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# Limit theorems for Curie-Weiss-Potts

## Model

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**Abstract**

In [2], there is a part of proof for theorem 1.3 that is not correct. So T.Garoni and I give a correct proof for that part in this report.

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**1 Introduction**

**1.1 The Cuire-Weiss-Potts model**

The Curie-Weiss-Potts model is an interaction of vertex variables on a complete graph  $K = (V, E)$ , with each of  $n$  vertices assigned a number  $i \in [q]$ . A colouring is a configuration of  $K$  with every vertex assigned a number. We define a colouring  $\sigma$  on  $K$  as an element of the state space  $\Omega = [q]^V$ , so that a colouring can be written as  $\sigma := (\sigma_j)_{j \in V} \in [q]^V$ . The essential quantity concerning a colouring  $\sigma$  is its total energy, given by Hamiltonian  $H : [q]^V \rightarrow \mathbb{R}$  via

$$H(\sigma) := \sum_{i \in V} -\frac{\beta}{|V|} \sum_{j \in V} \delta(\sigma_i, \sigma_j)$$

where  $\beta$  is the inverse temperature of the system. The Gibbs distribution of  $\sigma$  is given by

$$\mathbb{P}(\sigma) = \frac{\exp(-H(\sigma))}{\sum_{\sigma' \in [q]^V} \exp(-H(\sigma'))}$$

Refer to [3] for more details.

**1.2 Multivariate Stein’s method**

The main tool we will be using is the following stein’s method theorem.

**Theorem 1.1.** *Assume that  $(W, W')$  is an exchangeable pair of  $\mathbb{R}^q$ -valued random vectors such that*

$$\mathbb{E}[W] = 0, \quad \mathbb{E}[WW'] = \Sigma,$$

with  $\Sigma \in \mathbb{R}^{q \times q}$  symmetric and positive definite. If  $(W, W')$  satisfies

$$\mathbb{E}[W' - W | W] = -\Lambda W + R, \quad (1)$$

where  $R$  represents remainder, for an invertible  $\Lambda$  and a  $\sigma(W)$ -measurable random vector  $R$  and if  $Z$  has  $d$ -dimensional standard normal distribution, we have for every three times differentiable function  $g$ ,

$$|\mathbb{E}g(W) - \mathbb{E}g(\Sigma^{1/2}Z)| \leq \frac{|g|_2}{4}A + \frac{|g|_3}{12}B + \left(|g|_1 + \frac{1}{2}d\|\Sigma\|^{1/2}|g|_2\right)C \quad (2)$$

where  $\lambda^{(i)} := \sum_{m=1}^q |(\Lambda^{-1})|_{m,i}$ ,

$$A = \sum_{i=1}^q \sum_{j=1}^q \lambda^{(i)} \sqrt{\text{Var} \left[ \mathbb{E} \left[ (W'_i - W_i)(W'_j - W_j) | W \right] \right]}, \quad (3)$$

$$B = \sum_{i=1}^q \sum_{j=1}^q \sum_{k=1}^q \lambda^{(i)} \mathbb{E} |(W'_i - W_i)(W'_j - W_j)(W'_k - W_k)|, \quad (4)$$

$$C = \sum_{i=1}^q \lambda^{(i)} \sqrt{\mathbb{E}[R_i^2]}. \quad (5)$$

### 1.3 Main theorem

From [1], we know that, if inverse temperature  $\beta < \beta_c$ , where

$$\beta_c = \begin{cases} q, & q \leq 2, \\ 2 \frac{q-1}{q-2} \log(q-1), & q > 2, \end{cases}$$

then

$$N_i(\sigma) = (\delta(\sigma_i, 1), \dots, \delta(\sigma_i, q))$$

satisfies the law of large number and the central limit theorem, that is,

$$\frac{1}{n} \sum_{i=1}^n N_i = \frac{1}{n} \left( \sum_{i=1}^n \delta(\sigma_i, 1), \dots, \sum_{i=1}^n \delta(\sigma_i, q) \right) \rightarrow \left( \frac{1}{q}, \dots, \frac{1}{q} \right)$$

and

$$W := \frac{1}{\sqrt{n}} \left( \frac{1}{n} \sum_{i=1}^n N_i - \frac{1}{q} \right) \Rightarrow \mathcal{N} \left( 0, \mathbb{E}[WW^t] \right).$$

The following theorem from [2] gives the rate of convergence that  $W$  converges in distribution to normal distribution,  $\mathcal{N}(0, \mathbb{E}[WW^t])$ .

**Theorem 1.2.** Let  $\beta \in (0, \beta_c)$  with  $\beta \neq 4(q-1)/q$  and  $h = 0$ . Assume that there is a unique minimizer  $x_0$  of

$$G_{\beta,h}(u) := \frac{1}{2} \beta \langle u, u \rangle - \log \left( \sum_{i=1}^q \exp(\beta u_i + h \delta(i, 1)) \right), \quad u \in \mathbb{R}^q.$$

If  $Z$  has the  $q$ -dimensional standard normal distribution, we have for every three times differentiable function  $g$  with bounded derivatives,

$$|\mathbb{E}g(W) - \mathbb{E}g(\Sigma^{1/2}Z)| \leq C \cdot n^{-1/2},$$

for a constant  $C$  who depends on  $\beta, h, q$  and bounds on the derivatives of  $g$ , and  $\Sigma := \mathbb{E}[WW^t]$  where

$$W := \sqrt{n} \left( \frac{N}{n} - x_0 \right), \quad N := \left( \sum_{i=1}^n \delta(\sigma_i, 1), \dots, \sum_{i=1}^n \delta(\sigma_i, q) \right),$$

and

$$\Sigma^{1/2} Z \sim \mathcal{N}(0, \Sigma).$$

**Remark 1.3.** From theorem 1.1 of [2], we know if  $\beta < \beta_c$  and zero external field,  $x_0 = (1/q, \dots, 1/q)$ .

## 2 Proof of the theorem 1.2

In this section, we present the more detailed proof for theorem 1.2 from [2] using theorem 1.1.

Constructing an exchangeable pair in the Cuire-Weiss-Potts model to obtain an approximate linear regression property (1) leads us to the function  $G_{\beta, h}$ . Let  $q > 2$ ,  $h = 0$  and  $\beta < \beta_c$  and let  $x_0$  denote the unique global minimum point of  $G_{\beta, 0}$ .

Then we produce a spin collection  $\sigma' = (\sigma'_i)_{i \geq 1}$  via Glauber dynamics,. Let  $I$  be uniformly distributed over  $\{1, \dots, n\}$  and independent from all other random variable involved. The spin  $\sigma_j$  will now be replaced by  $\sigma'_j$  drawn from conditional distribution of the  $i$ -th coordinate given  $(\sigma_t)_{t \neq j}$ , independently from  $\sigma_j$ . We define

$$\begin{aligned} Y_{t,i} &:= Y_{t,i}(\sigma) := \delta(\sigma_t, i), \\ Y_j &:= (Y_{j,1}, \dots, Y_{j,q})^t := (\delta(\sigma_j, 1), \dots, \delta(\sigma_j, q))^t, \\ m_i &:= m_i(\sigma) := \frac{1}{n} \sum_{t=1}^n Y_{t,i}, \\ W_i &:= W_i(\sigma) := \sqrt{n} \left( m_i - \frac{1}{q} \right) = \frac{1}{\sqrt{n}} \sum_{t=1}^n Y_{t,i} - \frac{\sqrt{n}}{q}, \end{aligned}$$

and consider

$$W' := W - \frac{Y_I}{\sqrt{n}} + \frac{Y'_I}{\sqrt{n}}.$$

First, we verify that  $(W, W')$  is indeed an exchangeable pair and satisfies property (1). Let  $\mathcal{F} := \sigma(\sigma_1, \dots, \sigma_n)$  be a sigma algebra generated by spins  $\{\sigma_1, \dots, \sigma_n\}$ . Then by law of total probability, we have

$$\begin{aligned} \mathbb{E}[W'_i - W_i | \mathcal{F}] &= \frac{1}{\sqrt{n}} \mathbb{E}[Y'_{I,i} - Y_{I,i} | \mathcal{F}] \\ &= \frac{1}{\sqrt{n}} \frac{1}{n} \sum_{k=1}^n \mathbb{E}[Y'_{k,i} - Y_{k,i} | \mathcal{F}, I = k] \\ &= \frac{1}{\sqrt{n}} \frac{1}{n} \sum_{k=1}^n \mathbb{E}[Y'_{k,i} | \mathcal{F}, I = k] - \frac{1}{\sqrt{n}} \frac{1}{n} \sum_{k=1}^n \mathbb{E}[Y_{k,i} | \mathcal{F}, I = k] \\ &= -\frac{1}{\sqrt{n}} \frac{1}{n} \sum_{k=1}^n Y_{k,i} + \frac{1}{\sqrt{n}} \frac{1}{n} \sum_{k=1}^n \mathbb{E}[Y'_{k,i} | \mathcal{F}, I = k], \end{aligned}$$

the last equality holds since  $Y_{k,i}$  is  $\mathcal{F}$ -measurable and the second summand can be simplified,

$$\mathbb{E}[Y'_{j,i} | \mathcal{F}, I = j] = \mathbb{E}[\delta(\sigma'_j, i) | \mathcal{F}, I = j]$$

$$\begin{aligned}
&= \mathbb{E} [\delta(\sigma'_j, i) | (\sigma_k)_{1 \leq k \leq n}, I = j] \\
&= \mathbb{E} \left[ \delta(\sigma'_j, i) | (\sigma'_k)_{\substack{1 \leq k \leq n, \\ k \neq j}}, I = j \right] \\
&= \sum_{\sigma \in [q]^V} \delta(\tau, i) \mathbb{P}_{\beta, 0, n}(\sigma'_j = \tau | (\sigma_k)_{\substack{1 \leq k \leq n, \\ k \neq j}}, I = j) \\
&= \mathbb{P}_{\beta, 0, n}(\sigma'_j = i | (\sigma_k)_{\substack{1 \leq k \leq n, \\ k \neq j}}, I = j) \\
&= \frac{\exp(\beta m_{i,j}(\sigma))}{\sum_{l=1}^q \exp(\beta m_{l,j}(\sigma))}, \tag{6}
\end{aligned}$$

where

$$m_{i,j}(\sigma) := \frac{1}{n} \sum_{k \neq j}^n \delta(\sigma_k, i)$$

Hence,

$$\begin{aligned}
\mathbb{E}[W'_i - W_i | \mathcal{F}] &= -\frac{1}{\sqrt{n}} \frac{1}{n} \sum_{j=1}^n Y_{j,i} + \frac{1}{\sqrt{n}} \frac{1}{n} \sum_{j=1}^n \frac{\exp(\beta m_{i,j}(\sigma))}{\sum_{l=1}^q \exp(\beta m_{l,j}(\sigma))} \\
&= -\frac{1}{\sqrt{n}} \frac{1}{n} \sum_{j=1}^n \delta(\sigma_j, i) + \frac{1}{\sqrt{n}} \frac{1}{n} \sum_{j=1}^n \left[ \frac{\exp(\beta m_{i,j}(\sigma))}{\sum_{l=1}^q \exp(\beta m_{l,j}(\sigma))} - \frac{\exp(\beta m_i(\sigma))}{\sum_{l=1}^q \exp(\beta m_k(\sigma))} \right] \\
&\quad + \frac{1}{\sqrt{n}} \frac{1}{n} \sum_{j=1}^n \frac{\exp(\beta m_i(\sigma))}{\sum_{l=1}^q \exp(\beta m_k(\sigma))} \\
&= -\frac{1}{\sqrt{n}} m_i(\sigma) + \frac{1}{\sqrt{n}} \frac{1}{n} \sum_{j=1}^n \left[ \frac{\exp(\beta m_{i,j}(\sigma))}{\sum_{l=1}^q \exp(\beta m_{l,j}(\sigma))} - \frac{\exp(\beta m_i(\sigma))}{\sum_{l=1}^q \exp(\beta m_k(\sigma))} \right] \\
&\quad + \frac{1}{\sqrt{n}} \frac{\exp(\beta m_i(\sigma))}{\sum_{l=1}^q \exp(\beta m_k(\sigma))} \\
&= -\frac{1}{\sqrt{n}} m_i(\sigma) + \frac{1}{\sqrt{n}} \frac{\exp(\beta m_i(\sigma))}{\sum_{l=1}^q \exp(\beta m_k(\sigma))} + R_n^{(1)}(i) \\
&= -\frac{1}{\sqrt{n}} \left( m_i(\sigma) - \frac{\exp(\beta m_i(\sigma))}{\sum_{l=1}^q \exp(\beta m_k(\sigma))} \right) + R_n^{(1)}(i) \\
&= -\frac{1}{\sqrt{n}} \frac{1}{\beta} \frac{\partial}{\partial m_i} G_{\beta, 0}(m(\sigma)) + R_n^{(1)}(i), \tag{7}
\end{aligned}$$

where

$$R_n^{(1)}(i) := \frac{1}{\sqrt{n}} \frac{1}{n} \sum_{j=1}^n \left[ \frac{\exp(\beta m_{i,j}(\sigma))}{\sum_{l=1}^q \exp(\beta m_{l,j}(\sigma))} - \frac{\exp(\beta m_i(\sigma))}{\sum_{l=1}^q \exp(\beta m_k(\sigma))} \right]. \tag{8}$$

We are dealing with the case with zero exterior field, i.e.  $h = 0$ . So the inverse temperature  $\beta$  is less than the critical inverse temperature  $\beta_c$  and  $G_{\beta, 0}$  has the global minimum point  $x_0 = (1/q, \dots, 1/q)$ .

In (7), the first summand can be written as

$$\begin{aligned}
\frac{\partial}{\partial m_i} G_{\beta, 0}(m(\sigma)) &= \frac{\partial}{\partial m_i} G_{\beta, 0}(x_0) + \frac{\partial^2}{\partial^2 m_i} G_{\beta, 0}(x_0) (m_i(\sigma) - x_{0,i}) \\
&\quad + \sum_{\substack{k=1 \\ k \neq i}}^q \frac{\partial^2}{\partial u_k \partial m_i} G_{\beta, 0}(x_0) (m_k(\sigma) - x_{0,k}) \\
&\quad + 2 \sum_{|\alpha|=2} \frac{(m(\sigma) - x_0)^\alpha}{\alpha!} \int_0^1 (1-t)^2 \partial_{\alpha+1} G_{\beta, 0}[x_0 + t(m(\sigma) - x_0)] dt
\end{aligned}$$

$$= 0 + \frac{\partial^2}{\partial^2 m_i} G_{\beta,0}(x_0) \frac{W_i}{\sqrt{n}} + \sum_{\substack{k=1 \\ k \neq i}}^q \frac{\partial^2}{\partial u_k \partial m_i} G_{\beta,0}(x_0) \frac{W_k}{\sqrt{n}} \\ + 2 \sum_{|\alpha|=2} \frac{W^\alpha}{(\sqrt{n})^\alpha \alpha!} \int_0^1 (1-t)^2 \partial_{\alpha+1} G_{\beta,0}[x_0 + t(m(\sigma) - x_0)] dt$$

Then set

$$R_n^{(2)}(i) := 2 \sum_{|\alpha|=2} \frac{W^\alpha}{(\sqrt{n})^\alpha \alpha!} \int_0^1 (1-t)^2 \partial_{\alpha+1} G_{\beta,0}[x_0 + t(m(\sigma) - x_0)] dt,$$

and

$$R(i) := R_n^{(2)}(i) + R_n^{(1)}(i).$$

So,

$$\mathbb{E} [W'_i - W_i | \mathcal{F}] = -\frac{1}{\beta} \frac{1}{n} \left[ \frac{\partial^2}{\partial^2 m_i} G_{\beta,0}(x_0) W_i + \sum_{\substack{k=1 \\ k \neq i}}^q \frac{\partial^2}{\partial u_k \partial m_i} G_{\beta,0}(x_0) W_k \right] + R(i) \\ = -\frac{1}{\beta} \frac{1}{n} \sum_{k=1}^q \frac{\partial^2}{\partial m_i \partial m_k} G_{\beta,0}(x_0) W_k + R(i)$$

The first summand can be seen as cartesian dot product  $\langle \cdot, \cdot \rangle$  of  $i$ -th row of  $D^2 G_{\beta,0}(x_0)$  and  $W$

$$= -\frac{1}{\beta} \frac{1}{n} \left\langle [D^2 G_{\beta,0}(x_0)]_i, W \right\rangle + R(i).$$

Furthermore, we can have

$$\mathbb{E} [W - W' | \mathcal{F}] = -\frac{1}{\beta} \frac{1}{n} [D^2 G_{\beta,0}(x_0)] W + R(W),$$

where  $R(W) := (R(1), \dots, R(q))$ . Hence,  $(W, W')$  satisfies (1) with a  $n \times n$  matrix  $\Lambda := \frac{1}{\beta n} [D^2 G_{\beta,0}(x_0)]$  and a remainder term  $R(W)$ .

Therefore, theorem 1.2 is proved by theorem 1.1 if equations (3),(4), and (5) are bounded by  $n^{-1/2}$  up to some constant.

In [2], (4) and (5) are shown to be in  $\mathcal{O}(n^{-1/2})$ .

Now, we want to show for (3),

$$\sum_{i=1}^d \sum_{j=1}^d \lambda^{(i)} \sqrt{\text{Var} \left[ \mathbb{E} \left[ (W'_i - W_i)(W'_j - W_j) | W \right] \right]} = \mathcal{O}(n^{-\frac{1}{2}}) \\ \Leftrightarrow \text{Var} \left[ \mathbb{E} \left[ (W'_i - W_i)(W'_j - W_j) | W \right] \right] = \mathcal{O}(n^{-3}).$$

Note by law of total variance, and since sigma algebra of  $W$ ,  $\sigma(W)$  is a subset of  $\mathcal{F}$ , we have

$$\text{Var} \left( \mathbb{E} \left[ (W'_i - W_i)(W'_j - W_j) | W \right] \right) \\ = \text{Var} \left( \mathbb{E} \left[ \mathbb{E} \left[ (W'_i - W_i)(W'_j - W_j) | \mathcal{F} \right] | W \right] \right) \\ = \text{Var} \left( \mathbb{E} \left[ (W'_i - W_i)(W'_j - W_j) | \mathcal{F} \right] \right) - \mathbb{E} \left[ \text{Var} \left( \mathbb{E} \left[ (W'_i - W_i)(W'_j - W_j) | \mathcal{F} \right] | W \right) \right] \\ \leq \text{Var} \left( \mathbb{E} \left[ (W'_i - W_i)(W'_j - W_j) | \mathcal{F} \right] \right).$$

Therefore, it is enough to just show

$$\text{Var} \left( \mathbb{E} \left[ (W'_i - W_i)(W'_j - W_j) | \mathcal{F} \right] \right) = \mathcal{O}(n^{-3}).$$

Now,

$$\begin{aligned}
 & \mathbb{E} [(W'_i - W_i)(W'_j - W_j) | \mathcal{F}] \\
 &= \mathbb{E} \left[ \frac{1}{\sqrt{n}} (Y'_{t,i} - Y_{t,i}) \frac{1}{\sqrt{n}} (Y'_{t,j} - Y_{t,j}) \mid \mathcal{F} \right] \\
 &= \frac{1}{n} \frac{1}{n} \sum_{t=1}^n \mathbb{E} [(Y'_{t,i} - Y_{t,i})(Y'_{t,j} - Y_{t,j}) | \mathcal{F}, I = t] \\
 &= \frac{1}{n^2} \sum_{t=1}^n \mathbb{E} [Y'_{t,i} Y'_{t,j} - Y'_{t,i} Y_{t,j} - Y_{t,i} Y'_{t,j} + Y_{t,i} Y_{t,j} | \mathcal{F}, I = t] \\
 &= \frac{1}{n^2} \sum_{t=1}^n \mathbb{E} [Y'_{t,i} Y'_{t,j} | \mathcal{F}, I = t] - Y_{t,j} \mathbb{E} [Y'_{t,i} | \mathcal{F}, I = t] - Y_{t,i} \mathbb{E} [Y'_{t,j} | \mathcal{F}, I = t] + Y_{t,i} Y_{t,j} \\
 &= \frac{1}{n^{3/2}} \left\{ \frac{1}{\sqrt{n}} \sum_{t=1}^n \delta_{ij} \mathbb{E} [Y'_{t,i} | \mathcal{F}, I = t] - \frac{1}{\sqrt{n}} \sum_{t=1}^n Y_{t,j} \mathbb{E} [Y'_{t,i} | \mathcal{F}, I = t] \right. \\
 &\quad \left. - \frac{1}{\sqrt{n}} \sum_{t=1}^n Y_{t,i} \mathbb{E} [Y'_{t,j} | \mathcal{F}, I = t] + \frac{1}{\sqrt{n}} \sum_{t=1}^n \delta_{ij} Y_{t,i} \right\}.
 \end{aligned}$$

Then, consider the variance,

$$\begin{aligned}
 & \text{Var} (\mathbb{E} [(W'_i - W_i)(W'_j - W_j) | \mathcal{F}]) \\
 &= \frac{1}{n^3} \text{Var} \left( \frac{\delta_{ij}}{\sqrt{n}} \sum_{t=1}^n \mathbb{E} [Y'_{t,i} | \mathcal{F}, I = t] - \frac{1}{\sqrt{n}} \sum_{t=1}^n Y_{t,j} \mathbb{E} [Y'_{t,i} | \mathcal{F}, I = t] \right. \\
 &\quad \left. - \frac{1}{\sqrt{n}} \sum_{t=1}^n Y_{t,i} \mathbb{E} [Y'_{t,j} | \mathcal{F}, I = t] + \frac{\delta_{ij}}{\sqrt{n}} \sum_{t=1}^n Y_{t,i} \right) \\
 &= \frac{1}{n^3} \text{Var} \left( \frac{\delta_{ij}}{\sqrt{n}} \sum_{t=1}^n \frac{\exp(\beta m_{i,t})}{\sum_{l=1}^q \exp(\beta m_{l,t})} - \frac{1}{\sqrt{n}} \sum_{t=1}^n Y_{t,j} \frac{\exp(\beta m_{i,t})}{\sum_{l=1}^q \exp(\beta m_{l,t})} \right. \\
 &\quad \left. - \frac{1}{\sqrt{n}} \sum_{t=1}^n Y_{t,i} \frac{\exp(\beta m_{j,t})}{\sum_{l=1}^q \exp(\beta m_{l,t})} + \frac{\delta_{ij}}{\sqrt{n}} \sum_{t=1}^n Y_{t,i} \right) \\
 &= \frac{1}{n^3} \text{Var} \left[ \frac{\delta_{ij}}{\sqrt{n}} \sum_{t=1}^n \frac{\exp(\beta m_i)}{\sum_{l=1}^q \exp(\beta m_l)} - \frac{1}{\sqrt{n}} \sum_{t=1}^n Y_{t,j} \frac{\exp(\beta m_i)}{\sum_{l=1}^q \exp(\beta m_l)} \right. \\
 &\quad \left. - \frac{1}{\sqrt{n}} \sum_{t=1}^n Y_{t,i} \frac{\exp(\beta m_j)}{\sum_{l=1}^q \exp(\beta m_l)} + \frac{\delta_{ij}}{\sqrt{n}} \sum_{t=1}^n Y_{t,i} \right. \\
 &\quad \left. + \frac{\delta_{ij}}{\sqrt{n}} \sum_{t=1}^n \left( \frac{\exp(\beta m_{i,t})}{\sum_{l=1}^q \exp(\beta m_{l,t})} - \frac{\exp(\beta m_i)}{\sum_{l=1}^q \exp(\beta m_l)} \right) \right. \\
 &\quad \left. - \frac{1}{\sqrt{n}} \sum_{t=1}^n Y_{t,i} \left( \frac{\exp(\beta m_{j,t})}{\sum_{l=1}^q \exp(\beta m_{l,t})} - \frac{\exp(\beta m_j)}{\sum_{l=1}^q \exp(\beta m_l)} \right) \right. \\
 &\quad \left. - \frac{1}{\sqrt{n}} \sum_{t=1}^n Y_{t,j} \left( \frac{\exp(\beta m_{i,t})}{\sum_{l=1}^q \exp(\beta m_{l,t})} - \frac{\exp(\beta m_i)}{\sum_{l=1}^q \exp(\beta m_l)} \right) \right] \\
 &\leq \frac{49}{n^3} \max \left\{ n \text{Var} \left( \frac{\exp(\beta m_i)}{\sum_{l=1}^q \exp(\beta m_l)} \right), \text{Var}(\sqrt{nm_i}), n \text{Var} \left( m_j \frac{\exp(\beta m_i)}{\sum_{l=1}^q \exp(\beta m_l)} \right), \right. \\
 &\quad \left. \text{Var} \left[ \frac{1}{\sqrt{n}} \sum_{t=1}^n Y_{t,i} \left( \frac{\exp(\beta m_{j,t})}{\sum_{l=1}^q \exp(\beta m_{l,t})} - \frac{\exp(\beta m_j)}{\sum_{l=1}^q \exp(\beta m_l)} \right) \right] \right\},
 \end{aligned}$$

$$\text{Var} \left[ \frac{1}{\sqrt{n}} \sum_{t=1}^n Y_{t,j} \left( \frac{\exp(\beta m_{i,t})}{\sum_{l=1}^q \exp(\beta m_{l,t})} - \frac{\exp(\beta m_i)}{\sum_{l=1}^q \exp(\beta m_l)} \right) \right] \Bigg\}.$$

To find the maximum, we analysis term by term. We start with  $\text{Var}(\sqrt{nm_i})$ ,

$$\text{Var}(\sqrt{nm_i}) = \text{Var} \left( \sqrt{nm_i} - \frac{\sqrt{n}}{q} \right) = \text{Var}(W_i) \leq \mathbb{E}[W_i^2] = \mathcal{O}(1), \quad (9)$$

the last equality is true by a result from EM.

Next,

$$\mathbb{E} \left[ \frac{\exp(\beta m_i)}{\sum_{l=1}^q \exp(\beta m_l)} \right] = \mathbb{E} \left[ \frac{\exp \left( \frac{\beta}{n} \sum_{t=1}^n Y_{t,i} \right)}{\sum_{l=1}^q \exp \left( \frac{\beta}{n} \sum_{t=1}^n Y_{t,l} \right)} \right]$$

is independent of index  $i$ , then

$$1 = \mathbb{E} \left[ \frac{\sum_{j=1}^q \exp(\beta m_j)}{\sum_{l=1}^q \exp(\beta m_l)} \right] = \sum_{j=1}^q \mathbb{E} \left[ \frac{\exp(\beta m_j)}{\sum_{l=1}^q \exp(\beta m_l)} \right] = q \cdot \mathbb{E} \left[ \frac{\exp(\beta m_j)}{\sum_{l=1}^q \exp(\beta m_l)} \right].$$

Hence,

$$\mathbb{E} \left[ \frac{\exp(\beta m_i)}{\sum_{l=1}^q \exp(\beta m_l)} \right] = \frac{1}{q}.$$

Then consider the variance,

$$\begin{aligned} \text{Var} \left[ \frac{\exp(\beta m_i)}{\sum_{l=1}^q \exp(\beta m_l)} \right] &= \mathbb{E} \left[ \left( \frac{\exp(\beta m_i)}{\sum_{l=1}^q \exp(\beta m_l)} \right)^2 \right] - \frac{1}{q^2} \\ &= \mathbb{E} \left[ \left( \frac{\exp(\beta/q) \exp(\beta m_i)}{\exp(\beta/q) \sum_{l=1}^q \exp(\beta m_l)} \right)^2 \right] - \frac{1}{q^2} \\ &= \mathbb{E} \left[ \left( \frac{\exp(\beta(m_i - 1/q))}{\sum_{l=1}^q \exp(\beta(m_l - 1/q))} \right)^2 \right] - \frac{1}{q^2}. \end{aligned}$$

Let  $\Delta_q := \{x \in \mathbb{R} : x_i \geq 0 \forall i \in [q], \sum_{i \in [q]} x_i = 1\}$ , and define  $F : \Delta_q \rightarrow \mathbb{R}$  via

$$F(x) = \left( \frac{\exp(\beta(x_1 - 1/q))}{\sum_{l=1}^q \exp(\beta(x_l - 1/q))} \right)^2 - \frac{1}{q^2}.$$

Then we taylor expand  $F$  up to the second order around  $\frac{1}{q}$ ,

$$\begin{aligned} F(x) &= F(1/q) + \langle DF(1/q), x - 1/q \rangle + 2 \sum_{|\alpha|=2} \frac{(x - 1/q)^\alpha}{(\alpha - 1)!} \int_0^1 (1-t)^2 D^\alpha F(1/q + t(x - 1/q)) dt \\ &= 0 + \sum_{j \in [q]} \partial_j F(1/q) (x_j - 1/q) \\ &\quad + 2 \sum_{1 \leq i < j \leq q} (x_i - 1/q)(x_j - 1/q) \int_0^1 (1-t)^2 \partial_i \partial_j F(1/q + t(x - 1/q)) dt \\ &\quad + \sum_{i=1}^q (x_i - 1/q)^2 \int_0^1 (1-t)^2 \partial_i^2 F(1/q + t(x - 1/q)) dt. \end{aligned}$$

Now,

$$\text{Var} \left[ \frac{\exp(\beta m_i)}{\sum_{l=1}^q \exp(\beta m_l)} \right] = \mathbb{E}[F(m)]$$

$$\begin{aligned}
&= \sum_{j \in [q]} \partial_j F(1/q) \mathbb{E}[(m_j - 1/q)] \\
&\quad + 2 \sum_{1 \leq i < j \leq q} \mathbb{E}[(m_i - 1/q)(m_j - 1/q)] \int_0^1 (1-t)^2 \partial_i \partial_j F(1/q + t(m - 1/q)) dt \\
&\quad + \sum_{i=1}^q \mathbb{E}[(m_i - 1/q)^2] \int_0^1 (1-t)^2 \partial_i^2 F(1/q + t(m - 1/q)) dt.
\end{aligned}$$

But, since  $\mathbb{E}[m_i - \frac{1}{q}] = \mathbb{E}[\frac{1}{n} \sum_{k=1}^n \delta(\sigma_k, i) - \frac{1}{q}]$  is independent of  $i$ , so

$$\mathbb{E}\left[m_i - \frac{1}{q}\right] = 0 \quad \forall i. \quad (10)$$

Therefore,

$$\begin{aligned}
\mathbb{E}[F(m)] &\leq 2 \sum_{1 \leq i < j \leq q} \mathbb{E}[(m_i - 1/q)(m_j - 1/q)] \sup_{\substack{t \in [0,1] \\ x \in \Delta_q}} \partial_i \partial_j F(1/q + t(x - 1/q)) \int_0^1 (1-t)^2 dt \\
&\quad + \sum_{i=1}^q \mathbb{E}[(m_i - 1/q)^2] \sup_{\substack{t \in [0,1] \\ x \in \Delta_q}} \partial_i^2 F(1/q + t(m - 1/q)) \int_0^1 (1-t)^2 dt \\
&= \frac{2}{3} \sum_{1 \leq i < j \leq q} \mathbb{E}[(m_i - 1/q)(m_j - 1/q)] \sup_{\substack{t \in [0,1] \\ x \in \Delta_q}} \partial_i \partial_j F(1/q + t(x - 1/q)) \\
&\quad + \frac{1}{3} \sum_{i=1}^q \mathbb{E}[(m_i - 1/q)^2] \sup_{\substack{t \in [0,1] \\ x \in \Delta_q}} \partial_i^2 F(1/q + t(x - 1/q)).
\end{aligned}$$

But

$$\begin{aligned}
&\partial_1 \partial_j F(1/q + t(x - 1/q)) \\
&= \partial_1 \partial_j \mathbb{E} \left[ \left( \frac{\exp(\beta t(x_1 - 1/q))}{\sum_{l=1}^q \exp(\beta t(x_l - 1/q))} \right)^2 \right] - \frac{1}{q^2} \\
&= \mathbb{E} \left\{ \partial_1 \partial_j \frac{\exp(2\beta t(x_1 - 1/q))}{[\sum_{l=1}^q \exp(\beta t(x_l - 1/q))]^2} \right\} \\
&= \mathbb{E} \left\{ \partial_1 \frac{-2\beta t \exp(2\beta t(x_1 - 1/q)) \exp(\beta t(m_j - 1/q))}{[\sum_{l=1}^q \exp(\beta t(m_l - 1/q))]^3} \right\} \\
&= \mathbb{E} \left\{ \frac{-4\beta^2 t^2 \exp(\beta t(x_j - 1/q)) \exp(2\beta t(x_1 - 1/q))}{[\sum_{l=1}^q \exp(\beta t(x_l - 1/q))]^6} \right. \\
&\quad \left. + \frac{6\beta^2 t^2 \exp(2\beta t(x_1 - 1/q)) \exp(2\beta t(x_j - 1/q))}{[\sum_{l=1}^q \exp(\beta t(x_l - 1/q))]^4} \right\} \\
&\leq 6\beta^2 t^2 \mathbb{E} \left\{ \frac{\exp(2\beta t(x_1 - 1/q)) \exp(2\beta t(x_j - 1/q))}{[\sum_{l=1}^q \exp(\beta t(x_l - 1/q))]^4} \right\} \\
&\leq 6\beta^2 t^2 \mathbb{E}[1] \\
&= 6\beta^2 t^2.
\end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}
& \partial_i \partial_j F(1/q + t(x - 1/q)) \\
&= \partial_i \partial_j \mathbb{E} \left[ \left( \frac{\exp(\beta t(x_1 - 1/q))}{\sum_{l=1}^q \exp(\beta t(x_l - 1/q))} \right)^2 \right] - \frac{1}{q^2} \\
&= \mathbb{E} \left\{ \partial_i \partial_j \frac{\exp(2\beta t(x_1 - 1/q))}{\left[ \sum_{l=1}^q \exp(\beta t(x_l - 1/q)) \right]^2} \right\} \\
&= \mathbb{E} \left\{ \partial_i \frac{-2\beta t \exp(2\beta t(x_1 - 1/q)) \exp(\beta t(m_j - 1/q))}{\left[ \sum_{l=1}^q \exp(\beta t(m_l - 1/q)) \right]^3} \right\} \\
&= \mathbb{E} \left\{ \frac{6\beta^2 t^2 \exp(2\beta t(x_1 - 1/q)) \exp(\beta t(x_j - 1/q)) \exp(\beta t(x_i - 1/q))}{\left[ \sum_{l=1}^q \exp(\beta t(x_l - 1/q)) \right]^4} \right\} \\
&= 6\beta^2 t^2 \mathbb{E} \left\{ \frac{\exp(2\beta t(x_1 - 1/q)) \exp(\beta t(x_j - 1/q)) \exp(\beta t(x_i - 1/q))}{\left[ \sum_{l=1}^q \exp(\beta t(x_l - 1/q)) \right]^4} \right\} \\
&\leq 6\beta^2 t^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}[F(m)] &\leq \sup_{t \in [0,1]} \frac{2}{3} \cdot 6\beta^2 t^2 \sum_{1 \leq i < j \leq q} \mathbb{E}[(m_i - 1/q)(m_j - 1/q)] + \frac{1}{3} \cdot 6\beta^2 t^2 \sum_{i=1}^q \mathbb{E}[(m_i - 1/q)^2] \\
&\leq 6\beta^2 \sum_{1 \leq i < j \leq q} \mathbb{E}[(m_i - 1/q)(m_j - 1/q)] \\
&\leq 6\beta^2 \sum_{1 \leq i < j \leq q} \mathbb{E} \left[ \frac{W_i}{\sqrt{n}} \frac{W_j}{\sqrt{n}} \right].
\end{aligned}$$

But

$$\mathbb{E}[W_i W_j] \leq |\mathbb{E}[W_i W_j]| \leq \max_{i \in [q]} \mathbb{E}[W_i^2] = \mathcal{O}(1), \quad (11)$$

and so

$$\mathbb{E}[F(m)] = \mathcal{O}\left(\frac{1}{n}\right) \Rightarrow n \text{Var} \left( \frac{\exp(\beta m_i)}{\sum_{l=1}^q \exp(\beta m_l)} \right) = \mathcal{O}(1). \quad (12)$$

Now consider  $\text{Var} \left( m_j \frac{\exp(\beta m_i)}{\sum_{l=1}^q \exp(\beta m_l)} \right)$ . Define  $G : \Delta_q \rightarrow \mathbb{R}$  via

$$G(x) = \left( x_j \frac{\exp(\beta(x_i - 1/q))}{\sum_{l=1}^q \exp(\beta(x_l - 1/q))} \right)^2 - \frac{1}{q^4}, \quad \text{for some } i, j \in [q],$$

and  $g : \Delta_q \rightarrow \mathbb{R}$  via

$$g(x) = x_j \frac{\exp(\beta x_i)}{\sum_{l=1}^q \exp(\beta x_l)}$$

Then we Taylor expand  $G$  to the second order around  $\frac{1}{q}$ , so

$$G(x) = G(1/q) + \sum_{k \in [q]} \partial_k G(1/q)(x_k - 1/q) + 2 \sum_{|\alpha|=2} \frac{(x - 1/q)^\alpha}{(\alpha - 1)!} \int_0^1 (1-t)^2 \partial_\alpha G[1/q + t(x - 1/q)] dt$$

Then take the variance of  $g$ , we have

$$\text{Var}(g(m)) = \mathbb{E}[g^2(m)] - (\mathbb{E}[g(m)])^2$$

$$\begin{aligned}
&= \mathbb{E}[G(m)] - (\mathbb{E}[g(m)])^2 + \frac{1}{q^4} \\
&= G(1/q) + \sum_{j \in [q]} \partial_j G(1/q) \mathbb{E}[m_j - 1/q] + 2 \sum_{|\alpha|=2} \mathbb{E} \left[ \frac{(m-1/q)^\alpha}{(\alpha-1)!} \right] \int_0^1 (1-t)^2 \partial_\alpha G[1/q + t(m-1/q)] dt \\
&\quad - \mathbb{E}[g(m)]^2 + \frac{1}{q^4} \\
&= 2 \sum_{|\alpha|=2} \mathbb{E} \left[ \frac{(m-1/q)^\alpha}{(\alpha-1)!} \right] \int_0^1 (1-t)^2 \partial_\alpha G[1/q + t(m-1/q)] dt - \mathbb{E}[g(m)]^2 + \frac{1}{q^4} \\
&= \mathcal{O}(n^{-1}) - \mathbb{E}[g(m)]^2 + \frac{1}{q^4}
\end{aligned}$$

Now, to show

$$\mathbb{E}[g(m)] = \mathbb{E} \left[ m_j \frac{\exp(\beta m_i)}{\sum_{l=1}^q \exp(\beta m_l)} \right] = \frac{1}{q^2} + \mathcal{O}(n^{-1}).$$

First, taylor expand  $g(x)$  around  $\frac{1}{q}$  to the second order,

$$g(x) = g(1/q) + \sum_{k \in [q]} \partial_k g(1/q) (x_k - 1/q) + 2 \sum_{|\alpha|=2} \frac{(x-1/q)^\alpha}{(\alpha-1)!} \int_0^1 (1-t)^2 \partial_\alpha g(1/q + t(x-1/q)) dt.$$

Then consider the expectation, the second summand vanishes by (10),

$$\mathbb{E}[g(m)] = \frac{1}{q^2} + 2 \sum_{|\alpha|=2} \mathbb{E} \left[ \frac{(m-1/q)^\alpha}{(\alpha-1)!} \right] \int_0^1 (1-t)^2 \partial_\alpha g(tm + (1-t)1/q) dt.$$

But, for  $x \in \Delta_q$ , and  $k_1 \neq k_2$ ,

$$\begin{aligned}
\partial_{k_2} \partial_{k_1} g(x) &= \partial_{k_2} \left\{ \delta_{k_1 j} \frac{\exp(\beta x_i)}{\sum_{l=1}^q \exp(\beta x_l)} + x_j \frac{\beta \delta_{k_1 i} \exp(\beta x_i) (\sum_{l=1}^q \exp(\beta x_l)) - \beta \exp(\beta x_i + \beta x_{k_1})}{(\sum_{l=1}^q \exp(\beta m_l))^2} \right\} \\
&= \partial_{k_2} \left\{ \delta_{k_1 j} \frac{\exp(\beta x_i)}{\sum_{l=1}^q \exp(\beta x_l)} + x_j \frac{\beta \delta_{k_1 i} \exp(\beta x_i)}{\sum_{l=1}^q \exp(\beta m_l)} - x_j \frac{\beta \exp(\beta x_i + \beta x_{k_1})}{(\sum_{l=1}^q \exp(\beta m_l))^2} \right\} \\
&= \delta_{k_1 j} \frac{\beta \delta_{k_2 i} \exp(\beta x_i) (\sum_{l=1}^q \exp(\beta x_l)) - \beta \exp(\beta x_i + \beta x_{k_2})}{(\sum_{l=1}^q \exp(\beta x_l))^2} \\
&\quad + \delta_{k_2 j} \frac{\beta \delta_{k_1 i} \exp(\beta x_i) (\sum_{l=1}^q \exp(\beta x_l)) - \beta \exp(\beta x_i + \beta x_{k_1})}{(\sum_{l=1}^q \exp(\beta x_l))^2} \\
&\quad + x_j \frac{\beta \delta_{k_1 i} \exp(\beta x_i + \beta x_{k_2}) (\sum_{l=1}^q \exp(\beta x_l))^2}{(\sum_{l=1}^q \exp(\beta x_l))^4} \\
&\quad - x_j \frac{[\beta \delta_{k_1 i} \exp(\beta x_i) (\sum_{l=1}^q \exp(\beta x_l)) - \beta \exp(\beta x_i + \beta x_{k_2})] \cdot 2 (\sum_{l=1}^q \exp(\beta x_l)) \exp(\beta x_{k_2}) \beta}{(\sum_{l=1}^q \exp(\beta x_l))^4} \\
&\leq \delta_{k_1 j} \frac{\beta \delta_{k_2 i} \exp(\beta x_i)}{(\sum_{l=1}^q \exp(\beta x_l))} + \delta_{k_2 j} \frac{\beta \delta_{k_1 i} \exp(\beta x_i)}{(\sum_{l=1}^q \exp(\beta x_l))} + x_j \frac{\beta \delta_{k_1 i} \exp(\beta x_i + \beta k_2)}{(\sum_{l=1}^q \exp(\beta x_l))^2} \\
&= \beta \left[ \frac{\delta_{k_1 j} \delta_{k_2 i} \exp(\beta x_i) + \delta_{k_2 j} \delta_{k_1 i} \exp(\beta x_i)}{\sum_{l=1}^q \exp(\beta x_l)} + x_j \frac{\delta_{k_1 i} \exp(\beta x_i + \beta k_2)}{(\sum_{l=1}^q \exp(\beta x_l))^2} \right] \\
&\leq 2\beta;
\end{aligned}$$

and, for  $k_1 = k_2$ ,

$$\partial_k^2 g(x) = \partial_k \left\{ \delta_{k j} \frac{\exp(\beta x_i)}{\sum_{l=1}^q \exp(\beta x_l)} + x_j \frac{\beta \delta_{k i} \exp(\beta x_i) (\sum_{l=1}^q \exp(\beta x_l)) - \beta \exp(\beta x_i + \beta x_k)}{(\sum_{l=1}^q \exp(\beta m_l))^2} \right\}$$

$$\begin{aligned}
&= \partial_k \left\{ \delta_{kj} \frac{\exp(\beta x_i)}{\sum_{l=1}^q \exp(\beta x_l)} + x_j \frac{\beta \delta_{ki} \exp(\beta x_i)}{\sum_{l=1}^q \exp(\beta m_l)} - x_j \frac{\beta \exp(\beta x_i + \beta x_k)}{(\sum_{l=1}^q \exp(\beta m_l))^2} \right\} \\
&= \delta_{kj} \frac{\beta \delta_{ki} \exp(\beta x_i) (\sum_{l=1}^q \exp(\beta x_l)) - \beta \exp(\beta x_i + \beta x_k)}{(\sum_{l=1}^q \exp(\beta x_l))^2} \\
&\quad + \delta_{kj} \frac{\beta \delta_{ki} \exp(\beta x_i) (\sum_{l=1}^q \exp(\beta x_l)) - \beta \exp(\beta x_i + \beta x_k)}{(\sum_{l=1}^q \exp(\beta x_l))^2} \\
&\quad + x_j \frac{\beta \delta_{ki} \exp(\beta x_i + \beta x_k) (\sum_{l=1}^q \exp(\beta x_l))^2}{(\sum_{l=1}^q \exp(\beta x_l))^4} \\
&\quad - x_j \frac{[\beta \delta_{ki} \exp(\beta x_i) (\sum_{l=1}^q \exp(\beta x_l)) - \beta \exp(\beta x_i + \beta x_k)] \cdot 2 (\sum_{l=1}^q \exp(\beta x_l)) \exp(\beta x_k) \beta}{(\sum_{l=1}^q \exp(\beta x_l))^4} \\
&= -2 \frac{\beta \delta_{kj} \exp(\beta x_i + \beta x_k)}{(\sum_{l=1}^q \exp(\beta x_l))^2} + x_j \frac{\beta \delta_{ki} \exp(\beta x_i + \beta x_k)}{(\sum_{l=1}^q \exp(\beta x_l))^2} \\
&\quad - x_j \frac{[\beta \delta_{ki} \exp(\beta x_i) (\sum_{l=1}^q \exp(\beta x_l)) - \beta \exp(\beta x_i + \beta x_k)] \cdot 2 (\sum_{l=1}^q \exp(\beta x_l)) \exp(\beta x_k) \beta}{(\sum_{l=1}^q \exp(\beta x_l))^4} \\
&\leq \beta \left( x_j \frac{\delta_{ki} \exp(\beta x_i + \beta x_k)}{(\sum_{l=1}^q \exp(\beta x_l))^2} \right) \\
&\leq \beta < 2\beta.
\end{aligned}$$

So,

$$\begin{aligned}
\mathbb{E}[g(m)] &\leq \frac{1}{q^2} + 4\beta \sum_{|\alpha|=2} \mathbb{E}[(m-1/q)^\alpha] \int_0^1 (1-t)^2 dt \\
&= \frac{1}{q^2} + \frac{4\beta}{3} \left[ \sum_{1 \leq i < j \leq q} \mathbb{E}[(m_i - 1/q)(m_j - 1/q)] \right] + \frac{2\beta}{3} \sum_{i=1}^q \mathbb{E}[(m_i - 1/q)^2] \\
&\leq \frac{1}{q^2} + 4\beta \sum_{1 \leq i < j \leq q} \mathbb{E}[(m_i - 1/q)(m_j - 1/q)] \\
&\leq \frac{1}{q^2} + 4\beta \sum_{1 \leq i < j \leq q} \mathbb{E} \left[ \frac{W_i}{\sqrt{n}} \frac{W_j}{\sqrt{n}} \right] \\
&= \frac{1}{q^2} + \frac{4\beta}{n} \sum_{1 \leq i < j \leq q} \mathbb{E}[W_i W_j].
\end{aligned}$$

We also know from (11),  $\mathbb{E}[W_i W_j] = \mathcal{O}(1)$  for all  $i$  and  $j$ . So,

$$\mathbb{E}[g(m)] = \frac{1}{q^2} + \mathcal{O}\left(\frac{1}{n}\right).$$

Then we have

$$\text{Var}(g(m)) = \mathcal{O}(n^{-1}) \quad \Rightarrow \quad \text{Var}\left(\sqrt{nm_j} \frac{\exp(\beta m_i)}{\sum_{l=1}^q \exp(\beta m_l)}\right) = \mathcal{O}(1). \quad (13)$$

Now,

$$\begin{aligned}
&\text{Var} \left[ \frac{1}{\sqrt{n}} \sum_{i=1}^n Y_{t,i} \left( \frac{\exp(\beta m_{j,t})}{\sum_{l=1}^q \exp(\beta m_{l,t})} - \frac{\exp(\beta m_j)}{\sum_{l=1}^q \exp(\beta m_l)} \right) \right] \\
&\leq 4 \max \left\{ \text{Var} \left[ \frac{1}{\sqrt{n}} \sum_{i=1}^n \delta(\sigma_t, i) \left( \frac{\exp(\beta m_{j,t})}{\sum_{l=1}^q \exp(\beta m_{l,t})} \right) \right], \text{Var} \left[ \frac{1}{\sqrt{n}} \sum_{i=1}^n \delta(\sigma_t, i) \left( \frac{\exp(\beta m_j)}{\sum_{l=1}^q \exp(\beta m_l)} \right) \right] \right\}
\end{aligned}$$

$$\leq 4 \max \left\{ n \text{Var} \left[ \frac{\exp(\beta m_{j,t})}{\sum_{l=1}^q \exp(\beta m_{l,t})} \right], n \text{Var} \left[ \frac{\exp(\beta m_j)}{\sum_{l=1}^q \exp(\beta m_l)} \right] \right\}.$$

We know, from above, that

$$n \text{Var} \left[ \frac{\exp(\beta m_j)}{\sum_{l=1}^q \exp(\beta m_l)} \right] = \mathcal{O}(1).$$

Now consider  $n \text{Var} \left[ \frac{\exp(\beta m_{j,t})}{\sum_{l=1}^q \exp(\beta m_{l,t})} \right]$ . Note that

$$m_{j,t}(\sigma) = \frac{1}{n} \sum_{k \neq t}^n \delta(\sigma_k, j) = \frac{1}{n} \sum_{k=1}^n \delta(\sigma_k, j) - \frac{\delta(\sigma_t, j)}{n} = m_j(\sigma) - \frac{\delta(\sigma_t, j)}{n}.$$

Now, if  $\sigma_t = j$ , then  $\delta(\sigma_t, l) = 0$  for all  $l \neq j$ , and

$$\text{Var} \left[ \frac{\exp(\beta m_{j,t})}{\sum_{l=1}^q \exp(\beta m_{l,t})} \right] = \text{Var} \left[ \frac{\exp(\beta m_j) \exp\left(-\frac{\beta}{n} \delta(\sigma_t, j)\right)}{\sum_{l=1}^q \exp(\beta m_l) \exp\left(-\frac{\beta}{n} \delta(\sigma_t, l)\right)} \right] = \text{Var} \left[ \frac{\exp(\beta m_j) \exp\left(-\frac{\beta}{n}\right)}{\sum_{l \neq j}^q \exp(\beta m_l) + \exp\left(\beta m_j - \frac{\beta}{n}\right)} \right];$$

if  $\sigma_t = k \neq j$ , then  $\delta(\sigma_t, l) = 0$  for all  $l \neq j$ , and

$$\text{Var} \left[ \frac{\exp(\beta m_{j,t})}{\sum_{l=1}^q \exp(\beta m_{l,t})} \right] = \text{Var} \left[ \frac{\exp(\beta m_j) \exp\left(-\frac{\beta}{n} \delta(\sigma_t, j)\right)}{\sum_{l=1}^q \exp(\beta m_l) \exp\left(-\frac{\beta}{n} \delta(\sigma_t, l)\right)} \right] = \text{Var} \left[ \frac{\exp(\beta m_j)}{\sum_{l \neq j}^q \exp(\beta m_l) + \exp\left(\beta m_k - \frac{\beta}{n}\right)} \right].$$

So, we can see that,

$$\text{Var} \left[ \frac{\exp(\beta m_{j,t})}{\sum_{l=1}^q \exp(\beta m_{l,t})} \right] \rightarrow \text{Var} \left[ \frac{\exp(\beta m_j)}{\sum_{l=1}^q \exp(\beta m_l)} \right], \quad n \rightarrow \infty.$$

Therefore, by definition,

$$\text{Var} \left[ \frac{\exp(\beta m_{j,t})}{\sum_{l=1}^q \exp(\beta m_{l,t})} \right] = \mathcal{O} \left( \text{Var} \left[ \frac{\exp(\beta m_j)}{\sum_{l=1}^q \exp(\beta m_l)} \right] \right) = \mathcal{O} \left( \mathcal{O} \left( \frac{1}{n} \right) \right) = \mathcal{O} \left( \frac{1}{n} \right).$$

So,

$$n \text{Var} \left[ \frac{\exp(\beta m_{j,t})}{\sum_{l=1}^q \exp(\beta m_{l,t})} \right] = \mathcal{O}(1).$$

Therefore,

$$\text{Var} \left[ \frac{1}{\sqrt{n}} \sum_{t=1}^n Y_{t,i} \left( \frac{\exp(\beta m_{j,t})}{\sum_{l=1}^q \exp(\beta m_{l,t})} - \frac{\exp(\beta m_j)}{\sum_{l=1}^q \exp(\beta m_l)} \right) \right] = \mathcal{O}(1). \quad (14)$$

Hence, by (9), (12), (13), and (14), we have

$$\text{Var} (\mathbb{E} [(W'_i - W_i)(W'_j - W_j) | \mathcal{F}]) = \mathcal{O} \left( \frac{1}{n^3} \right).$$

Then, it follows that (3) is in  $\mathcal{O}(n^{-1/2})$ , which then concludes the proof.

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