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Cofreely adding Equality to Primary Fibrations

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Abstract

Primary doctrines provide a way to add structure presenting first order logic to an underlying category. An elementary doctrine is a primary doctrine that has a notion of equality. In [Pas12], Pasquali showed that there is a cofree construction adding elementary structure to primary doctrines. We can generalise to doctrines to fibrations to get a proof relevant presentation of first order logic. We show that a similar cofree construction adds elementary structure to primary fibrations.

Contents

Acknowledgements

Thank you to my supervisor, Richard Garner, for the extensive help with this project.

Statement of Authorship

The work presented in this report is largely a generalisation of the result presented in [Pas12]. As such this report draws heavily from the techniques presented in Pasquali's work. My supervisor, Richard Garner, highlighted where the differences would be between the proof of the result presented in [Pas12] and the result presented here. Additionally, he gave a sketch of the proof for the main result presented here.

1 Introduction

This report describes how equality can be handled in a proof relevant presentation of first order logic through category theory. The construction studied in this report was first explored by Pasquali in [Pas12], in which he showed that equality can be added cofreely to a presentation of first order logic through category theory, called a primary doctrine. We generalise the notion of a primary doctrine to a primary fibration in which the "witnesses" of proofs are part of the structure. We show that Pasquali's construction generalises naturally to this setting.

In section 2 we define primary doctrines and outline Pasquali's construction. In section 3 we define primary fibrations and generalise the notions of equality and equivalence relations. In section 4 we generalise Pasquali's construction to primary fibrations.

2 Background

2.1 Doctrines

A category can be regarded as a setting for some (restricted) mathematics to take place, by regarding our objects as sets and our morphisms as functions. We can further this view by adding extra structure to a category to describe possible predicates on each object in order to present a system of first order logic on our base category. Formally speaking

Definition 2.1. A *primary doctrine* is a product preserving contravariant functor $P : \mathcal{C}^{\mathrm{op}} \to \mathbf{ISL}$ from a category $\mathcal{C}$ with all finite products to the category of inf-semilattices [Pas12].

Example 2.2. The functor

$$\mathscr{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{ISL},$$

given by

$$\mathscr{P}(X) = \{x : x \subset X\},$$

and

$$\mathscr{P}(f) = f^{-1},$$

is a primary doctrine. For each set X we have an associated collection of sets $\mathscr{P}(X)$, which should be thought of as predicates on elements X. In particular if $q \in \mathscr{P}(X)$ then we say $q(y)$ is true for $y \in X$ if $y \in q$. There are many predicates over $\mathbb{R}$, one being $\mathbb{Q}$. The predicate $\mathbb{Q}(x)$ is true for $x \in \mathbb{R}$ exactly when x is rational.

Example 2.3. The functor

$$\mathscr{O} : \mathbf{Top}^{\mathrm{op}} \to \mathbf{ISL},$$

given by

$$\mathscr{O}((X, \mathcal{T})) = \mathcal{T},$$

and

$$\mathcal{O}(f) = f^{-1}$$

is a primary doctrine. We think about this in a similar way to last time: for every space X we have a collection of predicates on elements of X. However this time the predicates only exist with respect to the topology of our space X. If X is a discrete space then the predicates are the same as our last example. If X is indiscrete then there are two predicates, one which is always true and one which is always false. In the case of the usual topology on $\mathbb{R}$ there are many predicates, however there is no predicate that is true precisely when $x \in \mathbb{R}$ is rational, because $\mathbb{Q}$ is not open.

2.2 Equality in Doctrines

Definition 2.4. If $P : \mathcal{C}^{op} \to \mathbf{ISL}$ is a primary doctrine and A is an object in $\mathcal{C}$ then an *equality predicate* [Pas12][Jac98] over A is an object δ_A such that:

1. The functor $\Lambda : P(A) \to P(A \times A)$ given by $\Lambda(p) = \pi_1^* p \wedge \delta_A$ is a left adjoint to Δ_A^*
2. For every object X in $\mathcal{C}$, the functor $M_X : P(X \times A) \to P(X \times A \times A)$ given by $M_X(p) = \pi_{1,2}^* p \wedge \pi_{2,3}^* \delta_A$ is a left adjoint to $(\mathrm{Id}_X \times \Delta_A)^*$

To understand why we call δ_A an "equality" predicate lets first unpack condition 1 for the doctrine $\mathcal{P}$. As Λ is a left adjoint to Δ_A^*, for any two predicates $p \in P(A)$ and $q \in P(A \times A)$ we have $\Lambda(p) \leq q$ in $P(A \times A)$ iff $p \leq \Delta_A^*(q)$ in $P(A)$. Introducing variables we have that the first condition of the adjunction corresponds to

$$\forall x, y.\ \Lambda(p)(x, y) \implies q(x, y),$$

and the second is

$$\forall z.\ p(z) \implies \Delta_A^*(q)(z).$$

With our definition of Λ, the first condition is now

$$\forall x, y.\ (\pi_1^* p)(x, y) \wedge \delta_A(x, y) \implies q(x, y),$$

or more simply

$$\forall x, y.\ p(x) \wedge \delta_A(x, y) \implies q(x, y).$$

Similarly, our second condition becomes

$$\forall z.\ p(z) \implies q(z, z).$$

Hence condition 1 says

$$\forall x, y.\ p(x) \wedge \delta_A(x, y) \implies q(x, y) \Leftrightarrow \forall z.\ p(z) \implies q(z, z).$$

For this reason, we write $x =_A y$ for $\delta_A(x, y)$. In this notation, we have

$$\forall x, y.\ p(x) \wedge x =_A y \implies q(x, y) \Leftrightarrow \forall z.\ p(z) \implies q(z, z).$$

Writing this in natural deduction notation we get

$$\frac{z : A,\ p(z) \vdash q(z,z)}{x : A,\ y : A,\ p(x),\ x =_A y \vdash q(x,y)}.$$

The second condition describes a similar property. Starting similarly, for any two predicates $p \in P(X \times A)$ and $q \in P(X \times A \times A)$ we have $M_X(p) \leq q$ in $P(X \times A \times A)$ iff $p \leq (\mathrm{Id}_X \times \Delta_A)^*(q)$ in $P(X \times A)$. Once we introduce variables, $x \in X$ and $y, z \in A$ we have that the left hand side of the iff is equivalent to

$$\forall x, y, z.\ M_X(p)(x,y,z) \implies q(x,y,z),$$

and the right hand side is equivalent to

$$\forall x, y.\ p(x,y) \implies (\mathrm{Id}_X \times \Delta_A)^*(q)(x,y).$$

Unpacking the first condition we have

$$\forall x, y, z.\ (\langle \pi_1, \pi_2 \rangle^* p \wedge \langle \pi_2, \pi_3 \rangle^* \delta_A)(x,y,z) \implies q(x,y,z),$$

which simplifies down to

$$\forall x, y, z.\ p(x,y) \wedge y =_A z \implies q(x,y,z).$$

Similarly, the right hand side of the iff is simply

$$\forall x, y.\ p(x,y) \implies q(x,y,y).$$

In natural deduction notation, we have

$$\frac{x : X,\ y : A,\ p(x,y) \vdash q(x,y,y)}{x : X,\ y : A,\ z : A,\ p(x,y),\ y =_A z \vdash q(x,y,z)}.$$

We can see this condition is very similar to the previous, and if $\mathcal{C}$ has a terminal object then taking $X = 1$ gives us the previous condition.

Example 2.5. In the primary doctrine given by the power set functor $\mathscr{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{ISL}$, every set A has an equality predicate given by $\delta_A = \{(a,a) : a \in A\}$.

Example 2.6. In the primary doctrine given by the open set functor $\mathcal{O} : \mathbf{Top}^{\mathrm{op}} \to \mathbf{ISL}$, some objects do not have equality predicates [Pas12].

Prop 2.7. *If δ_A and δ'_A are two equality predicates for A in the doctrine P, then $\delta_A \cong \delta'_A$ in $P(A)$.*

Proof. Let δ_A and δ'_A be two equality predicates for A and Λ and Λ' given by $\Lambda(p) = \pi_1^* p \wedge \delta_A$ and $\Lambda'(p) = \pi_1^* p \wedge \delta'_A$ respectively. As we have $\Lambda \dashv \Delta_A^*$ and $\Lambda' \dashv \Delta_A^*$, and since adjoints are unique upto natural isomorphism, we have an isomorphism $\eta_X : \Lambda(X) \xrightarrow{\sim} \Lambda'(X)$. In particular $\Lambda(\top_A) \cong \Lambda'(\top_A)$ giving us $\delta_A \cong \delta'_A$. $\square$

Definition 2.8. A primary doctrine is an *elementary doctrine* when every object in the base category has an equality predicate.

2.3 Adding Equality Cofreely

In [Pas12] Pasquali showed we can cofreely add equality to a primary doctrine. Informally speaking the construction described adds equality by forming a new doctrine where the objects in the base are pairs (A, ρ) where A is an object of our original base category and ρ is an equivalence relation on A. The predicates are the predicates on A which respect ρ.

Definition 2.9. If P is a primary doctrine and A is an object of our base category, then ρ in $P(A \times A)$ is an *equivalence relation* if ρ satisfies

reflexivity $\top_A \leq \Delta_A^* \rho$

symmetry $\rho \leq \pi_{2,1}^* \rho$

transitivity $\pi_{1,2}^* \rho \wedge \pi_{2,3}^* \rho \leq \pi_{1,3}^* \rho$

Prop 2.10. *Equality predicates are equivalence relations.*

Proof. Identical to lemma 3.6. □

The cofree construction described in [Pas12] can then be defined.

Definition 2.11. If $P : \mathcal{C}^{\text{op}} \to \mathbf{ISL}$ is a primary doctrine, we define the category $\mathcal{Q}_P$. An object in $\mathcal{Q}_P$ is a pair (A, ρ) where A is an object of $\mathcal{C}$ and ρ is an equivalence relation on A. A morphism from (A, ρ) to (B, σ) is a morphism in $\mathcal{C}$ from A to B such that $\rho \leq (f \times f)^* \sigma$. For a given equivalence ρ on A we define a suborder $\mathcal{D}es_\rho$ of $P(A)$ by taking the predicates $\alpha \in P(A)$ such that $\pi_1^*(\alpha) \wedge \rho \leq \pi_2^*(\alpha)$. Finally we define the doctrine $P_{\mathcal{D}} : \mathcal{Q}_P^{\text{op}} \to \mathbf{ISL}$ by $P_{\mathcal{D}}((A, \rho)) = \mathcal{D}es_\rho$.

Theorem 2.12. *The doctrine $P_{\mathcal{D}}$ is an elementary doctrine. Furthermore the assignment $P \mapsto P_{\mathcal{D}}$ is the right adjoint to the forgetful functor from elementary doctrines to primary doctrines.*

Proof. See [Pas12]. □

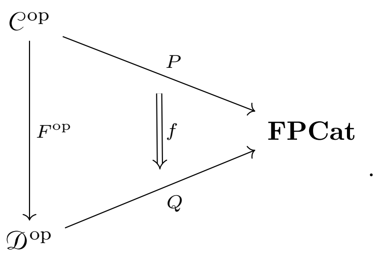
3 Primary Fibrations

For a Primary Doctrine $P : \mathcal{C}^{\text{op}} \to \mathbf{ISL}$, an object $A \in \mathcal{C}$, and predicates $\alpha, \beta \in P(A)$, there is at most one morphism between α and β, which is interpreted as an implication. Primary fibrations relax this condition to allow any number of morphisms between α and β, which are thought of as witnesses of proofs of the implications. Formally

Definition 3.1. A *primary fibration* $P : \mathcal{C}^{\text{op}} \to \mathbf{FPCat}$ is a product preserving contravariant functor from a base category $\mathcal{C}$ with all finite products to $\mathbf{FPCat}$, the category of categories with all finite products, and product preserving functors as morphisms.

We can then define the category $\mathbf{PF}$ of primary fibrations.

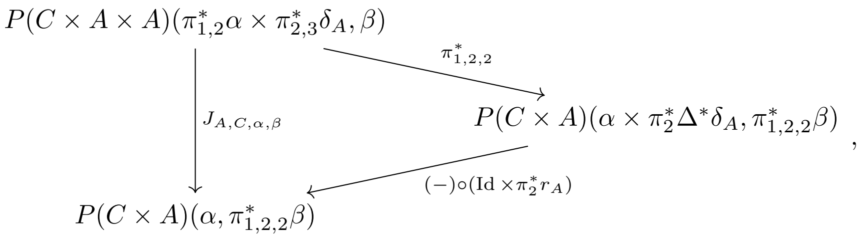
Definition 3.2. The *category of primary fibrations* has, as objects, primary fibrations, and as morphisms between $P : \mathcal{C}^{\text{op}} \to \textbf{FPCat}$ and $Q : \mathcal{D}^{\text{op}} \to \textbf{FPCat}$ pairs (F, f), where $F : \mathcal{C} \to \mathcal{D}$ is a product preserving functor and $f : P \to Q \circ F^{op}$ is a natural transformation,



3.1 Equality Predicates in Fibrations

We can define equality predicates analogously to doctrines.

Definition 3.3. If $P : \mathcal{C}^{\text{op}} \to \textbf{FPCat}$ is a primary fibration and A is an object of $\mathcal{C}$, then an object δ_A together with a map $r_A : 1 \to \Delta^*\delta_A$ form an *equality predicate* if for every object $C \in \mathcal{C}$, and propositions $\alpha \in P(C \times A)$ and $\beta \in P(C \times A \times A)$, the functions $J_{A,C,\alpha,\beta}$



are invertible (cf [EPR20] [HS98]).

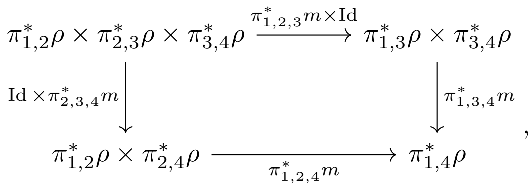
Definition 3.4. An *elementary fibration* is a primary fibration $P : \mathcal{C}^{\text{op}} \to \textbf{FPCat}$, together with an equality predicate for every object $A \in \mathcal{C}$. A morphism (F, f) of primary fibrations between elementary fibrations is *elementary* if $f(\delta_A) = \delta_{FA}$ and $f(r_A) = r_{FA}$. The *category of elementary fibrations* is the subcategory of the category of primary fibrations in which every object and every morphism is elementary.

3.2 Equivalence Relations in Fibrations

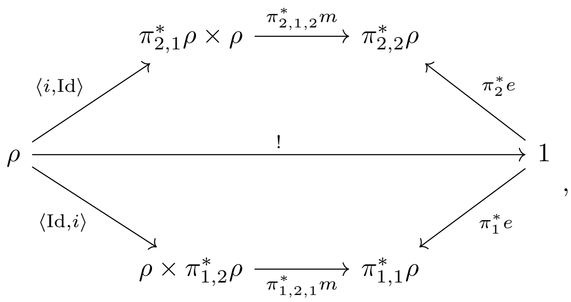
In the case of doctrines, we defined an object $\rho \in P(A \times A)$ to be an equivalence relation if $\top \leq \Delta^*\rho$, $\rho \leq \pi_{2,1}^*\rho$, and $\pi_{1,2}^*\rho \wedge \pi_{2,3}^*\rho \leq \pi_{1,3}^*\rho$. Equivalently we could have phrased these conditions as $P(A)(\top, \Delta^*\rho) = \{\leq\}$, $P(A \times A)(\rho, \pi_{2,1}^*\rho) = \{\leq\}$, and $P(A \times A)(\pi_{1,2}^*\rho \wedge \pi_{2,3}^*\rho, \pi_{1,3}^*\rho) = \{\leq\}$ where $\leq$ is the unique morphism between any two predicates over an object. However, in fibrations there is no such unique choice. As such, equivalence relations in fibrations require an explicit choice of morphisms (together with some coherency conditions).

Definition 3.5. If P is a primary fibration and A is an object in the base of P, then an equivalence relation on A is an object $\rho \in P(A \times A)$, together with a morphism called the *identity* $e : 1 \to \Delta^*\rho$, a morphism called the inverse map $i : \rho \to \pi^*_{2,1}\rho$, and a map called the multiplication[1] map $m : \pi^*_{1,2}\rho \times \pi^*_{2,3}\rho \to \pi^*_{1,3}\rho$, such that the following diagrams commute:

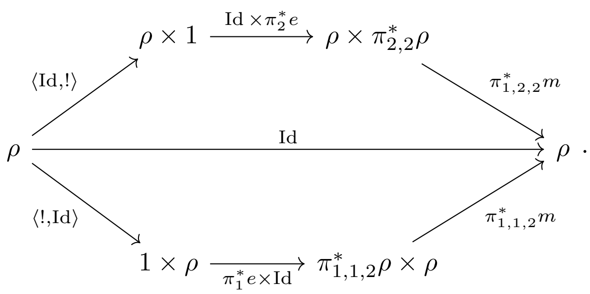
Associativity



Inverses



Identity



For conciseness we write the tuple (ρ, m, i, e) as $\vec{\rho}$, when the choice of m, i, and e is not ambiguous.

Lemma 3.6. *If $P : \mathcal{C}^{\mathrm{op}} \to \mathbf{FPCat}$ is a primary fibration, and A is an object with an equality predicate (δ_A, r_c), then $(\delta_a, J^{-1}_{A,A,\pi^*_{1,2}\delta_A,\pi^*_{1,3}\delta_A}(\mathrm{Id}), J^{-1}_{A,1,1,\pi^*_{2,1}\delta_A}(r_A), r_A)$ is an equivalence relation.*

Proof. The coherence conditions follow by the invertibility of $J_{A,C,\alpha,\beta}$. Associativity follows by taking $C = A \times A$, $\alpha = \pi^*_{1,2}\delta_A \times \pi^*_{2,3}\delta_a$, and $\beta = \pi^*_{1,4}\delta_A$. Other conditions follow similarly. As J is a bijection we have that this is the unique equivalence relation for our equality predicate. ☐

[1] Note that the order of variables is backward to what we would expect for mutliplication.

4 Fibrations of Descent Data

4.1 The Category of Quotients

Definition 4.1. For a primary fibration $P : \mathcal{C}^{\mathrm{op}} \to \mathbf{FPCat}$ we define the *category of quotients* $\mathcal{Q}_P$. Objects are given by pairs $(A, \vec{\rho})$ where $\vec{\rho}$ is an equivalence on A. Morphisms are given by pairs $(f, g) : (A, \vec{\rho}) \to (B, \vec{\sigma})$ where $f : A \to B$ in $\mathcal{C}$ and $g : \rho \to (f \times f)^*\sigma$ in $P(A \times A)$, such that the following diagrams commute:

Multiplication Preservation

$$\begin{array}{ccc} \pi_{1,2}^*\rho \times \pi_{2,3}^*\rho & \xrightarrow{m_\rho} & \pi_{1,3}^*\rho \\ {\scriptstyle \pi_{1,2}^*g \times \pi_{2,3}^*g}\downarrow & & \downarrow{\scriptstyle \pi_{1,3}^*g} \\ (f \times f \times f)^*\pi_{1,2}^*\sigma \times (f \times f \times f)^*\pi_{2,3}^*\sigma & \xrightarrow[(f \times f \times f)^* m_\sigma]{} & (f \times f \times f)^*\pi_{1,3}^*\sigma \end{array}$$

Identity Preservation

$$\begin{array}{ccc} 1_A & \xrightarrow{e_\rho} & \Delta^*\rho \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \Delta^*g} \\ f^*1_B & \xrightarrow[f^*e_\sigma]{} & f^*\Delta^*\sigma \end{array}.$$

4.2 Descent Data

Definition 4.2. If $P : \mathcal{C}^{\mathrm{op}} \to \mathbf{FPCat}$ is a primary fibration, we define the category of *descent data*, $\mathcal{Des}_{\vec{\rho}}$ for an object $A \in \mathcal{C}$ and an equivalence relation $\vec{\rho}$ on A. The objects of $\mathcal{Des}_{\vec{\rho}}$ are pairs (α, F) where $\alpha \in P(A)$ and $F : \pi_1^*\alpha \times \rho \to \pi_2^*\alpha$ such that the following diagrams commute:

Functor Identity

$$\begin{array}{ccccc} \alpha & \xrightarrow{\langle \mathrm{Id}, ! \rangle} & \alpha \times 1_A & \xrightarrow{\mathrm{Id} \times e} & \alpha \times \Delta^*\rho \\ & \searrow^{\mathrm{Id}} & & & \downarrow{\scriptstyle \Delta^*F} \\ & & & & \alpha \end{array}$$

Functor Composition

$$\begin{array}{ccc} \pi_1^*\alpha \times \pi_{1,2}^*\rho \times \pi_{2,3}^*\rho & \xrightarrow{\pi_{1,2}^*F \times \mathrm{Id}} & \pi_2^*\alpha \times \pi_{2,3}^*\rho \\ {\scriptstyle \mathrm{Id} \times m}\downarrow & & \downarrow{\scriptstyle \pi_{2,3}^*F} \\ \pi_1^*\alpha \times \pi_{1,3}^*\rho & \xrightarrow[\pi_{1,3}^*F]{} & \pi_3^*\alpha \end{array}.$$

Morphisms in $\mathcal{Des}_{\vec{\rho}}$ from (α, F) to (β, G) are given by maps $\eta : \alpha \to \beta$ in $P(A)$ such that

$$\begin{array}{ccc} \pi_1^*\alpha \times \rho & \xrightarrow{F} & \pi_2^*\alpha \\ {\scriptstyle \pi_1^*\eta \times \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle \pi_2^*\eta} \\ \pi_1^*\beta \times \rho & \xrightarrow[G]{} & \pi_2^*\beta \end{array}$$

commutes.

4.3 The Descent Fibration

Definition 4.3. If $P : \mathcal{C}^{\mathrm{op}} \to \mathbf{FPCat}$ is a primary fibration then we define the *descent fibration* of P to be the primary fibration $P_{\mathcal{D}} : \mathcal{Q}_P^{\mathrm{op}} \to \mathbf{FPCat}$ taking objects to their descent categories.

Prop 4.4. *The descent fibration is a primary fibration.*

Proof. If $P : \mathcal{C}^{\mathrm{op}} \to \mathbf{FPCat}$ is a primary fibration, then $\mathcal{Q}_P$ has products given by $(A \times B, \vec{\rho} \boxtimes \vec{\sigma})$ where $\vec{\rho}$ is an equivalence relation on A and $\vec{\sigma}$ is an equivalence relation on B. The equivalence relation

$$\vec{\rho} \boxtimes \vec{\sigma} = (\theta, m_\theta, i_\theta, e_\theta)$$

on $A \times B$ is given by

$$\theta = \pi_{1,3}^* \rho \times \pi_{2,4}^* \sigma,$$

$$m_\theta = \pi_{1,3,5}^* m_\rho \times \pi_{2,4,6}^* m_\sigma,$$

$$i_\theta = \pi_{1,3}^* i_\rho \times \pi_{2,4}^* i_\sigma,$$

and

$$e_\theta = \langle i_\rho, i_\sigma \rangle.$$

As $\mathcal{C}$ has a terminal object, and P preserves all finite products, there is a unique groupoid structure on $1_{\mathcal{C}}$, hence $\mathcal{Q}_P$ has a terminal object. Similarly, $\mathcal{D}es_\rho$ inherits products and terminal objects from the fibres of P. Finally, as P preserves products, $P_{\mathcal{D}}$ must as well. □

Prop 4.5. *The descent fibration is an elementary fibration and the equivalence relation induced by the equality predicate on an object $(A, \vec{\rho})$ is $\vec{\rho}$.*

Proof. We claim that the equality predicate on an object $(A, \vec{\rho})$, where $\vec{\rho} = (\rho, m, i, e)$ is $((\rho, c), e)$, where $c : \pi_1^* \rho \times \rho \boxtimes \rho \to \pi_2^* \rho$ is conjugation, given by

$$c(l, (f, g)) = m \langle i \circ \pi_2, m \circ \langle \pi_1, \pi_3 \rangle \rangle.$$

The function $J^{-1}_{A,C,(\alpha,F),(\beta,G)}$ is given by

$$J^{-1}_{A,C,(\alpha,F),(\beta,G)}(f) = \pi_{1,2,2,1,2,3}^* G \circ \langle f \circ \pi_1, \langle \pi_1^* e \circ !, \pi_2^* e \circ !, \pi_2 \rangle \rangle.$$

By uniqueness of the groupoid structure, we have that the groupoid for the equality predicate on $(A, \vec{\rho})$ is $\vec{\rho}$. □

Prop 4.6. *If P is an elementary fibration, $\alpha \in P(A)$ is a predicate on A, and $g : \pi_1^* \alpha \times \delta_A \to \pi_2^* \alpha$ is the unique map such that $J(g) = \mathrm{Id}_\alpha$, then $(\alpha, g) \in \mathcal{D}es_{\vec{\delta_A}}$.*

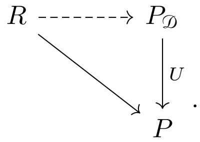
Proof. The coherency conditions follow immediately. □

Prop 4.7. *In the fibration $P_{\mathcal{D}}$, if $(\alpha, g) \in \mathcal{D}es_{\vec{\rho}}$ then $J^{-1}(\mathrm{Id}_{(\alpha,g)}) = g$.*

Proof. This follows from the fact that g preserves the identity. □

Corollary 4.8. *The fibration $P_{\mathcal{D}}$ is universal, in the sense that the assignment $P \mapsto P_{\mathcal{D}}$ determines the right adjoint to the forgetful functor from elementary fibrations to primary fibrations.*

Proof. If $U : P_{\mathcal{D}} \to P$ is the inclusion map, then we can observe from the previous lemmas that for any primary morphism $R \to P$ there is a unique morphism such that the following diagram commutes



□

5 Conclusions and Future Work

There are many possible ways to strengthen this result. The first is to generalise the objects studied. In this report we regard **FPCat** as a 1-category, when it is best regarded as a 2-category. Taking this view we should define primary fibrations as pseudo functors from a base category to the 2-category of categories with all finite products.

Another natural question to ask is what other structure can we (co)freely add to a fibration. If the base category has exponential objects we can naturally talk about higher order logic on the base category. For this reason we could investigate fibrations of the free cartesian closure of our base category. We can also investigate how to present other structure through fibrations. In [DR21] Dagnino and Rosolini describe modal logic through doctrines. A natural question to ask is if we can also present modal logic on fibrations, and if a modal structure can be added freely.

Finally we can try and use fibrations to study objects in a base category based on which axioms they satisfy. For example, the predicates for a space X in the open set doctrine $\mathcal{O}$ satisfy excluded middle iff X is discrete. We can also characterise connectivity conditions logically. An interesting question is if we can also study other geometric structures, such as iterated function systems, through fibrations.

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