Simplifying the first-order incidence axioms of plane hyperbolic geometry

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Abstract

In recent work, J. Bamberg and T. Penttila (2023) give a new system of nine first-order axioms for plane hyperbolic geometry by introducing a perpendicularity relation in terms of point-line incidence alone. Their ninth axiom (A9) is required for this relation to be well-defined and is the subject of this investigation. It is shown that two simpler assumptions are equivalent to (A9) given the first seven axioms. From this simplification, a computational method is presented for probing the dependence of (A9) using the computer algebra system Magma. The method is shown to be inconclusive, but other computational strategies may be possible.

Contents

1 Introduction

Since the first axiomatisation of plane hyperbolic geometry was given by David Hilbert in 1903 [5], significant progress has been made in simplifying the foundations of the subject. Notably, where Hilbert's system includes relations of betweenness on points and congruence on line segments along with point-line incidence, in 1938 Karl Menger [6] showed that the incidence relation alone is sufficient.[1] However, it is only since as recently as 1992 that a simple axiom system for the hyperbolic plane has been known. In that year, Helen Skala [8]. presented the first first-order axiom system for hyperbolic geometry in terms of incidence alone, requiring nine axioms.

Since Skala's findings, no further progress has been made towards giving a short list of simple incidence axioms for the hyperbolic plane in the same spirit as those for other incidence geometries – that is, until last year. In 2023, Bamberg & Penttila [2] put forward a new axiom system which replaces the last two of Skala's axioms with two new incidence axioms using fewer points and lines. This was accomplished by developing a new notion of perpendicularity in terms of incidence alone which satisfies Bachmann's axioms for a metric plane [1]. For the new perpendicularity relation to be well-defined, Bamberg & Penttila rely upon a somewhat unwieldy assumption (A9) which is the subject of this investigation.

It is the hope of Bamberg & Penttila, and the author of this report, that the assumption (A9) can be recovered from the previous eight axioms and hence proved redundant. To this end, building upon the results of [2], it is proven here that two simpler assumptions are equivalent to (A9) given the first seven axioms. This demonstrates the viability of constructive methods in future research. Furthermore, from this simplification, a computational method is presented for testing whether (A9) is in fact a dependency of the previous axioms using the computer algebra system Magma. This method is shown to be inconclusive without additional algebraic assumptions, but other computational approaches are possible.

1.1 Statement of Authorship

- Any ideas presented in sections labelled as 'preliminaries' are not original.
- Bamberg expected that (A9) and (A10) (i.e. Lemma 4.1 of [2]), however the mathematical argument to this end and the resulting simplification presented in Section 3.2 are original.
- The idea of encoding a geometric construction of the premise of (A9) in terms of the action of half-turns upon a rimpoint and attempting to leverage a computer algebra system to recover (A9) is due to Bamberg. The specific method presented is original.

[1] For a historical survey of the axiomatic development of the subject, see Greenberg, 2010 [4].

2 First-order incidence axioms for the hyperbolic plane

2.1 Preliminaries

Before beginning our investigation of Bamberg & Penttila's ninth axiom, we make a survey and comparison of first-order incidence axiomatisations for the hyperbolic plane.

It will be useful to have an example of a hyperbolic plane on hand. The easiest to imagine or draw is the Cayley-Klein disc model. Take an ellipse in an affine plane over a Euclidean field[2]. The points of our hyperbolic plane are the points within the ellipse and the lines of our hyperbolic plane are the interior line segments of secants to the ellipse. The points on the ellipse are called rimpoints, and though they are not in our geometry in the strict sense. Such a set of rimpoints exists for every hyperbolic plane.

The axioms presented are first-order in the sense that they do not quantify over predicates or functions. Note that the relation of parallelism and the set of rimpoints used in these axioms may be defined in terms of point-line incidence alone. We omit these definitions as they are not needed for our purposes and refer the curious reader to Section 1 of [8], Section 3.4.2 of [7] and Section 1 of [2], each of which offer an explanation and historical references.

The following *seven basic axioms* are held in common between Skala's and Bamberg & Penttila's systems and are credited to Menger and his students by Skala [[8], p. 256] – herself a student of Menger. It is yet to be shown whether these axioms alone form an axiomatisation for plane hyperbolic geometry.

(A1) *Every pair of distinct points is incident with a unique line.*

(A2) *Every line is incident with a point.*

(A3) *There are three non-collinear points and three collinear points.*

(A4) *Given three collinear points, one of these points is such that every line incident with it meets at least one of each pair of intersecting lines on the other two points.*

(A5) *If a point P is not on a line l, then there exist two distinct lines on P parallel to l.*

(A6) *Any two non-collinear rays have a common parallel line.*

(A7) *If A, B, C, D, E and F are six distinct rimpoints such that the three diagonal points $AB \cap DE$, $BC \cap EF$ and $CD \cap FA$ of the hexagon $ABCDEF$ exist, then these points are collinear.*

Note that (A7) is Pascal's hexagon theorem where the hexagon's vertices are rimpoints.

Due to Skala (cf. Theorem 3 of [8]), the seven basic axioms plus the projective forms of Pappus' and Desargues' theorems for the hyperbolic plane given below provide an axiomatisation for plane hyperbolic geometry. See Figure 4 in Appendix 1 for diagrams.

[2] A Euclidean field is an ordered field F such that for all $x \in F$ greater than 0 (the additive identity of F), there exists $y \in F$ such that $x = y^2$. The easiest example to state is the field of real numbers $\mathbb{R}$. That the field is Euclidean is required for the standard hyperbolic metric to be defined.

(PAPPUS) *If a and b are lines containing points A_1, A_2 and A_3, and B_1, B_2 and B_3 respectively, and the points $A_1B_2 \cap A_2B_1$, $A_1B_3 \cap A_3B_1$ and $A_2B_3 \cap A_3B_2$ exist, then these three points are collinear.*

(DESARGUES) *If a, b and c are concurrent lines containing points $\{A_1, A_2\}$, $\{B_1, B_2\}$ and $\{C_1, C_2\}$ respectively, and the points $A_1B_1 \cap A_2B_2$, $A_1C_1 \cap A_2C_2$ and $B_1C_1 \cap B_2C_2$ exist, then these three points are collinear.*

Due to Bamberg & Penttila (cf. Theorem 1.2 of [2]), the seven basic axioms plus the following two axioms also provide an axiomatisation for plane hyperbolic geometry. See Figure 5 in Appendix 1 for diagrams. (A8) gives rise to midpoints and (A9) allows for a new perpendicularity relation to be well-defined.[3]

(A8) *Every pair of distinct points P and Q is such that there exist two lines l and m on P and Q respectively such that a pair of mutual parallels to l and m meet on the line PQ.*

(A9) *If l is a line and X is a rimpoint not on l, there exists a unique point P on l such that for all points Q on l different to P, a pair of mutual parallels to l and $X^{PQ}X^{QP}$ meet on the line XP.*

In (A9), following [2], we have employed a shorthand where for some (interior) points A and B, and a rimpoint X, 'X^A' denotes the other rimpoint on the line AX, and $X^{AB} \coloneqq (X^A)^B$.

Finally, it is shown in Theorem 3.12 that (A9) may be simplified by either removing the word 'unique' to produce a statement we will call (A9'), or by removing the assumption about the existence of a point P altogether for the following statement:[4]

(A10) *If P, Q and R are distinct points on a line l and X is a rimpoint not on l, then X, $l \bowtie X^{PQ}X^{QP}$ and $l \bowtie X^{PR}X^{RP}$ are collinear.*

Here, again following [2], $l \bowtie m$ is a shorthand denoting the meet of a pair of mutual parallels to the lines l and m, which for non-intersecting lines must exist from the basic axioms (see Lemma 3.4 [2]).

2.2 Comparative complexity

We now compare the axioms introduced.

First, consider the axioms of Skala and Bamberg & Penttila. It is immediately clear that Bamberg & Penttila's system is a significant constructive simplification over Skala's in that the axioms require fewer points and lines. However, both (A8) and (A9) are of higher-quantifier complexity, supposing the existence of points and lines which satisfy certain conditions. (A9) is particularly complex in this regard.

Now, we consider the alternative axioms (A9') and (A10) which are shown to be equivalent to (A9) in this report. (A10) has the lowest quantifier complexity and is logically minimal in that it does not suppose the existence of a point of intersection which must exist by Lemma 3.11. Still, the construction of (A9') is roughly

[3] The details of the definition of this relation are not important for this report. It suffices for us to say that in the statement of (A9), XP and l are perpendicular. The curious reader is referred to Section 4 of [2].

[4] Note that this statement appears as Lemma 4.1 of [2] where it is proven with (A9). We prove in this report that this lemma also implies (A9).

half the size of (A10) when one takes into account the number of lines needed to construct an additional point of the form $l \bowtie X^{AB}X^{BA}$.

3 Simplification of (A9)

This section is a focused mathematical argument culminating in the proof of Theorem 3.12 which gives equivalent assumptions for the perpendicularity relation introduced above to be well-defined – namely (A9') and (A10) – therefore simplifying the ninth axiom of Bamberg & Penttila's system, as discussed in Section 2.2. The argument builds directly upon the results of [2] and makes use of the same mathematical tools defined briefly below. The reader is encouraged to read [2] for a more thorough treatment and historical references.

3.1 Preliminaries

Let $\mathcal{I}$ be a model of the seven basic axioms (see Section 2.1) and let $\mathcal{O}$ be the set of rimpoints associated with $\mathcal{I}$. Let $\mathcal{C}$ be a *cyclic ordering* on the rimpoints $\mathcal{O}$.

Definition 3.1. A ternary relation $\mathcal{C}$ on a set Ω is a cyclic ordering if for all $a, b, c, d \in \Omega$:

(C1) $\mathcal{C}(a, b, c) \Rightarrow a \neq b \neq c \neq a$;

(C2) $a \neq b \neq c \neq a \Rightarrow$ either $\mathcal{C}(a, b, c)$ or $\mathcal{C}(a, c, b)$;

(C3) $\mathcal{C}(a, b, c) \Rightarrow \mathcal{C}(b, c, a)$

(C4) $\mathcal{C}(a, b, c)$ and $\mathcal{C}(b, c, d) \Rightarrow \mathcal{C}(a, b, d)$.

Definition 3.2. Given four rimpoints $A, B, C, D \in \mathcal{O}$, the pair (A, B) *separates* the pair (B, C) and we write '$AB//CD$', if and only if A, B, C and D are distinct and the lines AB and CD meet in $\mathcal{I}$.

The following theorem gives a rule associating the two relations $\mathcal{C}$ and $//$.

Theorem 3.3. ([2]) *If* $A, B, C, D \in \mathcal{O}$, *then*

$$AB//CD \iff (\mathcal{C}(A, B, C) \wedge \mathcal{C}(C, D, A)) \vee (\mathcal{C}(A, D, C) \wedge \mathcal{C}(C, B, A)).$$

Proof. The rule is given for a general 'separation relation' $//$ and a cyclic ordering $\mathcal{C}$ on an arbitrary non-empty set in Section 2 of [2]. By Theorem 3.5 of [2], the relation defined in Definition 3.2 on $\mathcal{O}$ of $\mathcal{I}$ is such a 'separation relation'. □

Definition 3.4. Let l be a line with rimpoints L and L', and let A and B be finite points or rimpoints not on l. Then A and B *lie on the same side of* l if one of the following holds:

(L1) $A = B$;

(L2) A and B are rimpoints and $AB \not\parallel LL'$ (that is, 'AB does not meet LL'');

(L3) A and B are finite points and the rimpoints $\bar{A}$ and $\bar{B}$ of the line AB are such that $\bar{A}\bar{B} \not\parallel LL'$;

(L4) A is a finite point and B is a rimpoint, and (L2) holds for B and L^A (i.e. $BL^A \not\parallel LL'$).

(L5) A is a rimpoint and B is a finite point, and (L4) holds with the points A and B exchanged.

Note that the definition above has been slightly modified from that given in [2] for the sake of brevity and for the reader to exercise their understanding of the concepts introduced so far.

Lemma 3.5. *If l is a line with rimpoints L and L', and A and B are rimpoints not on l, then A and B lie on the same side of l if and only if either $\mathcal{C}(L,A,L')$ and $\mathcal{C}(L,B,L')$, or $\mathcal{C}(L',A,L)$ and $\mathcal{C}(L',B,L)$.*

Proof. This follows directly from (C2), (L2) and Theorem 3.3. □

Remark 3.6. The relation 'lie on the same side of l' defined in Definition 3.4 is an equivalence relation on the set of finite points and rimpoints not on l, with the equivalence classes being the sides of l.

Definition 3.7. A *half-turn* about a point P is a fixed-point free permutation P of the rimpoints $\mathcal{O}$ where for all $X \in \mathcal{O}$, if the other rimpoint of the line XP is Y, then $X^P = Y$.[5]

Lemma 3.8. (Lemma 3.2 of [2]) *A half-turn about a point P on a line l interchanges the sides of l.*

Lemma 3.9. ([2]) *Let P and Q be distinct points on a line l and consider the composition PQ of a half-turn about P followed by a half-turn about Q. If R is a point on l then there is a unique point S on l such that $RS = PQ$.*

Proof. This is a direct consequence of Lemmas 3.18 and 3.19 of [2]. Note that the proof given there does not make use of (A8). □

3.2 Argument

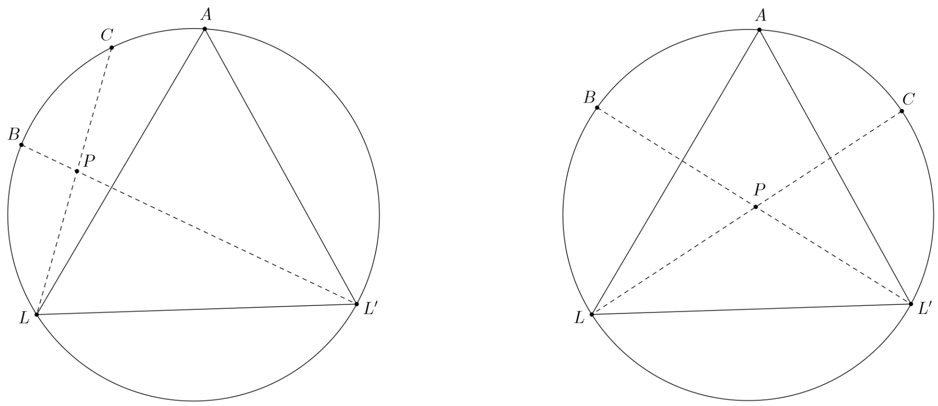
Using the concepts introduced up to this point, and building upon the results given in [2], we offer the following argument which culminates in a simplification of (A9) in Theorem 3.12.

Recall that our argument takes place within $\mathcal{I}$ where we only assume the seven basic axioms hold and we have equipped the set of rimpoints $\mathcal{O}$ with a cyclic ordering $\mathcal{C}$.

Theorem 3.10. *Let A, B and C be distinct rimpoints lying on the same side of a line l with rimpoints L and L' such that $\mathcal{C}(L,A,L')$ and $\mathcal{C}(L,B,C)$. If P is the meet of a pair of mutual parallels to l and BC (that is, $P := l \bowtie BC$), then AP meets l if and only if $\mathcal{C}(B,A,C)$.*

Proof. ***Set-up.*** By hypothesis, A and C are on the same side of l, and $\mathcal{C}(L,A,L')$, so $\mathcal{C}(L,C,L')$ by Lemma 3.5. Then $\mathcal{C}(C,L,L')$ by (C2) of Definition 3.1. Also by hypothesis, $\mathcal{C}(L,B,C)$, and together with $\mathcal{C}(C,L,L')$ we have $LC//L'B$ by Theorem 3.3.

[5] For clarity, 'X under the action of the half-turn about P' is written as 'X under the action of P': X^P.



(a) $\mathcal{C}(B, C, A)$ (b) $\mathcal{C}(B, A, C)$

Figure 1: Two cases for Theorem 3.10.

Thus, $\{LC, L'B\}$ is the pair of mutual parallels to l and BC which meet (cf. Lemma 3.4 of [2]). Let P be their point of intersection, and let $D \coloneqq A^P$. Recall from Definition 3.2 that the statement 'AP meets l' may be written '$AD//LL'$'.

Proof of $\Leftarrow$**.** We prove the contrapositive – see Figure 1(a). Suppose AP does not meet l – i.e. $AD \nparallel LL'$. Then either D is a rimpoint of l, or D and A are on the same side of l.

Suppose $D = L$. Then $L^P = A$. But by definition, P is on LC, and so $A = L^P = C$ – a contradiction, since A and C are distinct rimpoints. Thus, $D \neq L$, and by the same reasoning, $D \neq L'$. Hence, D and A are on the same side of l – i.e. $\mathcal{C}(L, D, L')$.

Additionally, $D \neq A$, since a half-turn is a fixed-point free permutation of the rimpoints (see Definition 3.7).

Hence, by (C2) in Definition 3.1, either $\mathcal{C}(L, A, D)$, or $\mathcal{C}(L, D, A)$. Without loss of generality, suppose $\mathcal{C}(L, D, A)$.

Towards a contradiction, suppose $\mathcal{C}(B, D, A)$. We wish to prove that $AD \nparallel L'B$ and that a contradiction follows. Well, $AD//L'B$ would imply $(\mathcal{C}(A, L', D) \wedge \mathcal{C}(D, B, A)) \vee (\mathcal{C}(A, B, D) \wedge \mathcal{C}(D, L', A))$ by Theorem 3.3. The first case is immediately ruled out since $\mathcal{C}(B, D, A) \overset{\text{(C3)}}{\Longrightarrow} \mathcal{C}(D, A, B) \overset{\text{(C2)}}{\Longrightarrow} \neg\mathcal{C}(D, B, A)$. In the second case, recall we have assumed $\mathcal{C}(L, D, A) \overset{\text{(C3)}}{\Longrightarrow} \mathcal{C}(A, L, D)$, and we have also shown $\mathcal{C}(L, D, L')$. Together, $\mathcal{C}(A, L, D) \wedge \mathcal{C}(L, D, L') \overset{\text{(C4)}}{\Longrightarrow} \mathcal{C}(A, L, L') \overset{\text{(C3)}}{\Longrightarrow} \mathcal{C}(L', A, L)$, which contradicts our hypothesis that $\mathcal{C}(L, A, L')$, and so the second case fails. Thus, $AD \nparallel L'B$. This entails a contradiction since $D = A^P$ and P is on the line $L'B$, so the half-turn about P exchanges the sides of $L'B$ (by Lemma 3.8), placing A and D on opposite sides of $L'B$.

Thus, we have shown $\neg\mathcal{C}(B, D, A) \overset{\text{(C2)}}{\Longrightarrow} \mathcal{C}(B, A, D)$.

There are now exactly two possible cyclic orderings for the six rimpoints, corresponding to the following lists (the cyclic permutations of each being identified): (L, D, B, A, C, L') or (L, D, B, C, A, L').

Suppose $\mathcal{C}(B, A, C)$ – that is, assume the first ordering holds. Then by an argument symmetric to that above (where $\mathcal{C}(B, D, A)$ was assumed), we must have $DA \nparallel LC$, which again entails a contradiction by the same reasoning.

Thus, $\neg\mathcal{C}(B, A, C)$ and so the contrapositive holds.

Proof of $\Rightarrow$. See Figure 1(b). Suppose AD meets l – i.e. $AD//LL'$. Then A and D are on opposite sides of l, and equally L and L' are on opposite sides of AD. Then since $\mathcal{C}(L, A, L')$, by two applications of Lemma 3.5 we have also $\mathcal{C}(L', D, L)$, $\mathcal{C}(A, L', D)$ and $\mathcal{C}(D, L, A)$.

Towards a contradiction, suppose $\mathcal{C}(A, D, C)$. We wish to prove that $AD \not\parallel LC$ and that a contradiction follows. Well, as before, $AD//LC$ would imply $(\mathcal{C}(A, L, D) \wedge \mathcal{C}(D, C, A)) \vee (\mathcal{C}(A, C, D) \wedge \mathcal{C}(D, L, A))$ by Theorem 3.3. The second case also fails immediately since $\mathcal{C}(A, D, C) \xRightarrow{(C2)} \neg\mathcal{C}(A, C, D)$. The first case also fails since we have shown $\mathcal{C}(D, L, A) \xRightarrow{\text{(C3)}} \mathcal{C}(A, D, L) \xRightarrow{\text{(C2)}} \neg\mathcal{C}(A, L, D)$. Thus, $AD \not\parallel LC$ – a contradiction since $D = A^P$ and P is on LC.

Then there are exactly two possible cyclic orderings for the six rimpoints, corresponding to the following lists (the cyclic permutations of each being identified): (L, B, A, C, L', D) or (L, A, B, C, L', D).

Suppose $\mathcal{C}(A, C, D)$ – that is, assume the first ordering holds. By an argument symmetric to that above, we must have that $AD \not\parallel L'B$, which again entails a contradiction by the same reasoning.

Thus, the second ordering must hold and so $\mathcal{C}(B, A, C)$. □

Lemma 3.11. *Let P and Q be distinct finite points on a line l and let X be a rimpoint not on l. If R is the meet of a pair of mutual parallels to l and $X^{PQ}X^{QP}$ (i.e. $R \coloneqq l \bowtie X^{PQ}X^{QP}$), then XR meets l.*

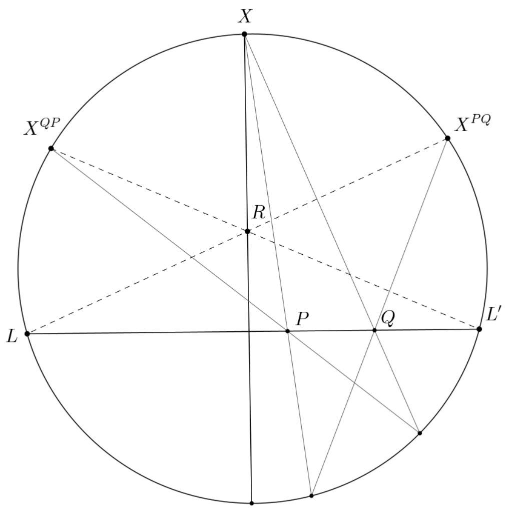


Figure 2: Lemma 3.11 in a Cayley-Klein hyperbolic plane.

Proof. See Figure 2 above. Let the rimpoints of l be L and L' such that $\mathcal{C}(L, X, L')$. Since P and Q are on l, a half-turn about P or Q exchanges the sides of l (by Lemma 3.8). Thus, X, X^{PQ} and X^{QP} are on the same side of l – i.e. $\mathcal{C}(L, X^{PQ}, L')$ and $\mathcal{C}(L, X^{QP}, L')$.

Now consider the line XP and the point Q. The two lines parallel to XP and through Q are XX^Q and X^PX^{PQ}. Thus, Q, X^Q and X^{PQ} lie on the same side of XP (recall (iv) of definition of 'same side of line'). Thus, X^{QP} is on the opposite side of XP as X^{Q} and X^{PQ}, since the half-turn about P exchanges the sides of XP (again by Lemma 3.8). Consequently, $X^{PQ}X^{QP}//XP$ and we have either $\mathcal{C}(X^{PQ}, X, X^{QP})$ or $\mathcal{C}(X^{QP}, X, X^{PQ})$.

Now, if P and Q are such that $\mathcal{C}(L, X^{PQ}, X^{QP})$, then $\mathcal{C}(X^{PQ}, L, X^{QP})$. Then by Theorem 3.10, XR meets l. Otherwise, P and Q are such that $\mathcal{C}(L, X^{QP}, X^{PQ})$, and thus $\mathcal{C}(X^{QP}, L, X^{PQ})$. Then again by Theorem 3.10, XR meets l. □

Recall that (A9) assumes the existence of an intersection point which has now been proven to exist from the first seven axioms alone. Hence, we have the following result.

Theorem 3.12. *Let l be a line and X a rimpoint not on l. The following are equivalent:*

(A9) *There exists a unique point P on l such that for all points Q on l different to P, $l \bowtie X^{PQ}X^{QP}$ lies on XP.*

(A9') *There exists a point P on l such that for all points Q on l different to P, $l \bowtie X^{PQ}X^{QP}$ lies on XP.*

(A10) *If P, Q and R are distinct points on l then X, $l \bowtie X^{PQ}X^{QP}$ and $l \bowtie X^{PR}X^{RP}$ are collinear.*

Proof. (A9) ⇒ (A10). A proof has already been given as the proof of Lemma 4.1 in [2]. For completeness' sake, I offer the following proof – the argument is equivelant.

Let P be the point on l stipulated by (A9). Then every point of the form $l \bowtie X^{AB}X^{BA}$ – where A and B are distinct points on l – lies on the line XP, since, making use of Lemma 3.9, $AB = PQ$ (and so $BA = QP$) for some point Q on l different to P. The points of (A10) whose collinearity is in question are of such a form and so all lie on the same line XP.

(A10) ⇒ (A9'). Let A and B be distinct points on l and let $C \coloneqq l \bowtie X^{AB}X^{BA}$. By Lemma 3.11, the line XC meets l in a point P, and by (A10), for all D on l different to A, $l \bowtie X^{AD}X^{DA}$ lies on $XC = XP$. Thus, this point P satisfies (A9') using the aforementioned property.

(A9') ⇒ (A9). Let P and P' be points on l stipulated by (A9'). That is: for all Q on l different to P, $l \bowtie X^{PQ}X^{QP}$ lies on XP, and; for all Q on l different to P', $l \bowtie X^{P'Q}X^{QP'}$ lies on XP'.

Let $R \coloneqq l \bowtie X^{P'Q}X^{QP'}$ for some Q on l different to P'. By Lemma 3.9, there is a unique Q' on l different to P such that $P'Q = PQ'$. Thus, $R = l \bowtie X^{PQ'}X^{Q'P}$ and lies on XP by the assumption on P. By the assumption on P', R also lies on XP'. Thus, the line $XP = XR = XP'$, and since P and P' both lie on l, we must have $P = P'$. □

4 A naive computational method to probe the dependence of (A9)

We present the following naive method to investigate the dependence of (A9) with Magma, a computer algebra system. To do so, we begin by constructing the premise of the statement A10 from Theorem 3.12. This construction is then codified into algebra by considering the action of half-turns upon a rimpoint. Note that this method is 'naive' in the sense that it knows nothing about (A8), even though (A8) may be necessary for (A9) to hold in $\mathcal{I}$.

4.1 Construction of (A10)

Recall (A10):

(A10) *If P, Q and R are distinct points on a line l and X is a rimpoint not on l, then X, $l \bowtie X^{PQ}X^{QP}$ and $l \bowtie X^{PR}X^{RP}$ are collinear.*

Consider the below construction of the premise of (A10). Figure 3 depicts this construction in a Cayley-Klein hyperbolic plane. Note that the points labelled R, S and V perform the roles of P, Q and R in the statement above, and that T and U perform the roles of $l \bowtie X^{PQ}X^{QP}$ and $l \bowtie X^{PR}X^{RP}$.

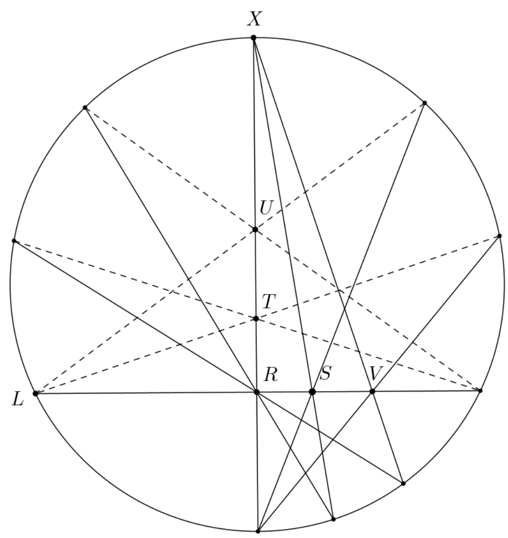


Figure 3: The construction 4.1 of (A10), represented in a Cayley-Klein hyperbolic plane.

Construction. *Let P and Q be distinct points and let l be the line joining them. Let X be a rimpoint not on l. Construct the point $T := l \bowtie X^{PQ}X^{QP}$ and let R be the meet of XT and l. Let S be a point on l different to R and construct the point $U := l \bowtie X^{RS}X^{SR}$. Note that U is on the line $XR = XT$ for all $S \neq R$ on l if and only if (A10) holds – we will call this collinearity property the 'testing conidition'.*

Now, eliminate one point by constructing T in terms of R. Let V be the unique point on l such that the composition of half-turns $PQ = RV$ (cf. Lemma 3.9). Thus, $T = l \bowtie X^{RV}X^{VR}$.

Finally, without loss of generality, suppose that S and V are on the same side of XR and take L to be the rimpoint on l on the opposite side of XR. That is, we have $L^T = X^{RV}$ and $L^U = X^{RS}$.

4.2 Towards an encoding

To encode the above construction algebraically, we seek to describe how the half-turns about the constructed points R, S, T, U and V act on the rimpoint X. We begin by enumerating equations which arise immediately from the construction.

First, by the definitions of T, U and L,

$$L = X^{RVT} = X^{VRT\star} \text{ and } L = X^{RSU} = X^{SRU\star}, \tag{1}$$

where $\star$ is a short-hand for any one of the points on l we have constructed: $\star \in \{R, S, V\}$. Also, by the definition of R, X is a rimpoint on the line RT, so $X^R = X^T$. Thus, from eq. (1) we have

$$L = X^{TVT} \text{ and } L = X^{TSU}. \tag{2}$$

Bringing together eqs. (1) and (2):

$$L = X^{RVT} = X^{TVT} = X^{RSU} = X^{TSU} = X^{VRT\star} = X^{SRU\star} \quad \text{where } \star \in \{R, S, V\}. \tag{3}$$

Since each half-turn is an involution, each of these equalities can be rearranged to give

$$X = L^{TVR} = L^{TVT} = L^{USR} = L^{UST} = L^{\star TRV} = L^{\star URS} \quad \text{where } \star \in \{R, S, V\}. \tag{4}$$

Now, to express the right-hand side (everything after the first = sign) of eq. (4) in terms of X, we substitute the expressions for L in terms of X of eq. (3) into eq. (4). This leaves a series of equalities of the form (where we have left the exponents unsimplified to illustrate a pattern):

$$X = X^{RVTTVR} = X^{RVTTVT} = \cdots = X^{TVTTVR} = X^{TVTTVT} = \cdots. \tag{5}$$

The exponents here are therefore elements of the group of half-turns which fix X, and so are elements of the stabiliser of X (under the action of the group of half-turns). The resulting exponents/elements in simplified form are given by the multiplication table below. That every element resulting from the table below fixes the rimpoint X may be verified with Figure 3.

$\cdot$	TVR	TVT	USR	UST	$\star TRV$	$\star URS$
RVT	1	RT	$RVTUSR$	$RVTUST$	$RVT\star TRV$	$RVT\star URS$
TVT	TR	1	$TVTUSR$	$TVTUST$	$TVT\star TRV$	$TVT\star URS$
RSU	$RSUTVR$	$RSUTVT$	1	RT	$RSU\star TRV$	$RSU\star URS$
TSU	$TSUTVR$	$TSUTVT$	TR	1	$TSU\star TRV$	$TSU\star URS$
$VRT\star$	$VRT\star TVR$	$VRT\star TVT$	$VRT\star USR$	$VRT\star UST$	1	$VRTURS$
$SRU\star$	$SRU\star TVR$	$SRU\star TVT$	$SRU\star USR$	$SRU\star UST$	$SRUTRV$	1

Table 1: The exponents in eq. (4) are listed down the first column and composed with the exponents in eq. (3), listed across the first row. Here again, $\star \in \{R, S, V\}$.

Notice that in Table 1, each word under the diagonal is the mirror-image of a corresponding word above the diagonal. Since half-turns are involutions, the elements of a mirror-image pair are inverse elements and so only one element in each pair is needed to generate the other. Then without loss, we may discard the elements below the diagonal.

We also include identities expressing the collinearity of R, S and V, which take the following form:

$$(ABC)^2 = 1, \tag{6}$$

where A, B and C are distinct elements of $\{R, S, V\}$. That these identities hold for three collinear points is a consequence of (A7) or 'Pascal's theorem on rimpoints' (cf. Lemma 3.7 of [2]).

Finally, we encode the testing condition. Recall that U is constructed from X, R and S and, as noted in Section 4.1, (A10) holds if and only if U is on the line $XR = XT$ for all $S \neq R$ on l. Well, U is on the line $XR = XT$ if and only if $X^R = X^T = X^U$, or equivalently, if and only if RU, UR, TU and UT are elements which fix X. Then we have the following, where G_X is the stabiliser of X under the action of the group of half-turns G:

$$\text{(A10) holds} \iff RU, UR, TU, UT \in G_X \text{ for all } S \neq R \text{ on } l. \tag{7}$$

4.3 The encoding formalised

We now formalise the identities we have produced.

Let G be a group with presentation $\langle\ r, s, t, u, v \mid R_1 \cup R_2\ \rangle$, where R_1 and R_2 are the following sets of relations among the generators of G:

$$R_1 = \{r^2 = s^2 = t^2 = u^2 = v^2 = 1\},$$

$$R_2 = \{(rsv)^2 = (rvs)^2 = (srv)^2 = 1\}.$$

With each of the generators of G we associate the corresponding point in our construction (r corresponds to R, etc.). We consider the action of G upon the set of rimpoints $\mathcal{O}$ with each generator acting upon $\mathcal{O}$ as the half-turn about its corresponding point. Here, R_1 expresses that the generators are involutions, and R_2 taken together with R_1 expresses the identities of (6) in the previous section.

Now, consider the subgroup H of G generated by the a of elements T, where we take T to be the set of elements which we know to fix the rimpoint $X \in \mathcal{O}$ – that is, the elements in Table 1 above its main diagonal (with $\star$ expanded into $\star = R$, $\star = S$ and $\star = V$):

$$\begin{aligned} T = \{&rt, rvtusr, rvtust, rvtrtrv, rvtstrv, rvtvtrv, rvtrurs, rvtsurs, rvtvurs,\\ &tvtusr, tvtust, tvtrtrv, tvtstrv, tvtvtrv, tvtrurs, tvtsurs, tvtvurs,\\ &rsurtrv, rsustrv, rsuvtrv, rsururs, rsusurs, rsuvurs,\\ &tsurtrv, tsustrv, tsuvtrv, tsururs, tsusurs, tsuvurs,\\ &vrturs\}.\end{aligned}$$

Then H is a subgroup of G_X (the stabiliser of X under the action of G).

The groups G and H above are then constructed in Magma, a computer algebra package. The code used is given in Appendix 2. The hope is that Magma can perform the manipulations necessary to say whether or not RU, UR, TU and/or UT are elements of H. If one of these elements is in H (we would expect then that they all are in H), then the element is in the stabiliser of X, the testing condition (7) holds, and (A9) holds and so is a redundant axiom. If, however, these elements are not in H, then we cannot say whether or not they are in the stabiliser of X, and the result is inconclusive.

4.4 Too big for Magma

Unfortunately, the groups G and H are too large for Magma to form closed coset tables and therefore answer questions about their structure. For instance, Magma replies in the affirmative to `R*U in G` and `R*T in H`, but `R*U in H` gives a runtime error: `Could not construct a closed coset table`. Likewise, asking Magma to give a permutation group isomorphic to our finitely-presented subgroup with the command `PermutationGroup(H)` gives the error, `Group too large`. The groups G and H are almost certainly infinite, but as large finitely-presented groups without obvious structure (e.g. they are not cyclic or Abelian), Magma does not possess the means to directly prove they are infinite with a command like `IsInfinite(G)` – here Magma could not answer either way within a practical length of time.

Hence, if one wishes to proceed using Magma, one must constrain the order of G, which requires imposing new relations upon the generators of G, and it is unclear what degree of constraint must be placed upon G for Magma to form a closed coset table of a given subgroup – likely the constraints required for our group are significant.

5 Discussion and conclusion

It is has been shown by constructive methods that Bamberg & Penttila's (A9) is equivalent to two simpler assumptions (see Theorem 3.12). This shows the viability of constructive approaches for further research.

Then, using the simplification shown, a geometric construction expressing the information contained in (A9) was encoded in terms of the action of half-turns on a rimpoint and this translated into Magma code. While this approach is unsuccessful due to the size of the group in question, it is noted that it has now been shown in principle that such an encoding can be made, and given additional constraints and/or an alternative algorithmic approach, the question may still be settled computationally.

That being so, further research need not resort to computational brute force. Promise lies in taking a group theoretic approach to better understand the group of half-turns in a model of the first eight axioms – however, the specifics of this line of inquiry lie beyond the scope of this report. If progress is made on the algebraic front and a dependence result for (A9) established, it will be interesting to see whether one can produce new corresponding relations for the group G in Section 4.3 and find the same dependence result using Magma and the method outlined.

As a final note, another line of simplificiation remains open in Skala's axiom system. For the projective plane, Pappus' theorem implies Desargues' theorem[6]. Victor Pambuccian noted ([7], p. 43) that no proof of the corresponding implication for Skala's (PAPPUS) and (DESARGUES) axioms has been given from the axioms of ordered incidence geometry (which includes hyperbolic geometry) – the author verified that this is presently still the case. Future research may look for constructive proofs of this implication, or attempt to analyse (PAPPUS) and (DESARGUES) computationally along the lines considered for (A9) in this report.

Thus, it remains to be shown whether or not (A9) may be removed from Bamberg & Penttila's system or (DESARGUES) may be removed from Skala's system (without loss, of course). Furthermore, since both systems have their first seven axioms in common, it is easy to suspect that these seven alone are needed.

6 Acknowledgements

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[6]The first proof of this, since deemed incomplete, was offered by Hessenberg in 1905. A completed proof of the so-called 'Hessenberg Theorem' was given by Cronheim [3]

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7 Appendices

7.1 Illustrations of Skala's and Bamberg & Penttila's axioms in a Cayley-Klein hyperbolic plane

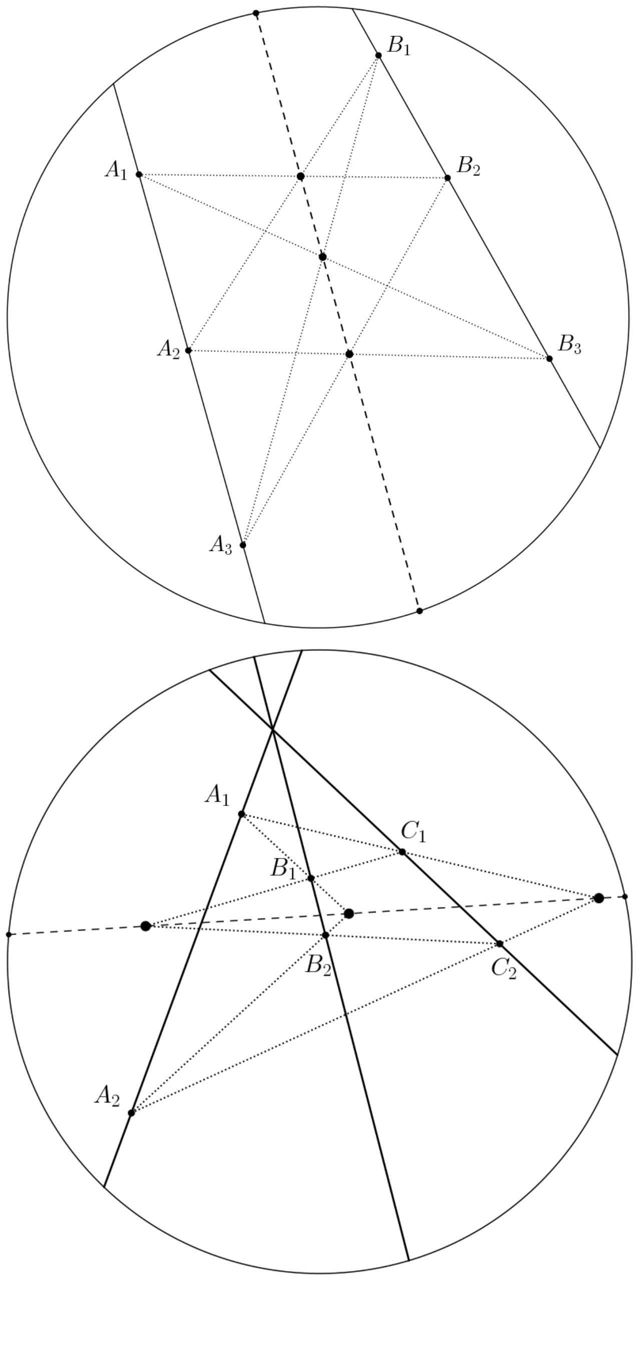

Figure 4: (PAPPUS) and (DESARGUES) in a Cayley-Klein hyperbolic plane.

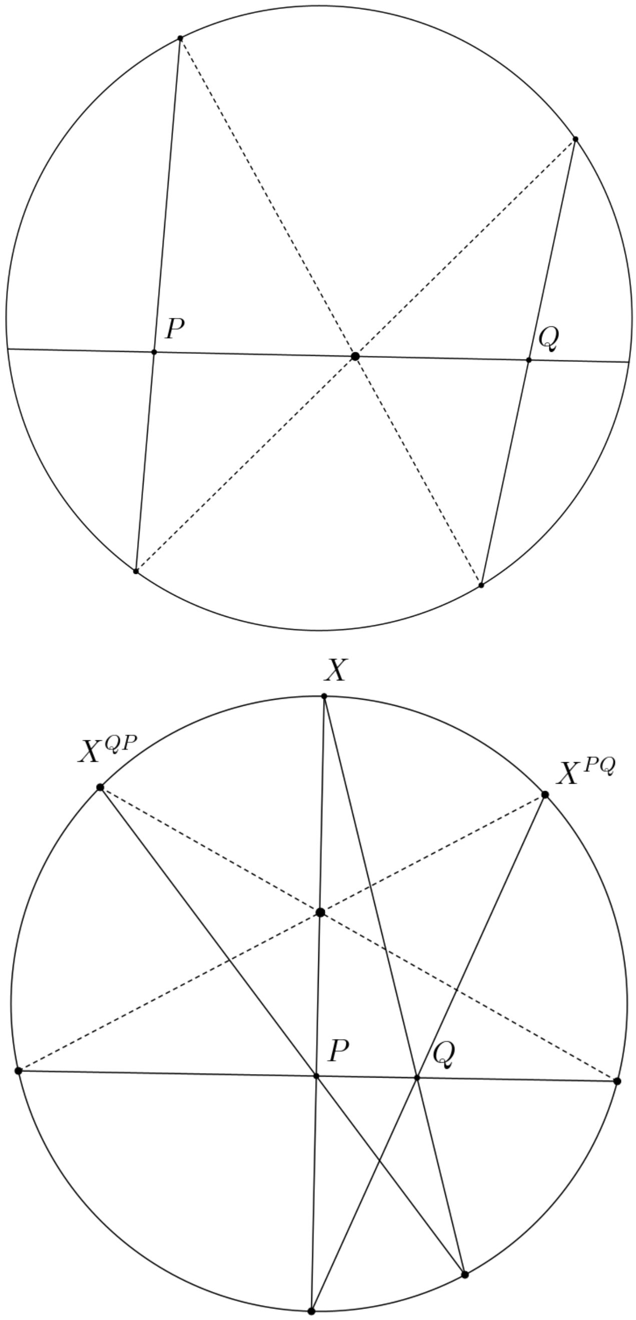

Figure 5: (A8) and (A9) in a Cayley-Klein hyperbolic plane.

7.2 Code for algebraic constructions in Magma

The finitely-presented group G and its subgroup H defined in Section 4.3 may be constructed in Magma with the following code.

```
> // G is finitely-presented group on 5 generators with relators from sets rels1 and rels2
> F< r, s, t, u, v > := FreeGroup(5);
> rels1 := { r^2, s^2, t^2, u^2, v^2 };
> rels2 := { (r*s*v)^2, (r*v*s)^2, (s*r*v)^2 };
> G< R, S, T, U, V > := quo< F | rels1 join rels2 >;
> //
> // H is subgroup of stabiliser of X (under G) generated by the elements of set T
> T := { r*t, r*v*t*u*s*r, r*v*t*u*s*t, r*v*t*r*t*r*v, r*v*t*s*t*r*v, r*v*t*v*t*r*v,
r*v*t*r*u*r*s, r*v*t*s*u*r*s, r*v*t*v*u*r*s,
> t*v*t*u*s*r, t*v*t*u*s*t, t*v*t*r*t*r*v, t*v*t*s*t*r*v, t*v*t*v*t*r*v, t*v*t*r*u*r*s,
t*v*t*s*u*r*s, t*v*t*v*u*r*s,
> r*s*u*r*t*r*v, r*s*u*s*t*r*v, r*s*u*v*t*r*v, r*s*u*r*u*r*s, r*s*u*s*u*r*s,
r*s*u*v*u*r*s,
> t*s*u*r*t*r*v,  t*s*u*s*t*r*v, t*s*u*v*t*r*v, t*s*u*r*u*r*s, t*s*u*s*u*r*s,
t*s*u*v*u*r*s,
> v*r*t*u*r*s };
> H := sub< G | T >;
```