropy subgroup at any other point is (isomorphic to) $SO(n-1)$.

**Example 2.16** ($SO(n)$ acting on $\mathbb{H}^n$)**.** The Lie group $SO(n)$ acts smoothly on the hyperboloid $\mathbb{H}^n$ as a smooth manifold (see Example 2.5) via $A \cdot (x, t) := (Ax, t)$, where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. The orbits are $P := (0, \dots, 0, 1)$ and $\mathbb{S}^{n-1}(\sqrt{t^2-1}) \times \{t\}$ for $t > 1$. The isotropy subgroup at $P$ is $SO(n)$, and the isotropy subgroup at any other point is (isomorphic to) $SO(n-1)$.

**Example 2.17** ($SO(n)$ acting on $I \times \mathbb{S}^{n-1}$)**.** Let $I$ be an open interval. Then $SO(n)$ acts smoothly on the product manifold $I \times \mathbb{S}^{n-1}$ via $A \cdot (t, s) := (t, As)$, where $t \in \mathbb{R}$ and $s \in \mathbb{S}^{n-1}$. The orbits are $\{t\} \times \mathbb{S}^{n-1}$ for $t \in I$. The isotropy subgroup at any point is (isomorphic to) $SO(n-1)$.

Let $\theta : G \to \mathrm{Diff}(M)$ be a smooth action, and let $N$ be an embedded submanifold of $M$ such that $\alpha \cdot p \in N$ for any $\alpha \in G$ and $p \in N$. Then $\theta$ restricts of a smooth action of $G$ on $N$. Thus, the action of $SO(n)$ on $\mathbb{R}^n$ can be restricted to $\mathbb{R}^n \backslash \{0\}$, $\mathbb{S}^{n-1}(R)$, $\mathbb{B}^n(R)$, and $\mathbb{B}^n(R)\backslash\{0\}$ for any $R > 0$. Here, $\mathbb{B}^n(R)$ denote the *open Euclidean ball of radius $R$ centred at the origin*, which consists of all points in $\mathbb{R}^n$ whose Euclidean norm is strictly less than $R$.

## 2.4 $SO(n)$-equivariant maps

Let $F : M \to N$ be a smooth map between smooth manifolds, and let $G$ be a Lie group acting smoothly on $M$ and $N$. We say that $F$ is *$G$-equivariant* if $F(\alpha \cdot p) = p \cdot F(p)$ for all $\alpha \in G$ and $p \in M$. If $F$ is a $G$-equivariant diffeomorphism, then $F^{-1}$ is also $G$-equivariant.

We give three examples of $SO(n)$-equivariant diffeomorphisms which are useful. Fix $n \geq 2$.

**Example 2.18** (Cylinder to punctured disk)**.** Let $I$ be an interval, and $R \in (0, \infty]$. Suppose $h : I \to (0, R)$ is a diffeomorphism. Then the diffeomorphism $\Phi : I \times \mathbb{S}^{n-1} \to \mathbb{B}^n(R)\backslash\{0\}$ given by

$$\Phi : (t, s) \mapsto h(t)s, \qquad \Phi^{-1} : x \mapsto \left( h(\|x\|), \frac{x}{\|x\|} \right)$$

is $SO(n)$-equivariant. Here, $t \in I$, $s \in \mathbb{S}^{n-1}$, $x \in \mathbb{B}^n(R)\backslash\{0\}$, and $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$.

**Example 2.19** (Punctured sphere to disk)**.** Consider the unit-sphere $\mathbb{S}^n$, and let $N$ and $S$ denote the north and south poles of $\mathbb{S}^n$, respectively. Let us identify $T_N\mathbb{S}^n$ with $\mathbb{R}^n$ via

$$\frac{\partial}{\partial x^i}\Big|_N \longleftrightarrow e_i, \quad i = 1, \ldots, n,$$

where $\partial/\partial x^1, \ldots, \partial/\partial x^{n+1}$ is the standard coordinate frame of $\mathbb{R}^{n+1}$ and $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$. Under this identification, let $\exp_N : \mathbb{R}^n \to \mathbb{S}^n$ denote the exponential map of the round metric $\mathring{g}$ at $N$. Then $\exp_N : \mathbb{R}^n \to \mathbb{S}^n$ is $SO(n)$-equivariant. Moreover, the restriction $\exp_N : \mathbb{B}^n(\pi) \to \mathbb{S}^n\backslash\{S\}$ is an $SO(n)$-equivariant diffeomorphism.

**Example 2.20** (Hyperboloid to plane)**.** Consider the hyperboloid $\mathbb{H}^n$ as a smooth manifold (see Example 2.5), and set $P := (0, \ldots, 0, 1)$. Let us identify $T_P\mathbb{H}^n$ with $\mathbb{R}^n$ via

$$\frac{\partial}{\partial x^i}\Big|_P \longleftrightarrow e_i, \quad i = 1, \ldots, n.$$

Under this identification, let $\exp_P : \mathbb{R}^n \to \mathbb{H}^n$ denote the exponential map of the hyperbolic metric $\breve{g}$ at $P$. Then $\exp_P : \mathbb{R}^n \to \mathbb{H}^n$ is an $SO(n)$-equivariant diffeomorphism.

### 2.5 $SO(n)$-invariant metrics

Let $G$ be a Lie group, $M$ a smooth manifold, and $\theta : G \to \mathrm{Diff}(M)$ a smooth action. A Riemannian metric $g$ on $M$ is called *$G$-invariant* if $\theta_\alpha : (M, g) \to (M, g)$ is an isometry for all $\alpha \in G$.

**Examples 2.21.** Fix $n \geq 2$. We list examples of $SO(n)$-invariant metrics:

(i) The Euclidean metric $\overline{g}$ on $\mathbb{R}^n$ is $SO(n)$-invariant with respect to the action $A \cdot x := Ax$.

(ii) The round metric $\mathring{g}$ on $\mathbb{S}^n$ is $SO(n)$-invariant with respect to the action $A \cdot (x, x^{n+1}) = (Ax, x^{n+1})$.

(iii) The hyperbolic metric $\breve{g}$ on the hyperboloid $\mathbb{H}^n$ is $SO(n)$-invariant with respect to the action $A \cdot (x, x^{n+1}) = (Ax, x^{n+1})$.

(iv) Let $I$ be an open interval, and let $f : I \to \mathbb{R}^+$ be a positive smooth function. Consider the warped product $I \times_f \mathbb{S}^{n-1}$ with warped product metric $g := dt^2 + f^2\mathring{g}$. Then $g$ is $SO(n)$-invariant with respect to the action $A \cdot (t, s) := (t, As)$.

*Remark* 2.22. Let $G$ be a Lie group acting smoothly on smooth manifolds $M$ and $N$. Let $F : M \to N$ be an $G$-equivariant diffeomorphism. If $g$ is an $G$-invariant metric on $N$, then $F^*g$ is an $G$-invariant metric on $M$.

The above remark implies that $SO(n)$-invariant metrics on the smooth manifold $\mathbb{H}^n$ are in bijection with $SO(n)$-invariant metrics on $\mathbb{R}^n$ via Example 2.20. Thus, there is no need to study $SO(n)$-invariant metrics on the smooth manifold $\mathbb{H}^n$.

Finally, suppose that $M = V$ is a vector space, and $G$ acts linearly on $V$. An inner product $b(\cdot, \cdot)$ on $V$ is called *$G$-invariant* if $\theta_\alpha : (V, b) \to (V, b)$ is a linear isometry for all $\alpha \in G$. For example, the standard inner product on $\mathbb{R}^n$ is $SO(n)$-invariant.

## 3 $SO(n)$-invariant Einstein metrics on $\mathbb{R}^n$ and $\mathbb{S}^n$

### 3.1 $SO(n)$-invariant metrics as warped product metrics

The aim of this subsection is to prove the following proposition:

**Proposition 3.1.** *Fix $n \geq 4$, and let $g$ be an $SO(n)$-invariant metric on $M = \mathbb{R}^n$ and $M = \mathbb{S}^n$. Then a dense open subset of $(M, g)$ is isometric to a warped product of the form $I \times_f \mathbb{S}^{n-1}$.*

Recall that Example 2.19 gives an $SO(n)$-equivariant diffeomorphism between $\mathbb{S}^n\backslash\{S\}$ and $\mathbb{B}^n(\pi)$. Since the pullback of an $SO(n)$-invariant metric under an $SO(n)$-equivariant diffeomorphism is again $SO(n)$-invariant (Remark 2.22), Proposition 3.1 follows immediately from the following proposition:

**Proposition 3.2.** *Fix $n \geq 4$, and let $g$ be an $SO(n)$-invariant metric on $\mathbb{B}^n(R)$ where $R \in (0, \infty]$. Then $(\mathbb{B}^n(R)\backslash\{0\}, g)$ is isometric to a warped product of the form $I \times_f \mathbb{S}^{n-1}$.*

We spend the rest of this subsection proving Proposition 3.2. Recall that $SO(n)$ acts smoothly and linearly on $\mathbb{R}^n$ via $A \cdot x := Ax$.

**Lemma 3.3.** *Fix $n \geq 1$. Let $b(\cdot, \cdot)$ be an $SO(n)$-invariant inner product on $\mathbb{R}^n$. Then $b$ is a scalar multiple of the standard inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$.*

*Proof.* Let $c > 0$ be a real number such that $b(e_1, e_1) = c^2$, where $e_1, \dots, e_n$ is the standard basis of $\mathbb{R}^n$. An arbitrary element of $\mathbb{R}^n$ can be written as $tv$, where $t \geq 0$ and $v \in \mathbb{S}^{n-1}$. Since $SO(n)$ acts transitively on $\mathbb{S}^{n-1}$, there exists $A \in SO(n)$ such that $v = Ae_1$. Therefore,

$$b(tv, tv) = t^2 b(Ae_1, Ae_1) = t^2 b(e_1, e_1) = t^2 c^2 = c^2 t^2 \langle v, v \rangle = c^2 \langle tv, tv \rangle,$$

so $b = c^2 \langle \cdot, \cdot \rangle$, as desired. □

**Lemma 3.4.** *Fix $n \geq 2$. Let $g$ be an $SO(n)$-invariant metric on $\mathbb{S}^{n-1}$. Then $g$ is a scalar multiple of the round metric $\mathring{g}$ on $\mathbb{S}^{n-1}$.*

*Proof.* Since $SO(n)$ acts transitively on $\mathbb{S}^{n-1}$ and both $g$ and $\mathring{g}$ are $SO(n)$-invariant, it suffices to show that $g_N$ is a scalar multiple of $\mathring{g}_N$, where $N := (0, \dots, 0, 1)$ is the north pole. Recall that the isotropy subgroup at $N$, denoted $SO(n)_N$, acts smoothly and linearly on $T_N\mathbb{S}^{n-1}$ via $A \cdot v := d\theta_A(v)$, where $\theta$ is the action of $SO(n)$ on $\mathbb{S}^{n-1}$. Moreover, we have a Lie group isomorphism

$$SO(n-1) \to SO(n)_N, \qquad A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix},$$

so $SO(n-1)$ acts on $T_N\mathbb{S}^{n-1}$, and $g_N$ is an $SO(n-1)$-invariant inner product on $T_N\mathbb{S}^{n-1}$. Moreover, the identification

$$\mathbb{R}^n \leftrightarrow T_N\mathbb{S}^{n-1}, \qquad e_i \leftrightarrow \frac{\partial}{\partial x^i}\Big|_N, \quad i = 1, \dots, n-1$$

so $SO(n)$-equivariant. Under this identification, $\mathring{g}_N$ is just the standard inner product on $\mathbb{R}^n$ and $g_N$ is an $SO(n-1)$-invariant inner product on $\mathbb{R}^n$. Thus, $g_N$ is a scalar multiple of $\mathring{g}_N$, by Lemma 3.3. This completes the proof. □

Consider the smooth manifold $\mathbb{R}^n$, and fix $p \in \mathbb{R}^n$. For the rest of this subsection, let us identify the tangent space $T_p\mathbb{R}^n$ with $\mathbb{R}^n$ itself through the identification

$$e_i \leftrightarrow \frac{\partial}{\partial x^i}\Big|_p, \quad i = 1, \dots, n.$$

Note that the identification $T_0\mathbb{R}^n \cong \mathbb{R}^n$ is $SO(n)$-equivariant.

**Lemma 3.5** (Geodesics passing through the origin)**.** *Fix $n \geq 4$, fix $R \in (0, \infty]$, and let $g$ be an $SO(n)$-invariant metric on $\mathbb{B}^n(R)$. Assume that $g_0$ is equal to the standard inner product on $\mathbb{R}^n \cong T_0\mathbb{R}^n$.*

*(i) Let $J$ be an open interval containing zero, and let $\alpha : J \to \mathbb{B}^n(R)$ be a non-constant geodesic such that $\alpha(0) \in \operatorname{span}\{\alpha'(0)\}$. Then $\alpha(t) \in \operatorname{span}\{\alpha'(0)\}$ for all $t \in J$.*

*(ii) There exists $T \in (0, \infty]$ and a diffeomorphism $h : (0, T) \to (0, R)$ such that $(-T, T)$ is the maximal interval of definition for a unit-speed geodesic on $(\mathbb{B}^n(R), g)$ which starts at the origin, and if $\gamma : (-T, T) \to \mathbb{B}^n(R)$ is such a geodesic, then*

$$\gamma(t) = h(t)\gamma'(0) \quad \forall t \in (0, T).$$

*(iii) Any geodesic passing through the origin intersects the Euclidean spheres $\mathbb{S}^{n-1}(r)$ orthogonally, where $r \in (0, R)$.*

*Proof of (i).* Let $\alpha$ be as in the statement of (i). Let $\theta$ denote the action of $SO(n)$ on $\mathbb{B}^n(R)$. By applying a rotation and/or reparameterising $\alpha$, we can (without loss of generality) assume that $\alpha'(0) = e_1$. By assumption, $\alpha(0) = ce_1$ for some $c \in \mathbb{R}$. For the sake of contradiction, suppose $\alpha(t_0) \notin \operatorname{span}\{e_1\}$ for some $t_0 \in J$. Write $\alpha(t_0) = (u_1, \ldots, u_n)$. Then $u_i \neq 0$ for some $i = 2, \ldots, n$. Without loss of generality, assume that $u_2 \neq 0$. Let $A \in SO(n)$ be the matrix

$$A := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & I_{n-3} \end{pmatrix}.$$

Then $\alpha(0) = ce_1 = \theta_A(ce_1) = (\theta_A \circ \alpha)(0)$ and $\alpha'(0) = e_1 = \theta_A(e_1) = (\theta_A \circ \alpha)'(0)$, so uniqueness of geodesics implies that $\theta_A \circ \alpha = \alpha$. However, this implies that

$$(u_1, u_2, \ldots, u_n) = \alpha(t_0) = (\theta_A \circ \alpha)(t_0) = (u_1, -u_2, -u_3, u_4, \ldots, u_n),$$

so $u_2 = 0$, a contradiction. $\square$

*Proof of (ii).* Let $\gamma : J \to (\mathbb{B}^n(R), g)$ be the maximal unit-speed geodesic such that $\gamma(0) = 0$ and $\gamma'(0) = e_1$. By $SO(n)$-invariance and uniqueness of geodesics, it follows that $J = (-T, T)$ for some $T \in (0, \infty]$. By (i), we know that for any $t \in (-T, T)$, there exists a real number $h(t)$ such that $\gamma(t) = h(t)e_1$. Let $h : (-T, T) \to \mathbb{R}$ denote the smooth function $t \mapsto h(t)$. Since $\gamma'(t) = h'(t)e_1$ is non-zero for any $t$ and $h'(0) = 1$, we know that $h'$ is strictly positive, and so $h$ is strictly increasing on $(-T, T)$ with $h(0) = 0$.

Now, consider the restriction of $h$ onto $(0, T)$. Henceforth, let us abuse notation and denote this restriction by $h : (0, T) \to \mathbb{R}$. Let us show that the image of $h$ is $(0, R)$. One inclusion is clear: observe that $h(t)$ is the Euclidean norm of $\gamma(t)$ for $t \in (0, T)$, so $h(t) \in (0, R)$. Next, set

$$r_0 := \sup_{t \in (0,T)} h(t) = \sup_{t \in (0,T)} \|\gamma(t)\|,$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$. If $r_0 = R$, then the other inclusion follows from the Intermediate Value Theorem.

For the sake of contradiction, suppose that $r_0 < R$. Let $\alpha : (-2\varepsilon, 2\varepsilon) \to (\mathbb{B}^n(R), g)$ be a unit-speed geodesic such that $\alpha(0) = r_0 e_1$, $\alpha'(0) = ce_1$ where $c > 0$, and $\varepsilon$ is chosen to be small enough so that $\alpha$ does not pass through the origin. By (i), there is a smooth function $a : (-2\varepsilon, 2\varepsilon) \to \mathbb{R}$ such that $\alpha(t) = a(t)e_1$ for all $t \in (-2\varepsilon, 2\varepsilon)$. Then by similar reasoning as before, we find that $a$ and $a'$ are strictly positive, so $a$ strictly increasing. Thus, $a(-\varepsilon) \in (0, r_0)$, so by the Intermediate Value Theorem, there is some $t_0 \in (0, T)$ such that $h(t_0) = a(-\varepsilon)$. Thus, $\gamma(t_0) = \alpha(-\varepsilon)$. Moreover, $\gamma'(t_0) = h'(t_0)e_1$ and $\alpha'(-\varepsilon) = \alpha'(-\varepsilon)e_1$ are linearly dependent unit vectors (with respect to $g$) whose first coordinate is positive. Thus, $\gamma'(t_0) = \alpha'(-\varepsilon)$. Next, let $\beta : (t_0 - \varepsilon, t_0 + 3\varepsilon) \to (\mathbb{B}^n(R), g)$ be the unit-speed geodesic defined by $\beta(t) := \alpha(t - t_0 - \varepsilon)$. Then observe that

$$\beta(t_0) = \alpha(-\varepsilon) = \gamma(t_0), \qquad \text{and} \qquad \beta'(t_0) = \alpha'(-\varepsilon) = \gamma'(t_0).$$

Thus, by uniqueness of geodesics, $\beta$ and $\gamma$ agree on their common domain. Since $\gamma$ is maximal, we know that $t_0 + 2\varepsilon \in (-T, T)$. However, this is a contradiction, because the Euclidean norm of $\gamma$ is bounded above by $r$, but

$$\|\gamma(t_0 + 2\varepsilon)\| = \|\alpha(\varepsilon)\| = a(\varepsilon) > r_0.$$

Thus, the image of $h$ is $(0, R)$. Since $h$ is injective, we know $h : (0, T) \to (0, R)$ is a bijection. Since $h'$ is positive, the Inverse Function Theorem implies that $h : (0, T) \to (0, R)$ is a diffeomorphism. Part (ii) now follows easily from $SO(n)$-invariance. $\square$

*Proof of (iii).* By (ii), we know that the Riemannian exponential map of $(\mathbb{B}^n(R), g)$ at the origin is given by

$$\exp_0 : \mathbb{B}^n(T) \to \mathbb{B}^n(R), \qquad 0 \mapsto 0, \qquad tv \mapsto h(t)v,$$

where $t \in (0, T)$ and $v \in \mathbb{S}^{n-1}$. Thus, $\exp_0$ is a diffeomorphism because its inverse is given by

$$(\exp_0)^{-1} : \mathbb{B}^n(R) \to \mathbb{B}^n(T), \qquad 0 \mapsto 0, \qquad rv \mapsto h^{-1}(r)v,$$

where $r \in (0, R)$ and $v \in \mathbb{S}^{n-1}$. Therefore, for each $r \in (0, R)$, the Euclidean sphere $\mathbb{S}^{n-1}(r)$ is a geodesic sphere of $(\mathbb{B}^n(R), g)$ centred at the origin. Part (iii) now follows immediately from the Gauss Lemma [Lee18, Theorem 6.9]. □

*Proof of Proposition 3.2.* Fix $n \geq 4$, and let $g$ be an $SO(n)$-invariant metric on $\mathbb{B}^n(R)$, where $R \in (0, \infty]$. Since rescaling a warped product of the form $I \times_f \mathbb{S}^{n-1}$ gives a warped product of the same form (see Remark, 2.13), we can assume that $g_0$ is the standard inner product on $\mathbb{R}^n \cong T_0\mathbb{R}^n$ by Lemma 3.3.

Let $T$ and $h$ be as in the statement of (ii) in Lemma 3.5. Set $I := (0, T)$, and let $\Phi : I \times \mathbb{S}^{n-1} \to \mathbb{B}^n(R)\backslash\{0\}$ denote the $SO(n)$-equivariant diffeomorphism given by $\Phi : (t, s) \mapsto h(t)s$. It remains to show that the $SO(n)$-invariant metric $\widehat{g} := \Phi^* g$ is a warped product metric.

For each $t \in I$, observe that the map $F_t : \{t\} \times \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ given by $(t, s) \mapsto s$ is an $SO(n)$-equivariant diffeomorphism. Therefore, since any $SO(n)$-invariant metric on $\mathbb{S}^{n-1}$ is a scalar multiple of the round metric (Lemma 3.4), we know that

$$\widehat{g}|_{\{t\} \times \mathbb{S}^{n-1}} = f(t)^2 (F_t)^* \mathring{g},$$

where $\widehat{g}|_{\{t\} \times \mathbb{S}^{n-1}}$ is the metric on $\{t\} \times \mathbb{S}^{n-1}$ induced by $\widehat{g}$, $f(t) > 0$ is some real number, and $\mathring{g}$ is the round metric on $\mathbb{S}^{n-1}$. Let $f : I \to \mathbb{R}$ be the function given by $t \mapsto f(t)$. Let $\alpha : I \to I \times \mathbb{S}^{n-1}$ be the smooth curve given by $t \mapsto (t, e_1)$. Fix a unit vector $v \in T_{e_1}\mathbb{S}^{n-1}$ with respect to the round metric $\mathring{g}$, and let $V$ be the smooth vector field along $\alpha$ given by $V(t) = (0, v)$. Then

$$\widehat{g}_{\alpha(t)}(V(t), V(t)) = \widehat{g}_{(t, e_1)}((0, v), (0, v)) = f(t)^2 \mathring{g}_{e_1}(v, v) = f(t)^2 \qquad \forall t \in I,$$

so $f$ is smooth.

Finally, fix $(t, s) \in I \times \mathbb{S}^{n-1}$. Let $\gamma : I \to I \times \mathbb{S}^{n-1}$ be the smooth curve given by $t' \mapsto (t', s)$, and observe that $\gamma'(t) = (d/dt|_t, 0)$. In fact, $\gamma$ is a unit-speed geodesic with respect to $\widehat{g}$ because $\Phi \circ \gamma$ is a unit-speed geodesic with respect to $g$ by (ii) of Lemma 3.5. Fix an arbitrary vector $(c\, d/dt|_t, v)$ in $T_{(t,s)}(I \times \mathbb{S}^{n-1}) \cong T_t\mathbb{R} \oplus T_s\mathbb{S}^{n-1}$, where $c \in \mathbb{R}$. Then by (iii) of Lemma 3.5, we find

$$\begin{aligned}\widehat{g}_{(t,s)}\left(\left(c\tfrac{d}{dt}\big|_t, v\right), \left(c\tfrac{d}{dt}\big|_t, v\right)\right) &= c^2 \widehat{g}_{(t,s)}\left(\left(\tfrac{d}{dt}\big|_t, 0\right), \left(\tfrac{d}{dt}\big|_t, 0\right)\right) + \widehat{g}_{(t,s)}\left((0, v), (0, v)\right) \\ &= c^2 + f(t)^2 \mathring{g}_s(v, v).\end{aligned}$$

Thus, $\widehat{g} = dt^2 + f^2 \mathring{g}$, as desired. □

## 3.2 The Einstein condition on $I \times_f \mathbb{S}^{n-1}$

The aim of this subsection is to prove the following proposition:

**Proposition 3.6.** *Let $I$ be an open interval, and let $f : I \to \mathbb{R}$ be a positive smooth function. Consider the warped product $I \times_f \mathbb{S}^{n-1}$ with warped product metric $g := dt^2 + f^2 \mathring{g}$. Suppose $g$ is Einstein. Then $g$ has constant sectional curvature.*

We proceed by studying the curvature formulae of $I \times_f \mathbb{S}^{n-1}$. Let us first fix some notation: let $\partial_t$ denote the smooth vector field on $I \times_f \mathbb{S}^{n-1}$ given by $\partial_t|_{(t,s)} := (d/dt|_t, 0)$. For each $X \in \mathfrak{X}(\mathbb{S}^{n-1})$, let $X^L$ denote the smooth vector field on $I \times_f \mathbb{S}^{n-1}$ given by $X^L|_{(t,s)} := (0, X_s)$. Let $R$, Rm, Ric, and sec denote the (1,3)-Riemann curvature of $g$, the (0,4)-Riemann curvature of $g$, the Ricci curvature of $g$, and the sectional curvature of $g$, respectively. Let $t : I \times \mathbb{S}^{n-1} \to I$ denote the usual projection map. If $h : I \to \mathbb{R}$ is a smooth function, let us write $h(t) := h \circ t$.

**Lemma 3.7.** *Fix $X, Y, Z \in \mathfrak{X}(\mathbb{S}^{n-1})$. The (1,3)-Riemann curvature of $I \times_f \mathbb{S}^{n-1}$ is given by*

$$\begin{aligned} R(\partial_t, X^L)\partial_t &= \frac{f''(t)}{f(t)} X^L, \\ R(\partial_t, X^L)Y^L &= -\frac{f''(t)}{f(t)} g(X^L, Y^L)\partial_t, \\ R(X^L, Y^L)\partial_t &= 0, \\ R(X^L, Y^L)Z^L &= \frac{1-(f'(t))^2}{f(t)^2}\bigg( g(Y^L, Z^L)X^L - g(X^L, Z^L)Y^L \bigg). \end{aligned}$$

*Proof.* See [Che17, Proposition 3.2] or [Pet16, Section 4.2.3]. □

**Lemma 3.8.** *Fix $X, Y \in \mathfrak{X}(\mathbb{S}^{n-1})$. The Ricci curvature of $I \times_f \mathbb{S}^{n-1}$ is given by*

$$\begin{aligned} \mathrm{Ric}(\partial_t, \partial_t) &= -(n-1)\frac{f''(t)}{f(t)}, \\ \mathrm{Ric}(\partial_t, X^L) &= 0, \\ \mathrm{Ric}(X^L, Y^L) &= \bigg( -\frac{f''(t)}{f(t)} + (n-2)\frac{1-(f'(t))^2}{f(t)^2}\bigg) g(X^L, Y^L). \end{aligned}$$

*Proof.* See [Che17, Proposition 3.3], [Pet16, Section 4.2.3], or [Bes87, Proposition 9.106]. □

*Proof of Proposition 3.6.* Fix a point $(t,s) \in I \times_f \mathbb{S}^{n-1}$. Recall that the sectional curvature at $(t,s)$ is given by

$$\sec(v,w) := \frac{\mathrm{Rm}(v,w,w,v)}{g(v,v)g(w,w) - g(v,w)^2},$$

where $v, w \in T_{(t,s)}(I \times \mathbb{S}^{n-1}) \cong T_t\mathbb{R} \oplus T_s\mathbb{S}^{n-1}$ are linearly independent vectors. Write

$$v = a\partial_t|_{(t,s)} + x^L, \qquad \text{and} \qquad w = b\partial_t|_{(t,s)} + y^L,$$

where $a, b \in \mathbb{R}$, $x, y \in T_s\mathbb{S}^{n-1}$, $x^L := (0,x)$, and $y^L := (0,y)$. Then, by a long but straightforward computation using the formulae for $R$ (Lemma 3.7), we find that

$$\begin{aligned} \mathrm{Rm}(v,w,w,v) = &-\frac{f''(t)}{f(t)}\bigg( a^2 g(y^L, y^L) + b^2 g(x^L, x^L) - 2ab\, g(x^L, y^L)\bigg) \\ &+ \frac{1-(f'(t))^2}{f(t)^2}\bigg( g(x^L, x^L)g(y^L, y^L) - g(x^L, y^L)^2\bigg). \end{aligned}$$

On the other hand,

$$\begin{aligned} g(v,v)g(w,w) - g(v,w)^2 = a^2 g(y^L, y^L) &+ b^2 g(x^L, x^L) - 2ab\, g(x^L, y^L) \\ &+ g(x^L, x^L)g(y^L, y^L) - g(x^L, y^L)^2. \end{aligned}$$

Thus, we are done if we show that $c = -\frac{f''}{f} = \frac{1-(f')^2}{f^2}$ for some constant $c \in \mathbb{R}$.

Since $g$ is Einstein, we know that $\mathrm{Ric} = \lambda g$ for some constant $\lambda \in \mathbb{R}$. The formulae for the Ricci curvature (Lemma 3.8) imply that

$$\lambda = -(n-1)\frac{f''}{f} = -\frac{f''}{f} + (n-2)\frac{1-(f')^2}{f^2}.$$

Rearranging shows that

$$\frac{\lambda}{n-1} = -\frac{f''}{f} = \frac{1-(f')^2}{f^2}.$$

This completes the proof. □

### 3.3 Proof of the main theorem

We are now ready to prove Theorem 1.1. We restate the theorem for the reader's convenience:

**Theorem 1.1.** *Fix $n \geq 2$, and let $g$ be a complete $SO(n)$-invariant Einstein metric on $M = \mathbb{R}^n$ or $M = \mathbb{S}^n$. Then, up to scaling, $(M, g)$ is isometric to one of the three model spaces.*

*Proof.* By assumption, we know that $(M, g)$ is complete and simply-connected. Thus, by the Killing-Hopf Theorem (Proposition 2.7), it remains to show that $(M, g)$ has constant sectional curvature. Since Einstein implies constant sectional curvature in dimension 2 and 3 (Proposition 2.8), we can assume that $n \geq 4$. Then by Proposition 3.1, there exists a dense open subset $U$ of $(M, g)$ such that $(U, g)$ is isometric to an Einstein warped product of the form $I \times_f \mathbb{S}^{n-1}$. On $I \times_f \mathbb{S}^{n-1}$, Einstein implies constant sectional curvature (Proposition 3.6). Thus, $g$ has constant sectional curvature $c \in \mathbb{R}$ on $U$. Proposition 2.2 implies that on $U$, we can write

$$\mathrm{Rm}_g = \frac{1}{2} c\, g \owedge g.$$

Now, since $\mathrm{Rm}_g$ and $\frac{1}{2}c\, g \owedge g$ are (in particular) continuous maps $M \to T^{(0,4)}TM$ between Hausdorff topological spaces which agree on a dense subset, they must be equal. Therefore, Proposition 2.2 implies that $g$ has constant sectional curvature on $M$. ☐

## References


[B$\ddot{9}$8] Christoph Böhm. Inhomogeneous Einstein metrics on low-dimensional spheres and other low-dimensional spaces. *Invent. Math.*, 134(1):145–176, 1998.

[Bes87] Arthur L. Besse. *Einstein manifolds*, volume 10 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1987.

[Che17] Bang-Yen Chen. *Differential geometry of warped product manifolds and submanifolds*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017. With a foreword by Leopold Verstraelen.

[Jen73] Gary R. Jensen. Einstein metrics on principal fibre bundles. *J. Differential Geometry*, 8:599–614, 1973.

[Lau09] Jorge Lauret. Einstein solvmanifolds and nilsolitons. In *New developments in Lie theory and geometry*, volume 491 of *Contemp. Math.*, pages 1–35. Amer. Math. Soc., Providence, RI, 2009.

[Lee18] John M. Lee. *Introduction to Riemannian manifolds*, volume 176 of *Graduate Texts in Mathematics*. Springer, Cham, 2018. Second edition of [ MR1468735].

[Pet16] Peter Petersen. *Riemannian geometry*, volume 171 of *Graduate Texts in Mathematics*. Springer, Cham, third edition, 2016.