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Random Walks and their Applications to Mathematical Physics

Liam Wood-Baker

Supervised by Andrea Collevecchio, Kais Hamza

Monash University

AMSI AUSTRALIAN MATHEMATICAL SCIENCES INSTITUTE

Abstract

This report is a summary of the first chapter of *Markov Processes: Theorems and Problems* by Dynkin and Yushkevich (1969). It introduces key ideas, notation, and theorems in the study of Random Walks in a fashion which is accessible to the undergraduate student.

1 Introduction

A random walk is a random process wherein a particle moves through a discrete lattice by making one-unit jumps. This paper will explore the behaviour of such random walks, culminating in a criterion which determines whether or not a set of points will receive infinitely many visits from the particle.

Statement of Authorship

This report is built on the work of Dynkin and Yushkevich. Special thanks to Andrea Collevecchio and Kais Hamza for assisting me with some of the particularly difficult parts of the text.

2 Symmetric Random Walk

Consider a particle moving along the integer-valued points of the x-axis by making one-unit jumps. If at each jump the probability of moving to the left or to the right are equal, then we say that the particle executes a symmetric random walk on a line.

We will show that a particle starting from an arbitrary position will with probability one eventually reach any other state. It is sufficient to show that a particle leaving any state will at some point reach 0. Let $\pi(x)$ be the probability of hitting 0 from a point x. Clearly $\pi(0) = 1$, and by the Law of Total Probability,

$$\pi(x) = \frac{1}{2}\pi(x-1) + \frac{1}{2}\pi(x+1) \tag{1}$$

for $x \neq 0$. Consider the graph of $\pi(x)$ on $x = 0, 1, 2, \ldots$. It follows from equation (1) that

$$\pi(x) - \pi(x-1) = \pi(x+1) - \pi(x),$$

In other words, the slope of the line joining $\pi(x-1)$ to $\pi(x)$ is the same as that of the line joining $\pi(x)$ to $\pi(x+1)$, for any $x = 1, 2, \ldots$. Therefore, all of the points on the graph $\pi(x)$ lie on a straight line. Since $\pi(0) = 1$, this line must go through the point $(0, 1)$. If the gradient of this line were positive (or negative), then for some sufficiently large x we would have $\pi(x) > 0$ (or $\pi(x) < 0$). But $\pi(x)$ is a probability, so cannot escape $[0, 1]$. Therefore, $\pi(x) = 1$ for all $x \geq 0$.

Due to the symmetry of the random walk, we can make the same argument for $x = 0, -1, -2, \ldots$, and find likewise that $\pi(x) = 1$ for all $x \leq 0$. We conclude that for any initial state, the probability of eventually reaching zero is one.

We now consider a generalisation of the random walk on a line: a random walk on an l-dimensional integer valued lattice H^l. If $\boldsymbol{e_1}, \ldots, \boldsymbol{e_l}$ is an orthonormal basis of an l-dimensional space, and $x_1, \ldots, x_l$ are integers, then the lattice H^l consists of points of the form

$$\boldsymbol{x} = x_1\boldsymbol{e_1} + \cdots + x_n\boldsymbol{e_n}$$

Any point which neighbours $\boldsymbol{x}$ will be of the same form, but with a single x value incremented or decremented by one. Thus, each point on the lattice has $2l$ neighbours, and at each step of a random walk the particle has a $\frac{1}{2l}$ chance of moving to any one of those neighbours.

3 The Transition Function

Let $X(0)$ represent the initial position of a particle in a random walk, and $X(n)$ represent its position after n steps ($n = 1, 2, 3, \ldots$). In the course of a random walk, the probability of some event A taking place depends on the point $\boldsymbol{x}$ from which the walk began. Call this probability $\mathbb{P}_{\boldsymbol{x}}(A)$. If ξ is the random variable corresponding to the distribution $\mathbb{P}_{\boldsymbol{x}}$, then call the expected value of this random variable $\mathbb{E}_{\boldsymbol{x}}[\xi]$.

Let $p(n, \boldsymbol{x}, \boldsymbol{y})$ be the probability that a particle leaving $\boldsymbol{x}$ will, in precisely n steps, reach $\boldsymbol{y}$:

$$p(n, \boldsymbol{x}, \boldsymbol{y}) = \mathbb{P}_{\boldsymbol{x}}(X(n) = \boldsymbol{y}).$$

$p(n, \boldsymbol{x}, \boldsymbol{y})$ is called the transition function. Some obvious properties are that $p(0, \boldsymbol{x}, \boldsymbol{x}) = 1,\ p(0, \boldsymbol{x}, \boldsymbol{y}) = 0$ for $\boldsymbol{x} \neq \boldsymbol{y}$, and $\sum_{y \in H^l} p(n, \boldsymbol{x}, \boldsymbol{y}) = 1$. The quantity

$$\sum_{y \in B} p(n, \boldsymbol{x}, \boldsymbol{y}) = \mathbb{P}_{\boldsymbol{x}}(X(n) \in B)$$

where B is some set in l-dimensional space, is called the transition probability from $\boldsymbol{x}$ to B in n steps.

An important property of the random walk is that the the jumps $\boldsymbol{\xi}_k = X(k) - X(k-1)$ $(k = 1, 2, \ldots)$ are independent (with each other and with the starting position $X(0)$) and identically distributed. In particular, any of the vectors $\boldsymbol{\xi}_k$ assumes with probability $\frac{1}{2l}$ one of the values $\pm\boldsymbol{e_1}, \ldots, \pm\boldsymbol{e_l}$. We use this fact to derive an integral representation for $p(n, \boldsymbol{x}, \boldsymbol{y})$.

Let $\theta(x)$ be defined as follows: if the vector $\boldsymbol{x} = x_1\boldsymbol{e}_1 + \cdots + x_l\boldsymbol{e}_l$, then $\theta(\boldsymbol{x}) = \theta_1 x_1 + \cdots + \theta_l x_l$ ($\theta(\boldsymbol{x})$ is called a linear form). We make use of the characteristic function of the random vector $X(n)$, which we will call $F(\theta)$:

$$F(\theta) = \mathbb{E}_{\boldsymbol{x}}[e^{i\theta(X(n))}] = \sum_{\boldsymbol{y}} p(n, \boldsymbol{x}, \boldsymbol{y}) e^{i\theta(\boldsymbol{x})}. \tag{2}$$

Our strategy is to multiply both sides of equation (2) by $e^{-i\theta(\boldsymbol{z})}$, where $\boldsymbol{z}$ is a point on H^l, and then integrate over an appropriate region $Q = \{\theta(\boldsymbol{z}) : |\theta_i| \leq \pi \text{ for all } 1 \leq i \leq l\}$. Consider the following integral,

$$\begin{aligned}
\int_Q e^{i\theta(\boldsymbol{y}) - i\theta(\boldsymbol{z})} d\boldsymbol{\theta} &= \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} e^{i[\theta_1(y_1 - z_1) + \cdots + \theta_l(y_l - z_l)]} d\theta_1 \ldots d\theta_l \\
&= \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \prod_{k=1}^{l} e^{i\theta_k(y_k - z_k)} d\theta_1 \ldots d\theta_l \\
&= \prod_{k=1}^{l} \int_{-\pi}^{\pi} e^{i\theta_k(y_k - z_k)} d\theta_k \text{ , as each } \theta_k \text{ is independent of the others.}
\end{aligned}$$

If $\boldsymbol{y} = \boldsymbol{z}$, then $y_k = z_k$ for each k between 1 and l, and so each integrand is equal to 1. Integrating 1 over the region $[-\pi, \pi]$ yields 2π, so we are left with

$$\prod_{k=1}^{l} (2\pi) = (2\pi)^l.$$

On the other hand, if $\boldsymbol{y} \neq \boldsymbol{z}$, then

$$\begin{aligned}\prod_{k=1}^{l} \int_{-\pi}^{\pi} e^{i\theta_k(y_k - z_k)} d\theta_k &= \prod_{k=1}^{l} \int_{-\pi}^{\pi} \cos((y_k - z_k)\theta_k) + i\sin((y_k - z_k)\theta_k) d\theta_k \\ &= \prod_{k=1}^{l} \left[\frac{1}{y_k - z_k}\sin((y_k - z_k)\theta_k) - \frac{i}{y_k - z_k}\cos((y_k - z_k)\theta_k)\right]_{-\pi}^{\pi},\end{aligned}$$

notice that because $\boldsymbol{y}$ and $\boldsymbol{z}$ are points on the integer-valued lattice H^l, the value of $y_k - z_k$ must be an integer. Therefore, after the substitution $\theta_k = \pm\pi$, we can guarantee that $\sin((y_k - z_k)\theta_k) = 0$. Since $\cos((y_k - z_k)\theta_k) = \cos(-(y_k - z_k)\theta_k)$, we conclude that

$$\int_Q e^{i\theta(y) - i\theta(z)} d\boldsymbol{\theta} = \begin{cases} (2\pi)^l & \text{for } \boldsymbol{y} = \boldsymbol{z} \\ 0 & \text{for } \boldsymbol{y} \neq \boldsymbol{z} \end{cases}.$$

Returning to equation (2), we multiply by $e^{-i\theta(\boldsymbol{z})}$ and integrate over Q. Keep in mind that the series $\sum_{\boldsymbol{y}} p(n, \boldsymbol{x}, \boldsymbol{y}) e^{i\theta(\boldsymbol{y})}$ contains only finitely many nonzero terms, since in n steps a random walk can visit no more than $(2l)^n$ states. Thus,

$$\begin{aligned}\int_Q F(\theta) e^{-i\theta(\boldsymbol{z})} d\boldsymbol{\theta} &= \int_Q \sum_{\boldsymbol{y}} p(n, \boldsymbol{x}, \boldsymbol{y}) e^{i\theta(\boldsymbol{y})} e^{-i\theta(\boldsymbol{z})} d\boldsymbol{\theta} \\ &= \sum_{\boldsymbol{y}} p(n, \boldsymbol{x}, \boldsymbol{y}) \int_Q e^{i\theta(\boldsymbol{y}) - i\theta(\boldsymbol{z})} d\boldsymbol{\theta} \\ &= \sum_{\boldsymbol{y}} p(n, \boldsymbol{x}, \boldsymbol{y}) \cdot \begin{cases} (2\pi)^l & \text{for } \boldsymbol{y} = \boldsymbol{z} \\ 0 & \text{for } \boldsymbol{y} \neq \boldsymbol{z}. \end{cases}\end{aligned}$$

Since the sum vanishes for any $\boldsymbol{y} \neq \boldsymbol{z}$, we see that

$$p(n, \boldsymbol{x}, \boldsymbol{z}) = \frac{1}{(2\pi)^l} \int_Q F(\theta) e^{-i\theta(\boldsymbol{z})} d\boldsymbol{\theta}. \tag{3}$$

Let $\boldsymbol{\xi}_k$ be the jump at the kth step. Then $X(n) = X(0) + \sum_{k=1}^{n} \boldsymbol{\xi}_k$, and

$$F(\theta) = \mathbb{E}_{\boldsymbol{x}}[e^{i\theta(X(n))}] = \mathbb{E}_{\boldsymbol{x}}[e^{i\theta(X(0) + \sum_{k=1}^{n} \boldsymbol{\xi}_k)}] = \mathbb{E}_{\boldsymbol{x}}\left[e^{i\theta(X(0))} \prod_{k=1}^{n} e^{i\theta \boldsymbol{\xi}_k}\right].$$

Since $X(0) = \boldsymbol{x}$ with probability one, and the random vectors $\boldsymbol{\xi}_k$ are independent and identically distributed,

$$F(\theta) = e^{i\theta(\boldsymbol{x})} \mathbb{E}_x[e^{i\theta(\boldsymbol{\xi}_1)}]^n. \tag{4}$$

Consider $\Phi(\theta) = \mathbb{E}_x[e^{i\theta(\boldsymbol{\xi}_1)}]$. Since $\boldsymbol{\xi}_1$ takes any of the values $\pm\boldsymbol{e}_1, \ldots, \pm\boldsymbol{e}_l$ with probability $\frac{1}{2l}$,

$$\begin{aligned}\Phi(\theta) &= \frac{1}{2l} \sum_{m=1}^{l} \left(e^{i\theta_m} + e^{-i\theta_m}\right) = \frac{1}{2l} \sum_{m=1}^{l} (\cos(i\theta_m) + i\sin(i\theta_m) + \cos(-i\theta_m) + i\sin(-i\theta_m)) \\ &= \frac{1}{l} \sum_{m=1}^{l} cos(\theta_m)\end{aligned} \tag{5}$$

Substituting these results into equation (3) and replacing $\boldsymbol{z}$ with $\boldsymbol{y}$, we obtain

$$p(n, \boldsymbol{x}, \boldsymbol{y}) = \frac{1}{(2\pi)^l} \int_Q e^{i\theta(\boldsymbol{x}-\boldsymbol{y})} \Phi^n(\theta) d\boldsymbol{\theta}. \tag{6}$$

4 Behaviour of the walk as $n \to \infty$

We now assume that $l \geq 3$. We will show that the length of the vector $X(n)$ tends to infinity with probability one as $n \to \infty$. A set B in the lattice H^l is *recurrent* if $\pi_B(\boldsymbol{x}) = 1$ for all $\boldsymbol{x} \in H^l$, i.e., if the probability of reaching H from $\boldsymbol{x}$ is one, and *nonrecurrent* if $\pi_B(\boldsymbol{x}) < 1$. We will show that any bounded set is nonrecurrent.

Consider a sequence of trials consisting of random walks initiated from a point $\boldsymbol{x}$, in which the nth trial is a success if $X(n) = \boldsymbol{y}$, and a failure otherwise. Then the probability of success in the nth trial is $p(n, \boldsymbol{x}, \boldsymbol{y})$, and

$$g(\boldsymbol{x}, \boldsymbol{y}) = \sum_{n=0}^{\infty} p(n, \boldsymbol{x}, \boldsymbol{y}) \tag{7}$$

represents the expected number of visits to the point $\boldsymbol{y}$.

We will prove that

$$g(\boldsymbol{x}, \boldsymbol{y}) < \infty. \tag{8}$$

The function $\Phi(\theta)$ defined by equation (5) is continuous, and $|\Phi(\theta)| < 1$ on all of Q except at points of the form $(\pm n\pi, \ldots, \pm n\pi)$ where $n = 0$ or 1. Therefore, starting with equation (7),

$$\begin{aligned} g(\boldsymbol{x}, \boldsymbol{y}) &= \sum_{n=0}^{\infty} p(n, \boldsymbol{x}, \boldsymbol{y}) \\ g(\boldsymbol{x}, \boldsymbol{y}) &= \sum_{n=0}^{\infty} \frac{1}{(2\pi)^l} \int_Q e^{i\theta(\boldsymbol{x}-\boldsymbol{y})} \Phi^n(\theta) d\boldsymbol{\theta} \text{ from (6), and therefore} \\ (2\pi)^l g(\boldsymbol{x}, \boldsymbol{y}) &= \sum_{n=0}^{\infty} \int_Q e^{i\theta(\boldsymbol{x}-\boldsymbol{y})} \Phi^n(\theta) d\boldsymbol{\theta}. \end{aligned}$$

Since $|e^{i\alpha}| \leq 1$ for any real α,

$$(2\pi)^l g(\boldsymbol{x}, \boldsymbol{y}) \leq \sum_{n=0}^{\infty} \int_Q |\Phi^n(\theta)| d\boldsymbol{\theta}.$$

We now make use of the fact that $|\Phi(\theta)| < 1$. The sum $\sum_{n=0}^{\infty} |\Phi^n(\theta)|$ is a geometric series which converges absolutely, and thus

$$(2\pi)^l g(\boldsymbol{x}, \boldsymbol{y}) \leq \int_Q \frac{1}{1 - |\Phi(\theta)|} d\theta. \tag{9}$$

To show the convergence of this integral, consider a neighbourhood, U, of the point $\theta = (0, \ldots, 0)$, in which

$$0 < \cos\theta_m \leq 1 - \frac{\theta_m^2}{4}$$

(To see that such a neighbourhood must exist, consider the Taylor Series expansion of cos). Plugging this into equation (5) we see that

$$\begin{aligned} |\Phi(\theta)| = \Phi(\theta) &= \frac{1}{l}\sum_{m=1}^{l}\cos(\theta_m) \\ &\leq \frac{1}{l}\sum_{m=1}^{l}\left(1-\frac{\theta_m^2}{4}\right) \\ &= 1-\frac{1}{4l}\sum_{m=1}^{l}\theta_m^2 \end{aligned}$$

Consequently,

$$\int_U \frac{1}{1-|\Phi(\theta)|}d\theta < \int_U \frac{4l}{\theta_1^2+\cdots+\theta_l^2}d\theta$$

We show the convergence of this integral by switching to (l-dimensional) polar co-ordinates,

$$\int_U \frac{4l}{\theta_1^2+\cdots+\theta_l^2}d\theta = \int_U \frac{4l}{r^2}r^{l-1}\sin^{l-2}(\varphi_1)\sin^{l-3}(\varphi_2)\ldots\sin(\varphi_{l-2})drd\varphi_1 d\varphi_2\ldots d\varphi_{l-1}.$$

The integral can be split into one-dimensional integrals in each variable. Each integral with respect to one of the angles φ is finite, which leaves only the integral with respect to r. If a is the radius of a sphere which completely contains U, then

$$\int_0^a \frac{4l}{r^2}r^{l-1}dr = \int_0^a 4lr^{l-3}dr < \infty,$$

so long as $l \geq 3$, which we have assumed earlier. That this integral is finite implies that the integral

$$\int_U \frac{4l}{\theta_1^2+\cdots+\theta_l^2}d\theta$$

is also finite. An analogous argument shows the convergence of the integral (9) in the neighbourhoods of the points $\theta = (\pm\pi,\ldots,\pm\pi)$. Thus,

$$\int_Q \frac{1}{1-|\Phi(\theta)|}d\theta < \infty. \tag{10}$$

This proves the inequality (8), i.e., that we will with probability one visit the point $\boldsymbol{y}$ only finitely many times. Since the choice of $\boldsymbol{y}$ is arbitrary, a particle on a random walk therefore has a probability one of occupying any given point on the lattice only finitely many times. The probability is one, therefore, that for any bounded set of lattice points, there will come a time after which the particle will never visit that set again.

We now prove the nonrecurrence of any bounded set B. First, suppose that B is recurrent. For any initial state $\boldsymbol{x}$, and any n, the probability of the event $A_n = \{\text{The particle visits } B \text{ after the nth step}\}$ is

$$\sum_y p(n,\boldsymbol{x},\boldsymbol{y})\pi_B(\boldsymbol{y}),$$

the probability of reaching some other point $\boldsymbol{y}$ in n steps, and from there visiting B. But B is recurrent, so $\pi_B(\boldsymbol{y}) = 1$ for any $\boldsymbol{y}$ on the lattice. So this probability is simply

$$\sum_y p(n,\boldsymbol{x},\boldsymbol{y}),$$

which is equal to one by the law of total probability. Therefore, the particle is guaranteed to visit B after time n, for any n. But this contradicts the fact that the particle will with probability one at some time leave B and not return, and so we have by contradiction that any bounded set must be nonrecurrent.

It follows from equations (9) and (10) that the series

$$e^{i\theta(\boldsymbol{x}-\boldsymbol{y})}\sum_{n=0}^{\infty}\Phi^n(\theta)$$

can be integrated term-by-term over Q. Therefore,

$$g(\boldsymbol{x},\boldsymbol{y})=\sum_{n=0}^{\infty}\frac{1}{(2\pi)^l}\int_Q e^{i\theta(\boldsymbol{x}-\boldsymbol{y})}\Phi^n(\theta)d\theta=\frac{1}{(2\pi)^l}\int_Q e^{i\theta(\boldsymbol{x}-\boldsymbol{y})}\sum_{n=0}^{\infty}\Phi^n(\theta)d\theta=\frac{1}{(2\pi)^l}\int_Q\frac{e^{i\theta(\boldsymbol{x}-\boldsymbol{y})}}{1-\Phi(\theta)}d\theta \tag{11}$$

This result can be used to prove that

$$g(\boldsymbol{x},\boldsymbol{y})\sim\frac{c}{|\boldsymbol{x}-\boldsymbol{y}|}\text{ for }|\boldsymbol{x}-\boldsymbol{y}|\to\infty, c>0. \tag{12}$$

However, the proof is too lengthy to include here. The full proof is available in Appendix 1 of Dynkin and Yushkevich.

5 Harmonic Functions and another Definition of Recurrence

Let f be a function on the points of the lattice H^l. We define the operator, P, as follows:

$$Pf(\boldsymbol{x})=\mathbb{E}_{\boldsymbol{x}}\left[f(X(1)\right]=\sum_{\boldsymbol{y}}p(1,\boldsymbol{x},\boldsymbol{y})f(\boldsymbol{y}). \tag{13}$$

P is called the “(one-step) shift operator”, or, since moving a step in each direction is equally likely, the “averaging operator”

$$Pf(x)=\frac{1}{2l}\sum_k f(x+\boldsymbol{e}_k),$$

where $k=\pm1,\pm2,\ldots,\pm l$, and $\boldsymbol{e}_{-k}=-\boldsymbol{e}_k$. We call a function f harmonic if $Pf=f$, and superharmonic if $Pf\leq f$. It is clear from the definition that any constant function will be harmonic. We will show that any bounded harmonic function is constant.

Let f be a bounded, harmonic function on H^l. It is quite easy to show that f is constant if f achieves a maximum at some point $\boldsymbol{y}_0$. Call the $2l$ neighbouring points $\boldsymbol{y}_1,\ldots,\boldsymbol{y}_{2l}$. Since f is harmonic,

$$Pf(\boldsymbol{y}_0)=f(\boldsymbol{y}_0)=\frac{1}{2l}\sum_{k=1}^{2l}f(\boldsymbol{y}_k)$$

$$\frac{1}{2l}\sum_{k=1}^{2l}(f(\boldsymbol{y}_k)-f(\boldsymbol{y}_0))=0.$$

In other words, the mean of the numbers $f(\boldsymbol{y}_0)-f(\boldsymbol{y}_k)$ is equal to zero. Now, consider the quantity $f(\boldsymbol{y}_0)-f(\boldsymbol{y}_k)$ for a particular k. It cannot be that $f(\boldsymbol{y}_0)-f(\boldsymbol{y}_k)<0$, as $f(\boldsymbol{y}_0)$ is the maximum value of f. But if none of these numbers can be negative, and their average is zero, then none of them can be positive either. Thus, $f(\boldsymbol{y}_0) = f(\boldsymbol{y}_k)$. Therefore, the set of points at which f reaches its maximum includes not only the points in that set, but also all of their neighbours. But then it must also include all of those neighbours' neighbours, and so on. The set of points at which f reaches its maximum is therefore the entirety of H^l, and so f is constant.

But what if f does not have a maximum? Since f is bounded, it has a least upper bound M, and for any $\varepsilon > 0$ there is a point $\boldsymbol{y}$ at which $f(\boldsymbol{y}) \geq M - \varepsilon$. If $\boldsymbol{y'}$ is a point neighbouring $\boldsymbol{y}$, then we can show that $f(\boldsymbol{y'}) \geq M - 2l\varepsilon$. Since f is harmonic and bounded,

$$f(\boldsymbol{y}) = \frac{1}{2l}\sum_k f(\boldsymbol{y} + \boldsymbol{e}_k) \geq M - \varepsilon$$

In order to minimise $f(\boldsymbol{y'})$, we set the value of f at each other neighbour of $\boldsymbol{y}$ to the largest value it can possibly take: M. Then,

$$f(\boldsymbol{y}) = \frac{1}{2l}(f(\boldsymbol{y'}) + (2l-1)M) \geq M - \varepsilon$$
$$f(\boldsymbol{y'}) \geq M - 2l\varepsilon$$

Hence, if $M > 0$, then we can pick a chain of points $\boldsymbol{y}_0,\ \boldsymbol{y}_1 = \boldsymbol{y}_0 + \boldsymbol{e}_1,\ \boldsymbol{y}_2 = \boldsymbol{y}_1 + \boldsymbol{e}_1, \dots,\ \boldsymbol{y}_n = \boldsymbol{y}_{n-1} + \boldsymbol{e}_1$ such that, for any number N,

$$s = f(\boldsymbol{y}_0) + f(\boldsymbol{y}_1) + \cdots + f(\boldsymbol{y}_n) \geq N$$

Keeping f as an arbitrary bounded harmonic function, define $\varphi(\boldsymbol{x}) = f(\boldsymbol{x} + \boldsymbol{e}_1) - f(\boldsymbol{x})$. φ is also harmonic and bounded, but for φ the sum

$$\begin{aligned}
s &= \varphi(\boldsymbol{y}_0) + \varphi(\boldsymbol{y}_1) + \cdots + \varphi(\boldsymbol{y}_n) \\
&= f(\boldsymbol{y}_0 + \boldsymbol{e}_1) - f(\boldsymbol{y}_0) + f(\boldsymbol{y}_1 + \boldsymbol{e}_1) - f(\boldsymbol{y}_1) + \cdots + f(\boldsymbol{y}_n + \boldsymbol{e}_1) - f(\boldsymbol{y}_n) \\
&= f(\boldsymbol{y}_1) - f(\boldsymbol{y}_0) + f(\boldsymbol{y}_2) - f(\boldsymbol{y}_1) + \cdots + f(\boldsymbol{y}_n + \boldsymbol{e}_1) - f(\boldsymbol{y}_n) \\
&= f(\boldsymbol{y}_n + \boldsymbol{e}_1) - f(\boldsymbol{y}_0)
\end{aligned}$$

cannot exceed twice the upper bound of f. But we have showed that for any harmonic function with a positive least upper bound, a chain of points can be chosen such that s is greater than any number. Therefore, the exact upper bound of φ cannot be positive. This implies that, for any $\boldsymbol{x}$,

$$\varphi(\boldsymbol{x}) = f(\boldsymbol{x} + \boldsymbol{e}_1) - f(\boldsymbol{x}) \leq 0.$$

An analogous argument (considering the greatest lower bound instead of the least upper bound) shows that

$$f(\boldsymbol{x} + \boldsymbol{e}_1) - f(\boldsymbol{x}) \geq 0.$$

From which we conclude that $f(\boldsymbol{x}+\boldsymbol{e}_1) = f(\boldsymbol{x})$. As our choice of $\boldsymbol{e}_1$ is arbitrary, it follows that $f(\boldsymbol{x}+\boldsymbol{e}_k) = f(\boldsymbol{x})$ for any k.

Let $\overline{\pi}_B(\boldsymbol{x})$ be the probability, starting from $\boldsymbol{x}$, of visiting the set B infinitely often. This probability must be equal to the probability of, starting from $\boldsymbol{x}$, taking one step, and then visiting B infinitely often. That is to say,

$$\overline{\pi}_B(\boldsymbol{x}) = \sum_y p(1, \boldsymbol{x}, \boldsymbol{y})\overline{\pi}_B(\boldsymbol{y}),$$

i.e., that $\overline{\pi}_B$ is harmonic. $\overline{\pi}_B$ is bounded, so by the above proof it must be constant. We will show that it is equal to one or zero depending on whether B is recurrent or nonrecurrent. First, suppose B is nonrecurrent. Let $q(n, \boldsymbol{y})$ be the probability that, starting from $\boldsymbol{x}$, the particle making its first visit to B by arriving at a point $\boldsymbol{y}$ in B at time n. If $\pi_B(\boldsymbol{x})$ is the probability of, starting from $\boldsymbol{x}$, visiting B (at any time), then

$$\pi_B(\boldsymbol{x}) = \sum_{n=0}^{\infty} \sum_{\boldsymbol{y} \in B} q(n, y).$$

In order to visit B infinitely many times, the particle must first visit B once, and then visit B infinitely often after that. Thus,

$$\overline{\pi}_B = \overline{\pi}_B(\boldsymbol{x}) = \sum_{n=0}^{\infty} \sum_{\boldsymbol{y} \in B} q(n, y)\overline{\pi}_B(\boldsymbol{y}) = \sum_{n=0}^{\infty} \sum_{\boldsymbol{y} \in B} q(n, y)\overline{\pi}_B = \pi_B(\boldsymbol{x})\overline{\pi}_B. \tag{14}$$

Recall that B is nonrecurrent, i.e., that there exists an $\boldsymbol{x}$ for which $\pi_B(\boldsymbol{x}) < 1$. Since $\pi_B(\boldsymbol{x})$ and $\overline{\pi}_B$ are both between zero and one, and $\overline{\pi}_B = \pi_B(\boldsymbol{x})\overline{\pi}_B$, the only possible value of $\overline{\pi}_B$ is zero.

Conversely, suppose that B is recurrent. Then the probability of the event $C_N = \{$the particle never visits B after the nth step$\}$ is equal to zero for any $n \geq 0$ and any point $\boldsymbol{x}$. Therefore,

$$\begin{aligned} 1 - \overline{\pi}_B(\boldsymbol{x}) &= \mathbb{P}_{\boldsymbol{x}}\{\text{The particle will visit B only a finite number of times}\} \\ &= \mathbb{P}_{\boldsymbol{x}}\{C_0 \cup C_1 \cup \dots\} \\ &\leq \mathbb{P}_{\boldsymbol{x}}\{C_0\} + \mathbb{P}_{\boldsymbol{x}}\{C_1\} + \dots \\ &= 0 \end{aligned}$$

Therefore, $\overline{\pi}_B = 1$. We can draw from this an equivalent characterisation of recurrence. A set B is recurrent if a particle starting from any point of the lattice visits B infinitely often with probability one. If, however, the probability of this event is less than one for some $\boldsymbol{x}$, then it must be equal to zero for all $\boldsymbol{x}$, and B is nonrecurrent.

6 Potential

Recall the operator P from Section 5, and let φ be a positive function on the lattice H^l. We define the potential of the function φ as

$$G\varphi = \varphi + P\varphi + P^2\varphi + \dots. \tag{15}$$

As well as being the discrete analogue of Newtonian potential, this quantity also has a straightforward probabilistic interpretation. In fact,

$$P^n\varphi(\boldsymbol{x}) = \sum_y p(n, \boldsymbol{x}, \boldsymbol{y})\varphi(\boldsymbol{y}) = \mathbb{E}_{\boldsymbol{x}}[\varphi(Xn))]. \tag{16}$$

We prove this by induction. For $n = 0$, equation (16) reduces to $\varphi(\boldsymbol{x}) = \varphi(\boldsymbol{x})$, and for $n = 1$ it reduces to the definition of the operator P (recall equation (13)). Suppose that equation (16) has been proved for some $n \in \mathbb{N}$. Then,

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{x}}[\varphi(X(n+1)] &= \sum_{\boldsymbol{y}} p(n+1, \boldsymbol{x}, \boldsymbol{y})\varphi(\boldsymbol{y}) \text{ , by definition,} \\
&= \sum_{\boldsymbol{z}} p(1, \boldsymbol{x}, \boldsymbol{z}) \left[\sum_{\boldsymbol{y}} p(n, \boldsymbol{z}, \boldsymbol{y})\varphi(\boldsymbol{y})\right] \\
&= \sum_{\boldsymbol{z}} p(1, \boldsymbol{x}, \boldsymbol{z}) \left[P^n\varphi(\boldsymbol{z})\right] \text{ , by the induction hypothesis,} \\
&= P^{n+1}\varphi(\boldsymbol{x}).
\end{aligned}$$

So, equation (16) is true for $n + 1$ as well. It follows that

$$G\varphi(\boldsymbol{x}) = \sum_{n=1}^{\infty} \mathbb{E}_{\boldsymbol{x}}[\varphi(X(n))] = \mathbb{E}_{\boldsymbol{x}} \sum_{n=1}^{\infty} [\varphi(X(n))] \tag{17}$$

This equation leads us to the following important interpretation of potential: Let every hit at the point $\boldsymbol{y}$ bring a payoff $\varphi(\boldsymbol{y})$. Then $G\varphi(\boldsymbol{x})$ is the mean value of the payoff obtained during a random walk of a particle with starting point $\boldsymbol{x}$.

We now show that if $f = G\varphi$ and τ is the time of the first visit of the particle to B, where B is a set in the lattice H^l, then

$$f(\boldsymbol{x}) - \mathbb{E}_{\boldsymbol{x}}[f(X(\tau))] = \mathbb{E}_{\boldsymbol{x}} \left[\sum_{k=0}^{\tau-1} \varphi(X(k))\right]. \tag{18}$$

If we divide the path of the particle up in to the part before time τ and the part after time τ, then we can write that

$$f(\boldsymbol{x}) = \mathbb{E}_{\boldsymbol{x}}[\varphi(X(0)) + X(1)) + \cdots + X(\tau - 1))] + \mathbb{E}_{\boldsymbol{x}}[\varphi(X(\tau)) + X(\tau + 1)) + \ldots]. \tag{19}$$

In this expression, the first term represents the average payoff during the random walk prior to visiting B, and the second term the average payoff after the first visit. To obtain (18), we need to show that

$$\mathbb{E}_{\boldsymbol{x}}[\varphi(X(\tau)) + \varphi(X(\tau + 1)) + \ldots] = \mathbb{E}_{\boldsymbol{x}} f(X(\tau)).$$

We will make use of the probability $q(n, \boldsymbol{y}) = \mathbb{P}(\tau = n, X(n) = \boldsymbol{y})$ (that is, the probability that the first visit to B takes place at time n, when the particle visits the point $\boldsymbol{y} \in B$). We have that

$$\mathbb{E}_{\boldsymbol{x}}[\varphi(X(\tau + k))] = \sum_{n,\boldsymbol{y}} q(n, \boldsymbol{y})\mathbb{E}_{\boldsymbol{y}}[\varphi(X(k))]$$

$$\sum_{n} q(n, \boldsymbol{y}) = \mathbb{P}_{\boldsymbol{x}}(X(\tau) = \boldsymbol{y}),$$

where n ranges from 0 to ∞, and $\boldsymbol{y}$ spans all the points in B. Starting from the left hand side,

$$\mathbb{E}_{\boldsymbol{x}}[\varphi(X(\tau)) + \varphi(X(\tau + 1)) + \ldots] = \mathbb{E}_{\boldsymbol{x}} \left[\sum_{k=0}^{\infty} \varphi(X(\tau + k))\right] = \sum_{k=0}^{\infty} \mathbb{E}_{\boldsymbol{x}}[\varphi(X(\tau + k))].$$

Using the first property of $q(n, \boldsymbol{y})$,

$$\sum_{k=0}^{\infty} \mathbb{E}_{\boldsymbol{x}}[\varphi(X(\tau + k))] = \sum_{k=0}^{\infty} \sum_{n,\boldsymbol{y}} q(n, \boldsymbol{y}) \mathbb{E}_{\boldsymbol{y}}[\varphi(X(k))] = \sum_{n,\boldsymbol{y}} q(n, \boldsymbol{y}) \sum_{k=0}^{\infty} \mathbb{E}_{\boldsymbol{y}}[\varphi(X(k))].$$

Recalling equation (17), and then applying the second property of $q(n, \boldsymbol{y})$, we see that

$$\sum_{n,\boldsymbol{y}} q(n, \boldsymbol{y}) \sum_{k=0}^{\infty} \mathbb{E}_{\boldsymbol{y}}[\varphi(X(k))] = \sum_{n,\boldsymbol{y}} q(n, \boldsymbol{y}) f(\boldsymbol{y}) = \sum_{\boldsymbol{y}} \mathbb{P}_{\boldsymbol{x}}(X(\tau) = \boldsymbol{y}) f(\boldsymbol{y}) = \mathbb{E}_{\boldsymbol{x}}\left[f(X(\tau))\right],$$

which is precisely the result we require.

7 Excessive Functions

Recall that a function f is called superharmonic if $Pf \leq f$. An excessive function is a superharmonic function that is non-negative. Furthermore, if φ is a non-negative function and $f = G\varphi$, then

$$f - Pf = (\varphi + P\varphi + P^2\varphi + \dots) + P(\varphi + P\varphi + P^2\varphi + \dots) = \varphi \geq 0$$

If $f - Pf \geq 0$, then f is superharmonic, thus the potential of any non-negative function is superharmonic

We will show that any excessive function is equal to the sum of a non-negative harmonic function and the potential of a non-negative function. Let f be an excessive function, and let $\varphi = f - Pf$, noting that φ is non-negative. It follows that

$$\begin{aligned} f &= \varphi + Pf \\ &= \varphi + P(\varphi + Pf) \\ &= \varphi + P\varphi + P^2(\varphi + Pf) \\ &\dots \\ &= \varphi + P\varphi + P^2\varphi + \dots + P^{n-1}\varphi + Pf. \end{aligned} \tag{20}$$

Notice that

$$\varphi + P\varphi + P^2\varphi + \dots + P^{n-1}\varphi = f - P^n f \leq f,$$

in other words, that

$$G\varphi = \varphi + P\varphi + P^2\varphi + \dots + P^{n-1}\varphi + \dots < \infty.$$

Equation (20) implies that $h = \lim_{n\to\infty} P^n f$ exists and that

$$f = G\varphi + h. \tag{21}$$

Since $Ph = h$, i.e., h is harmonic, we are done.

An example of an excessive function is $\pi_B(x)$, the probability of visiting a set B. Consider the sequence of events given by

$$A_n = \{\text{The particle visits the set } B \text{ after the nth step}\}.$$

Clearly $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \ldots$. Notice also that $\mathbb{P}_{\boldsymbol{x}}(A_0) = \pi_B(\boldsymbol{x})$. It follows from Equation (16) that

$$\mathbb{P}_{\boldsymbol{x}}(A_n) = \sum_{\boldsymbol{y}} p(n, \boldsymbol{x}, \boldsymbol{y})\pi_B(\boldsymbol{y}) = P^n \pi_B(\boldsymbol{x}). \tag{22}$$

In particular, $P\pi_B(\boldsymbol{x}) = \mathbb{P}_{\boldsymbol{x}}(A_1) \leq \mathbb{P}_{\boldsymbol{x}}(A_0) = \pi_B(\boldsymbol{x})$, hence π_B is excessive. We can therefore apply the expansion (20) to the function $\pi_B(\boldsymbol{x})$. Let $\overline{\pi}_B(\boldsymbol{x}) = \lim_{n\to\infty} P^n\pi_B(\boldsymbol{x})$, and $\varphi_B(\boldsymbol{x}) = \pi_B(\boldsymbol{x}) - P\pi_B(\boldsymbol{x})$, then

$$\pi_B(\boldsymbol{x}) = G\varphi_B(\boldsymbol{x}) + \overline{\pi}_B(\boldsymbol{x}). \tag{23}$$

According to Equation (22),

$$\overline{\pi}_B(\boldsymbol{x}) = \lim_{n\to\infty} \mathbb{P}_{\boldsymbol{x}}(A_n) = \mathbb{P}_{\boldsymbol{x}}\left(\bigcap_{n=0}^{\infty} A_n\right).$$

Thus, $\overline{\pi}_B(\boldsymbol{x})$ is the probability that the particle will visit B at arbitrarily remote times, i.e., that it will visit B infinitely often. We have already encountered this probability in Section 5, where we showed that it is identically equal to zero if B is nonrecurrent, and identically equal to one if B is recurrent. Consequently, if B is nonrecurrent, then $\pi_B(\boldsymbol{x}) = G\varphi_B(\boldsymbol{x})$. It follows from Equation (22) that

$$\varphi_B(\boldsymbol{x}) = \pi_B(\boldsymbol{x}) - P\pi_B(\boldsymbol{x}) = \mathbb{P}_{\boldsymbol{x}}(A_0) - \mathbb{P}_{\boldsymbol{x}}(A_1) = \mathbb{P}_{\boldsymbol{x}}(A_0 \setminus A_1).$$

That is to say that $\varphi_B(\boldsymbol{x})$ is the probability, starting from $\boldsymbol{x}$, of the particle initially being inside B, and then leaving B) in the first step. This probability can only be nonzero if $\boldsymbol{x} \in B$, and similarly, outside of B the function φ_B is equal to zero.

The same argument, only starting with $P^n\varphi_B(\boldsymbol{x})$, shows that $P^n\varphi_B(\boldsymbol{x}) = \mathbb{P}_{\boldsymbol{x}}(A_n \setminus A_{n+1})$. Therefore,

$$\pi_B(\boldsymbol{x}) = G\varphi_B = \sum_{n=0}^{\infty} P^n \varphi_B,$$

i.e., that the probability of visiting B finitely many times is equal to the sum of the probabilities of visiting B for the last time in the nth step.

It follows from Equation (18) that if $f = G\varphi$ ($\varphi \geq 0$) and τ is the time of the first visit to the set B, then

$$\mathbb{E}_{\boldsymbol{x}}[f(X(\tau))] \leq f(\boldsymbol{x}). \tag{24}$$

We can see from Equation (21) that this inequality is valid for any bounded excessive function

8 Capacity

The Newtonian potential is closely related to the notion of capacity. We develop an analogous formulation for functions defined on the lattice H^l. Let $f = G\varphi$, B be a subset of H^l, and τ be the time of the first visit to B. We investigate the class K_B of all functions $\varphi \geq 0$ that are equal to zero outside of B and such that $G\varphi \leq 1$.

For the function $f = G\varphi$, where $\varphi \in K_B$, Equation (18) takes the form

$$f(\boldsymbol{x}) = \mathbb{E}_{\boldsymbol{x}}[f(\boldsymbol{x}(\tau))] \tag{25}$$

(since φ is equal to zero outside of B). It follows from the inequality $f \leq 1$ that $\mathbb{E}_{\boldsymbol{x}}[f(\boldsymbol{x}(\tau))] \leq \mathbb{P}_{\boldsymbol{x}}(\tau < \infty) = \pi_B(\boldsymbol{x})$. Therefore,

$$f(\boldsymbol{x}) \leq \pi_B(\boldsymbol{x}) \tag{26}$$

As we saw in the previous section, if B is nonrecurrent, then $\pi_B = G\varphi_B$, where $\varphi_B = \pi_B - P\pi_B \in K_B$. We call π_B the equilibrium potential, φ_B the equilibrium distribution, and define the capacity of the set B as

$$C(B) = \sum_{\boldsymbol{y}} \varphi_B(\boldsymbol{y}). \tag{27}$$

If B is recurrent, then the concept of capacity cannot be met (since if $\pi_B = 1$, then $\varphi_B = 1 - 1 = 0$). Recall that all finite sets are nonrecurrent.

We now state an extremal property of the equilibrium distribution φ_B (the discrete analog of Gauss' Theorem). Let a set B be nonrecurrent. We show that for any function $\varphi \in K_B$,

$$\sum_{\boldsymbol{y}} \varphi(\boldsymbol{y}) \leq \sum_{\boldsymbol{y}} \varphi_B(\boldsymbol{y}) = C(B). \tag{28}$$

$\sum_{\boldsymbol{y}} \varphi(\boldsymbol{y})$ is called the total charge corresponding to the distribution φ. This inequality tells us that the capacity of a nonrecurrent set B is definable as the maximum total charge concentrated on B whose potential does not exceed one.

For the proof of (28), we introduce the notation

$$(f_1, f_2) = \sum_{\boldsymbol{y} \in B} f_1(\boldsymbol{y}) f_2(\boldsymbol{y}).$$

It can be shown fairly straightforwardly that

$$(Gf_1, f_2) = (f_1, Gf_2).$$

Utilizing this, together with the fact that $\pi_B(\boldsymbol{x}) = 1$ for $\boldsymbol{x} \in B$, Equation (23), and the fact that $G\varphi \leq 1$ for $\varphi \in B_K$, we show that

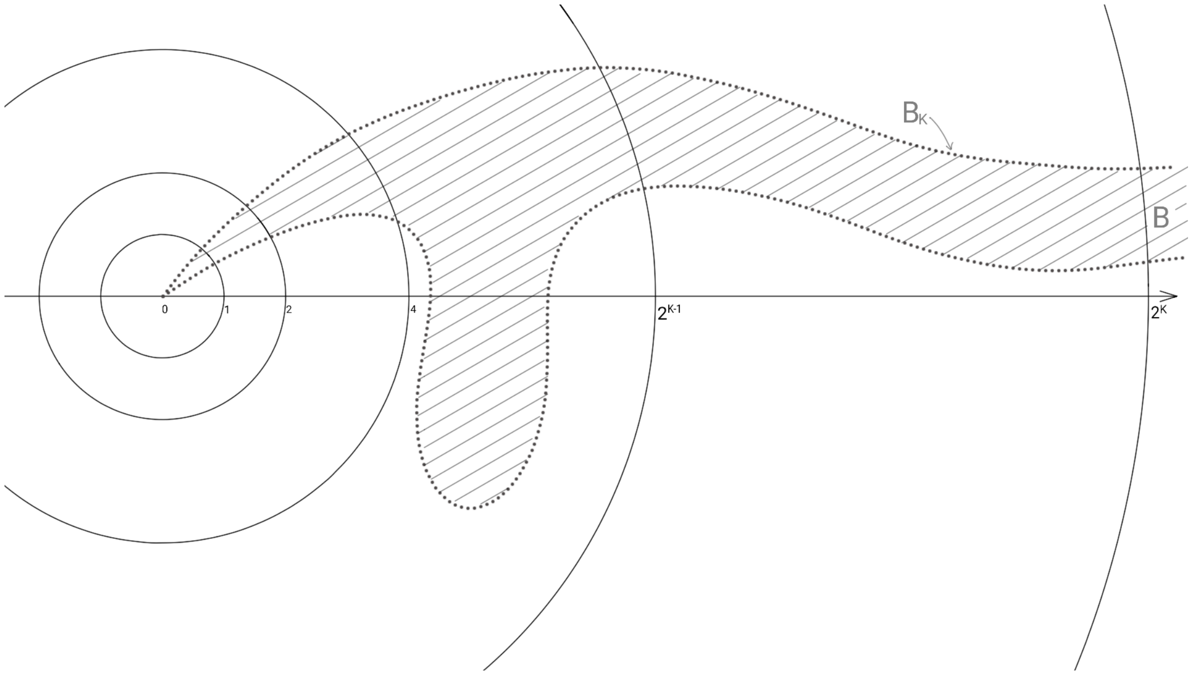
$$\sum_{\boldsymbol{y}} \varphi(\boldsymbol{y}) = (\varphi, 1) = (\varphi, \pi_B) = (\varphi, G\varphi_B) = (G\varphi, \varphi_B) \leq (1, \varphi_B) = C(B).$$

9 The Recurrence Criterion

We now establish a necessary and sufficient condition for the recurrence of a subset, B, of a three-dimensional lattice. This condition is formulated in terms of capacity, and can be extended with little difficultly to $l > 3$.

It will turn out that the recurrence of B depends on how the number of points of B falling within a sphere of radius r grows as $r \to \infty$. Consider an expanding sequence of spheres with radii $r = 1, 2, 2^2, \ldots, 2^k, \ldots$. We denote by B_k the part of the set B which falls between the kth and the (k+1)th spheres (more precisely, the set of those $\boldsymbol{x} \in B$ for which $2^{k-1} < |\boldsymbol{x}| \leq 2^k$).

The set B_k is finite, hence for it the capacity $C(B_k)$ is defined. The following criterion holds:



The set B is recurrent if and only if the following series diverges

$$\sum_{k=0}^{\infty} \frac{C(B_k)}{2^k}. \tag{29}$$

First, we prove the necessity of this condition, i.e., that if the series (29) converges, then B_k is nonrecurrent. So, suppose that the series (29) converges. We first show that, along with the series (29), the following series also converges:

$$\sum_{k=0}^{\infty} \pi_{B_k}(0) \tag{30}$$

To do this, we will use the asymptotic estimate given at the end of Section 4,

$$g(\boldsymbol{x}, \boldsymbol{y}) \sim \frac{Q}{|\boldsymbol{x}-\boldsymbol{y}|} \quad (|\boldsymbol{x}-\boldsymbol{y}| \to \infty), \tag{31}$$

where $Q \in \mathbb{R}$. It follows from (31) that there exists an $N > 0$ such that for all $\boldsymbol{y} \in B_k, k > N$,

$$g(\boldsymbol{0}, \boldsymbol{y}) \leq \frac{2Q}{|\boldsymbol{y}|}. \tag{32}$$

π_{B_k} is the equilibrium potential of the set B_k. Recalling that B_k is finite, we make use of our finding in Section 7 that for a finite set A, $\pi_A = G\varphi_A$, where φ_A is the equilibrium distribution on the set A. Therefore,

$$\pi_{B_k}(\boldsymbol{0}) = G\varphi_{B_k}(\boldsymbol{0}) = \sum_{n=0}^{\infty} P^n \varphi_{B_k}(0).$$

Then,

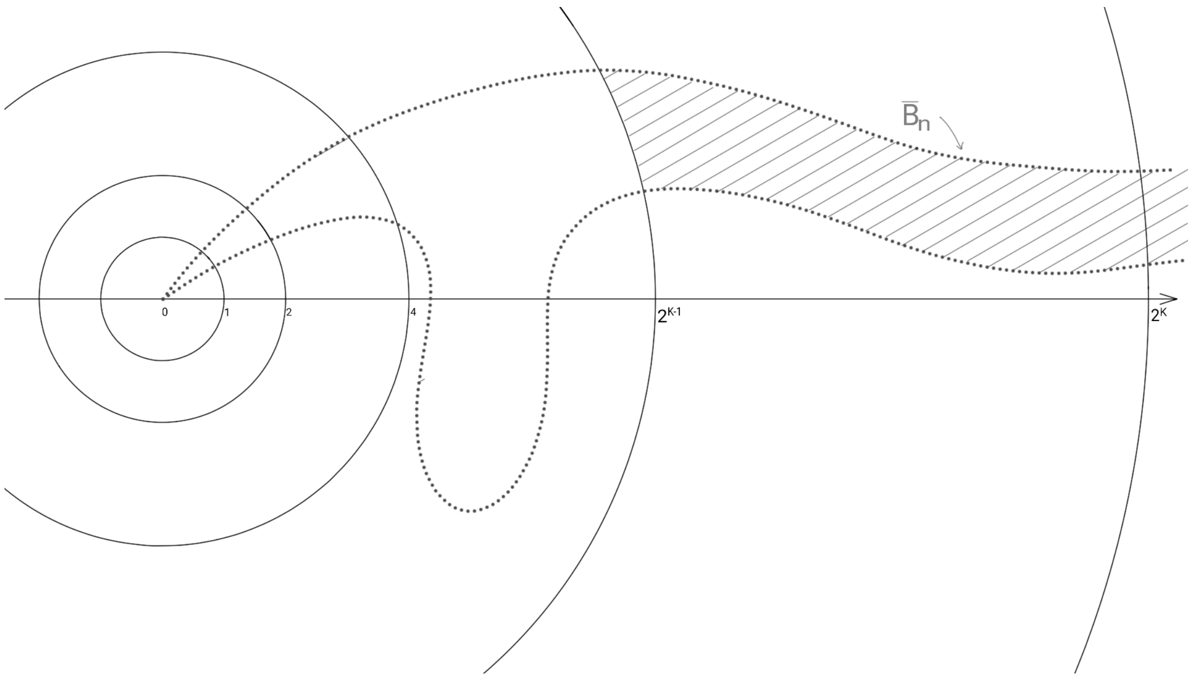
$$\sum_{n=0}^{\infty} P^n \varphi_{B_k}(0) = \sum_{n=0}^{\infty} \sum_{\boldsymbol{y}} p(n, \boldsymbol{0}, \boldsymbol{y}) \varphi_{B_k}(\boldsymbol{y}) = \sum_{\boldsymbol{y}} \left(\sum_{n=0}^{\infty} p(n, \boldsymbol{0}, \boldsymbol{y}) \right) \varphi_{B_k}(\boldsymbol{y}) = \sum_{\boldsymbol{y}} g(\boldsymbol{0}, \boldsymbol{y}) \varphi_{B_k}(\boldsymbol{y}).$$

We now show that the series (29) dominates the series (30):

$$\begin{aligned}
\pi_{B_k}(\mathbf{0}) &= \sum_{\boldsymbol{y}} g(\mathbf{0}, \boldsymbol{y})\varphi_{B_k}(\boldsymbol{y}) \\
&\le \sum_{\boldsymbol{y}} \frac{2Q}{|\boldsymbol{y}|}\varphi_{B_k}(\boldsymbol{y}) \text{ , by (32)} \\
&\le \sum_{\boldsymbol{y}} \frac{2Q}{2^{k-1}}\varphi_{B_k}(\boldsymbol{y}) \text{ , since } |\boldsymbol{y}| > 2^{k-1} \forall \boldsymbol{y} \in B_k, \text{ and } \varphi_{B_k}(\boldsymbol{y}) = 0 \forall \boldsymbol{y} \notin B_k \\
&= \frac{Q}{2^{k-2}} \sum_{\boldsymbol{y}} \varphi_{B_k}(\boldsymbol{y}) \\
&= 4Q\frac{C(B_k)}{2^k} \text{ , since } \sum_{\boldsymbol{y}} \varphi_{B_k}(\boldsymbol{y}) \text{ is just the definition of the capacity of } B_k
\end{aligned}$$

We conclude that the series (29) dominates the series (30), correct to a constant factor, and therefore that the series (30) converges. We now make use of the set

$$\overline{B}_n = \bigcup_{k=n}^{\infty} B_k$$



Note that the event {The particle visits $\overline{B}_n$} is exactly the union of the events {The particle visits B_k}, $k = n, n+1, \ldots$. Therefore,

$$\pi_{\overline{B}_n}(\mathbf{0}) \le \sum_{k=n}^{\infty} \pi_{B_k}(\mathbf{0})$$

Then for sufficiently large n we have that $\pi_{\overline{B}_n}(\mathbf{0}) < 1$, so the set $\overline{B} = \overline{B}_n$ is nonrecurrent. Since the set $\overline{\overline{B}} = B \setminus \overline{B}$ is finite, it is also nonrecurrent. Therefore, to prove that B is nonrecurrent, we need only to prove that the union of two nonrecurrent sets is itself nonrecurrent.

For this purpose, we recall the alternate definition of recurrence given in Section 5, wherein a set is recurrent if a particle visits that set a finite number of times with probability one. Since the intersection of two events of probability one is itself an event of probability one, and the particle has probability one of visiting both $\overline{B}$ and $\overline{\overline{B}}$ a finite number of times, it follows that the same is true of their union $B = \overline{\overline{B}} \cup \overline{B}$. Therefore, the set B is nonrecurrent.

This concludes the proof of necessity, so we now move on to proving the sufficiency of the recurrence condition, i.e., that if the series (29) diverges, then B is nonrecurrent.

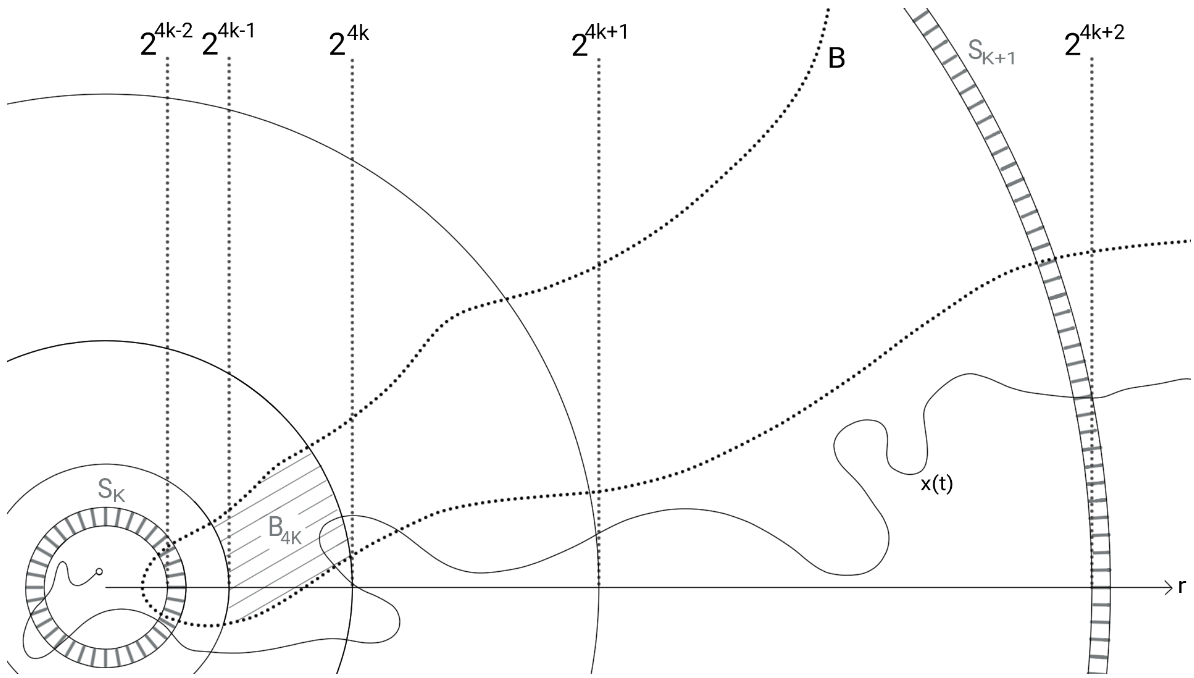
Suppose the series (29) diverges. We decompose it into four series, each with indices that are equal modulo four,

$$\sum_k \frac{C(B_{4k})}{2^{4k}}, \sum_k \frac{C(B_{4k+1})}{2^{4k+1}}, \sum_k \frac{C(B_{4k+2})}{2^{4k+2}}, \sum_k \frac{C(B_{4k+3})}{2^{4k+3}}.$$

Since the sum of these series diverges, at least one of them must also diverge. Let us suppose for the sake of definiteness that the following series does so:

$$\sum_k \frac{C(B_{4k})}{2^{4k}}. \tag{33}$$

Let S_k be the set of points between the two spheres of radius 2^{4k-2} and $2^{4k-2}+1$ (that is, the set of all $\boldsymbol{y} \in H^l$ such that $2^{4k-2} \leq |\boldsymbol{y}| \leq 2^{4k-2}+1$).



Notice that the set B_{4k} is contained between the layers S_k and S_{k+1}. Since the distance of the particle from the origin cannot change by more than one in a single step, the particle cannot pass from one side of S_k to the other without occupying a state within S_k at least once. The length of $X(n)$ goes to infinity with probability one, so the particle must pass through all the layers S_k enclosing the initial state $X(0)$. We investigate the event $A_k = \{$The particle visits B_{4k} between the time of the first visit to S_k and the time of the first visit to $S_{k+1}\}$.

We show that for all sufficiently large k,

$$\mathbb{P}_{\boldsymbol{y}}(A_k) \geq Q_1 \frac{C(B_{4k})}{2^{4k}} \text{ , for } \boldsymbol{y} \in S_k, \tag{34}$$

where $Q_1 > 0$ does not depend on $\boldsymbol{y}$ or k.

Let $D_k = \{$The particle visits B_{4k} after the time of the first visit to S_{k+1}.$\}$. If a particle on a random walk leaves $\boldsymbol{y} \in S_k$ and later visits B_{4k}, then either A_k or D_k must occur. Therefore,

$$\pi_{B_{4k}}(\boldsymbol{y}) \leq \mathbb{P}_{\boldsymbol{y}}(A_k) + \mathbb{P}_{\boldsymbol{y}}(D_k).$$

Additionally,

$$\begin{aligned}
\mathbb{P}(D_k) &= \mathbb{P}(\text{The particle visits } B_{4k} \text{ after the time of the first visit to } S_{k+1}) \\
&= \mathbb{P}(\text{The particle visits } S_{k+1} \text{ without visiting } B_{4k}) \cdot \mathbb{P}(\text{The particle visits } B_{4k} \text{ from } S_{k+1}) \\
&\leq \mathbb{P}(\text{The particle visits } B_{4k} \text{ from } S_{k+1}) \\
&\leq \max_{\boldsymbol{z} \in S_{k+1}} \pi_{B_4 k}(\boldsymbol{z}).
\end{aligned}$$

It follows from the inequality $\mathbb{P}(D_k) \leq \max_{\boldsymbol{z} \in S_{k+1}} \pi_{B_4 k}(\boldsymbol{z})$ that

$$\mathbb{P}_{\boldsymbol{y}}(A_k) \geq \pi_{B_{4k}}(\boldsymbol{y}) - \max_{\boldsymbol{z} \in S_{k+1}} \pi_{B_4 k}(\boldsymbol{z}). \tag{35}$$

Recall once again that $\pi_{B_{4k}}$ is the equilibrium potential of B_{4k}, and that $\pi_{B_{4k}} = G\varphi_{B_{4k}}$, where $\varphi_{B_{4k}} = \pi_{B_{4k}} - P\pi_{B_{4k}}$. Note that, for $\boldsymbol{y} \in H^l$,

$$\pi_{B_{4k}}(\boldsymbol{y}) = G\varphi_{B_{4k}}(\boldsymbol{y}) = \sum_{n=0}^{\infty} P^n \varphi_{B_{4k}}(\boldsymbol{y}) = \sum_{n=0}^{\infty} \sum_{\boldsymbol{u} \in H^l} p(n, \boldsymbol{y}, \boldsymbol{u}) \varphi_{B_{4k}}(\boldsymbol{u}) = \sum_{\boldsymbol{u} \in H^l} g(\boldsymbol{y}, \boldsymbol{u}) \varphi_{B_{4k}}(\boldsymbol{u}).$$

Substituting this in to Equation (35),

$$\mathbb{P}_{\boldsymbol{y}}(A_k) \geq \sum_{\boldsymbol{u} \in H^l} g(\boldsymbol{y}, \boldsymbol{u}) \varphi_{B_{4k}}(\boldsymbol{u}) - \max_{\boldsymbol{z} \in S_{k+1}} \sum_{\boldsymbol{u} \in H^l} g(\boldsymbol{z}, \boldsymbol{u}) \varphi_{B_{4k}}(\boldsymbol{u})$$

By taking out appropriate minima and maxima as common factors from each sum, we can maintain the inequality (keep in mind that $\varphi_{B_4 k} = 0$ everywhere outside of B_{4k}):

$$\begin{aligned}
\mathbb{P}_{\boldsymbol{y}}(A_k) &\geq \left(\min_{\boldsymbol{y} \in S_k, \boldsymbol{u} \in B_{4_k}} g(\boldsymbol{y}, \boldsymbol{u}) \right) \sum_{\boldsymbol{u} \in H^l} \varphi_{B_{4k}}(\boldsymbol{u}) - \left(\max_{\boldsymbol{z} \in S_{k+1}, \boldsymbol{u} \in B_{4k}} g(\boldsymbol{z}, \boldsymbol{u}) \right) \max_{\boldsymbol{z} \in S_{k+1}} \sum_{\boldsymbol{u} \in H^l} \varphi_{B_{4k}}(\boldsymbol{u}) \\
&= \left(\min_{\boldsymbol{y} \in S_k, \boldsymbol{u} \in B_{4_k}} g(\boldsymbol{y}, \boldsymbol{u}) \right) C(B_{4k}) - \left(\max_{\boldsymbol{z} \in S_{k+1}, \boldsymbol{u} \in B_{4k}} g(\boldsymbol{z}, \boldsymbol{u}) \right) C(B_{4k}) \text{ , since } C(B_{4k}) = \sum_{\boldsymbol{y}} \varphi_{B_{4k}}(\boldsymbol{y}) \\
&= C(B_{4k}) \left(\min_{\boldsymbol{y} \in S_k, \boldsymbol{u} \in B_{4_k}} g(\boldsymbol{y}, \boldsymbol{u}) - \max_{\boldsymbol{z} \in S_{k+1}, \boldsymbol{u} \in B_{4k}} g(\boldsymbol{z}, \boldsymbol{u}) \right).
\end{aligned}$$

We now apply the asymptotic estimate of $g(\boldsymbol{x}, \boldsymbol{y})$ given by (31). For sufficiently large k, and $\boldsymbol{y} \in S_k$,

$$\frac{5}{6} \cdot \frac{Q}{|\boldsymbol{x} - \boldsymbol{y}|} \leq g(\boldsymbol{x}, \boldsymbol{y}) \leq \frac{7}{6} \cdot \frac{Q}{|\boldsymbol{x} - \boldsymbol{y}|}.$$

It follows that

$$\mathbb{P}_{\boldsymbol{y}}(A_k) \geq C(B_{4k}) \left(\min_{\boldsymbol{y} \in S_k, \boldsymbol{u} \in B_{4_k}} \frac{5}{6} \cdot \frac{Q}{|\boldsymbol{y} - \boldsymbol{u}|} - \max_{\boldsymbol{z} \in S_{k+1}, \boldsymbol{u} \in B_{4k}} \frac{7}{6} \cdot \frac{Q}{|\boldsymbol{z} - \boldsymbol{u}|} \right).$$

We see from the relative position of the sets $S_k, B_{4k}, \text{and} S_{k+1}$ that

$$|\boldsymbol{y}-\boldsymbol{u}| \leq 2^{4k-2}+1+2^{4k} \leq 2 \cdot 2^{4k}$$
$$|\boldsymbol{z}-\boldsymbol{u}| \geq 2^{4k+2}-2^{4k} = 3 \cdot 2^{4k}.$$

Consequently, for sufficiently large k

$$\begin{aligned}\mathbb{P}_{\boldsymbol{y}}(A_k) &\geq C(B_{4k})\left(\frac{5}{6}\cdot\frac{Q}{2\cdot 2^{4k}}-\frac{7}{6}\cdot\frac{Q}{3\cdot 2^{4k}|}\right)\\ &= \frac{1}{36}\cdot\frac{C(B_{4k})}{2^{4k}}.\end{aligned}$$

This completes our proof of the inequality (34).

We pick a number m such that the initial state $\boldsymbol{x}$ lies inside the layer S_m and the inequality (34) is satisfied for all $k \geq m$. We denote by τ_k the time of the first visit to the layer S_k. The opposite of the event A_k is the event $\overline{A}_k = \{$In the interval of time $[\tau_k, \tau_{k+1}]$, the particle does not visit B_{4k} $\}$. It follows from Equation (34) that

$$\mathbb{P}_{\boldsymbol{x}}\left(\overline{A}_k\right) \leq 1-Q_1\frac{C(B_{4k})}{2^{4k}},$$

and that this bound holds regardless of the values of τ_k or $X(\tau_k)$, or of the path taken by the particle before τ_k. Therefore,

$$\mathbb{P}_{\boldsymbol{x}}\left(\overline{A}_m\cap\overline{A}_{m+1}\cap\cdots\cap\overline{A}_{m+s}\right) \leq \prod_{k=m}^{m+s}\left(1-Q_1\frac{C(B_{4k})}{2^{4k}}\right),$$

where s is any positive integer. We define

$$q_k(n,\boldsymbol{y}) = \mathbb{P}_{\boldsymbol{x}}(\tau_k = n, X(\tau_k)=\boldsymbol{y}, \overline{A}_m\cap\overline{A}_{m+1}\cap\cdots\cap\overline{A}_{k-1}),$$

i.e., that the first visit to S_m occurs at time n, where $X(n)=\boldsymbol{y}\in S_m$, and that between its first visit to any two layers S_i and S_{i+1}, the particle does not visit B_{4i} for all $i=m, m+1, \ldots, k-1$. Then,

$$\begin{aligned}\mathbb{P}_{\boldsymbol{x}}\left(\overline{A}_m\cap\overline{A}_{m+1}\cap\cdots\cap\overline{A}_k\right) &= \sum_{n,\boldsymbol{y}} q_k(n,\boldsymbol{y})\mathbb{P}_{\boldsymbol{y}}(\overline{A}_k)\\ &\leq \sum_{n,\boldsymbol{y}} q_k(n,\boldsymbol{y})\left(1-Q_1\frac{C(B_{4k})}{2^{4k}}\right)\\ &= \mathbb{P}_{\boldsymbol{x}}(\overline{A}_m\cap\overline{A}_{m+1}\cap\cdots\cap\overline{A}_{k-1})\left(1-Q_1\frac{C(B_{4k})}{2^{4k}}\right).\end{aligned} \tag{36}$$

We are now ready to prove the recurrence of B. We start by making use of the inequality (36).

$$\begin{aligned}\mathbb{P}_{\boldsymbol{x}}\left(\bigcap_{i=m}^{\infty}\overline{A}_i\right) &= \lim_{k\to\infty}\left(\mathbb{P}_{\boldsymbol{x}}\left(\overline{A}_m\cap\overline{A}_{m+1}\cap\cdots\cap\overline{A}_k\right)\right)\\ &\leq \lim_{k\to\infty}\left(\mathbb{P}_{\boldsymbol{x}}(\overline{A}_m\cap\overline{A}_{m+1}\cap\cdots\cap\overline{A}_{k-1})\left(1-Q_1\frac{C(B_{4k})}{2^{4k}}\right)\right)\\ &= \mathbb{P}_{\boldsymbol{x}}\left(\bigcap_{i=m}^{\infty}\overline{A}_i\right)\lim_{k\to\infty}\left(1-Q_1\frac{C(B_{4k})}{2^{4k}}\right).\end{aligned}$$

Recall that the series $\sum Q_1\frac{C(B_{4k})}{2^{4k}}$ diverges, so $Q_1\frac{C(B_{4k})}{2^{4k}} > 0$ for any k, no matter how large. Therefore,

$$\lim_{k\to\infty}\left(1-Q_1\frac{C(B_{4k})}{2^{4k}}\right) < 1.$$

But then the only possible value of $\mathbb{P}_{\boldsymbol{x}}\left(\bigcap_{i=m}^{\infty} \overline{A}_i\right)$ is zero. From this it follows that

$$\mathbb{P}_{\boldsymbol{x}}\left(\bigcup_{i=m}^{\infty} A_i\right) = 1 - \mathbb{P}_{\boldsymbol{x}}\left(\bigcap_{i=m}^{\infty} \overline{A}_i\right) = 1 - 0 = 1.$$

Hence, the particle with probability one visits one of the sets B_4k, each of which belongs to B. Thus, B is recurrent.

10 Recurrence of a Set Situated on the Axis

We now make use of the recurrence criterion developed in the previous section to try and imagine what recurrent and nonrecurrent sets of a three-dimensional lattice look like.

Firstly, any subset of a nonrecurrent set must itself be nonrecurrent, and if a set contains a recurrent subset, then that set must also be recurrent. Furthermore, we know that any bounded set is nonrecurrent.

Let the position of the random walk $X(n) = (X_1(n), X_2(n), X_3(n))$. We will show that the co-ordinate plane $X_3 = 0$ is a recurrent set. At any point during the random walk, there is a $\frac{1}{6}$ chance of the particle moving in a particular direction. Therefore, the probability that, at any step, X_3 remains unchanged is $\frac{2}{3}$. So, the probability that X_3 will remain unchanged for k steps in a row is $\left(\frac{2}{3}\right)^k$. Since

$$\lim_{k\to\infty} \left(\frac{2}{3}\right)^k = 0,$$

there is zero probability that X_3 remains unchanged forever. In other words, X_3 must eventually change. Due to the symmetry of the random walk, when X_3 does change it will be incremented by one with probability $\frac{1}{2}$, and decremented by one with probability $\frac{1}{2}$. Therefore, the random process X_3 differs from the one-dimensional symmetric random walk described at the beginning of Section 2 only in the fact that X_3 can get stuck in any state for a finite amount of time. While this property will change the expected time X_3 takes to reach some other state, it will not affect the probability that X_3 will reach that state sooner or later. Since particle in a one-dimensional random walk reaches zero with probability one, X_3 is also guaranteed to reach zero. Therefore, the co-ordinate plane $X_3 = 0$ is a recurrent set.

It can be proved by a similar argument to the one employed in Section 4 that the point $\mathbf{0}$ will be reached from any other point on a 2D lattice with probability one. We can, by combining these insights, prove that the co-ordinate axis $x_2 = x_3 = 0$ is recurrent. We can obtain the following test of recurrence of the set B consisting of the points $\{b_n, 0, 0\}$, where $0 < b_1 < b_2 < b_3 < \ldots$ (and the b_n are integers):

If the series $\sum \frac{1}{b_n}$ converges, the set B is recurrent. If the series $\sum \frac{1}{b_n}$ diverges and for large n,

$$b_{n+1} - b_n \geq c \log_2(b_n), \tag{37}$$

where c is a positive constant, then B is recurrent.

The relationship between the recurrence of B and the divergence of $\sum \frac{1}{b_n}$ is fairly intuitive; the divergence of this series tells us that the points $\{b_n, 0, 0\}$ are close together, whereas the divergence tells us that they are far apart. The condition (37) is related to the method we use to estimate capacities. It requires the series to diverge very slowly. For example, if $b_n = \lfloor n \log_2 n \rfloor$, then for large n

$$\begin{aligned} b_{n+1} - b_n &\geq (n+1)\log_2(n+1) - 1 - n\log_2 n \geq \log_2 n - 1 \\ &= \log \frac{n}{2} \geq \log_2 \sqrt{n \log_2 n} \geq \frac{1}{2}\log_2 b_n. \end{aligned}$$

So, the inequality (37) is satisfied, and therefore B is recurrent. However, if $b_n = \lfloor n \log_2^\alpha n \rfloor$, where $\alpha > 1$, then $\sum \frac{1}{b_n}$ converges, and B is nonrecurrent.

We now prove that this criterion holds. Let the series $\sum \frac{1}{b_n}$ converge. Note that the capacity of a finite set is bounded above by the number of elements in that set. This follows from the definition of capacity,

$$C(B) = \sum_{\boldsymbol{x} \in B} \varphi_B(\boldsymbol{x}),$$

where $\varphi_B \leq 1$. Let us estimate the number of elements in the set B_k involved in the recurrence criterion. Call this number $|B_k|$. If $\boldsymbol{b_n} \in B_k$, then $2^{k+1} < b_n \leq 2^k$, from which it follows that $\frac{1}{2^k} \leq \frac{1}{b_n}$. If we sum both sides of this inequality over B, then

$$\frac{|B_k|}{2^k} \leq \sum_{\boldsymbol{b_n} \in B_k} \frac{1}{b_n}$$
$$\text{so } \frac{C(B_k)}{2^k} \leq \sum_{\boldsymbol{b_n} \in B_k} \frac{1}{b_n}.$$

Consequently,

$$\sum_k \frac{C(B_k)}{2^k} \leq \sum_k \sum_{\boldsymbol{b_n} \in B_k} \frac{1}{b_n} = \sum_n \frac{1}{b_n}.$$

So, $\sum \frac{C(B_k)}{2^k}$ converges, and according to the recurrence criterion given in the previous section B is nonrecurrent.

Now, suppose that the series $\sum \frac{1}{b_n}$ diverges and the inequalities (37) are fulfilled. We show that, for any $\boldsymbol{x} \in H^3$, there is M such that

$$\sum_{\boldsymbol{y} \in B_k} g(\boldsymbol{x}, \boldsymbol{y}) \leq M. \tag{38}$$

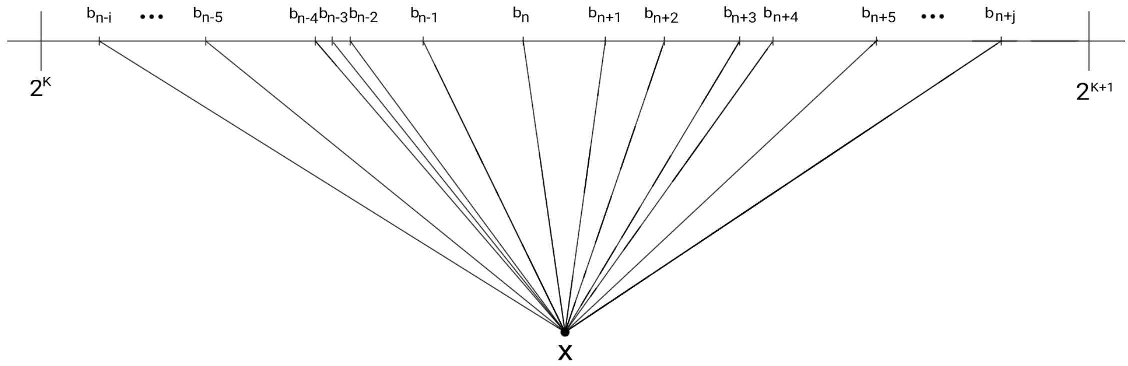
We may assume that $k \geq 2$ (the cases $k = 0, 1$ are covered by the work in Section 4). It follows from the estimate (31) that $g(\boldsymbol{x}, \boldsymbol{y})$ and $|\boldsymbol{x} - \boldsymbol{y}| g(\boldsymbol{x}, \boldsymbol{y})$ are bounded by some constant Q. Let b_n and b_{n+1} be the two points in B_k closest to $\boldsymbol{x}$, and let $b_{n-1}, b_{n-2}, \ldots, b_{n-i}$ and $b_{n+2}, b_{n+3}, \ldots, b_{n+j}$ be all the remaining points in the set B:

We have that

$$\begin{aligned} \sum_{\boldsymbol{y} \in B_k} g(\boldsymbol{x}, \boldsymbol{y}) &\leq \sum_{\boldsymbol{y} \in B_k} \frac{Q}{|\boldsymbol{x} - \boldsymbol{y}|} \\ &\leq 2Q + Q \sum_{r=1}^{i} \left(\frac{1}{|\boldsymbol{x} - \boldsymbol{b_{n-r}}|} \right) + Q \sum_{r=2}^{j} \left(\frac{1}{|\boldsymbol{x} - \boldsymbol{b_{n+r}}|} \right) \end{aligned}$$

It follows from the condition (37) that within B_k,

$$b_{l+1} - b_l \geq c \log_2 b_l \geq c \log_2 2^{k-1} = c(k-1).$$



So, for any $\boldsymbol{b_{n-l}} \in B_k$ (that is, a point in B to the left of $\boldsymbol{x}$),

$$|\boldsymbol{x} - \boldsymbol{b_{n-l}}| \geq |\boldsymbol{b_n} - \boldsymbol{b_{n-l}}| = \sum_{r=1}^{l} |\boldsymbol{b_{n-r+1}} - \boldsymbol{b_{n-r}}| = \sum_{r=1}^{l} b_{n-r+1} - b_{n-r} \geq \sum_{r=1}^{l} c(k-1) = lc(k-1).$$

An almost identical argument shows that $|\boldsymbol{x} - \boldsymbol{b_{n+l}}| \geq (j-1)c(k-1)$, the only difference being that we replace $\boldsymbol{x}$ with $\boldsymbol{b_{n+1}}$ instead of $\boldsymbol{b_n}$. Applying these estimates to our earlier upper bound on $\sum g(\boldsymbol{x}, \boldsymbol{y})$, we see that

$$\begin{aligned}\sum_{\boldsymbol{y} \in B_k} g(\boldsymbol{x}, \boldsymbol{y}) &\leq 2Q + Q \sum_{r=1}^{i} \left(\frac{1}{rc(k-1)} \right) + Q \sum_{r=2}^{j} \left(\frac{1}{(r-1)c(k-1)} \right) \\ &= 2Q + \frac{Q}{c(k-1)} \left(\sum_{r=1}^{i} \frac{1}{r} + \sum_{r=2}^{j} \frac{1}{j-1} \right)\end{aligned}$$

Since there are only 2^{k-1} integer-valued points on the x-axis in B_k, i and j cannot exceed 2^{k-1}. Therefore,

$$\begin{aligned}\sum_{\boldsymbol{y} \in B_k} g(\boldsymbol{x}, \boldsymbol{y}) &\leq 2Q + \frac{2Q}{c(k-1)} \sum_{r=1}^{2^k-1} \frac{1}{r} \\ &\leq 4Q + \frac{2Q}{c(k-1)} \sum_{r=2}^{2^k-1} \frac{1}{r}\end{aligned}$$

Using the fact that

$$\int_1^l \frac{1}{x} dx \geq \sum_{r=2}^{l} \frac{1}{r}$$

We have that

$$\sum_{\boldsymbol{y} \in B_k} g(\boldsymbol{x}, \boldsymbol{y}) \leq 4Q + \frac{2Q}{c(k-1)} \int_1^{2^{k-1}} \frac{1}{x} dx = 4Q + \frac{2Q}{c(k-1)} \ln 2^{k-1} = 2Q \left(2 + \frac{\ln 2}{c} \right),$$

thus completing the proof of (38). We can use the inequality that we have just proven to find a lower estimate for $C(B_k)$. Consider the function $\varphi(\boldsymbol{y}) = \frac{1}{M}$ for $\boldsymbol{y} \in B_k$, and zero at all other points. The potential of this function is

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{y}} g(\boldsymbol{x}, \boldsymbol{y}) \varphi(\boldsymbol{y}) = \frac{1}{M} \sum_{\boldsymbol{y} \in B_k} g(\boldsymbol{x}, \boldsymbol{y}).$$

Due to (38), f cannot exceed one. Recalling from Section 8 that the capacity is the maximum total charge whose potential does not exceed one,

$$C(B_k) \geq \sum_{\boldsymbol{y}} \varphi(\boldsymbol{y}) = \sum_{\boldsymbol{y} \in B_k} \frac{1}{M} = \frac{|B_k|}{M}.$$

If $\boldsymbol{b_n} \in B_k$, then $2^{k+1} < b_n \leq 2^k$, so $\frac{1}{b_n} < \frac{1}{2^{k-1}}$. If we sum this inequality over all the $\boldsymbol{b_n} \in B_k$, then

$$\sum_{\boldsymbol{b_n} \in B_k} \frac{1}{b_n} < \sum_{\boldsymbol{b_n} \in B_k} = \frac{|B_k|}{2^{k-1}} \leq 2M \frac{C(B_k)}{2^k}.$$

We see that the series $\sum \frac{1}{b_n}$ is dominated by the series (29). The divergence of $\sum \frac{1}{b_n}$ implies the divergence of the series (29), which by our recurrence criterion implies the recurrence of the set B.

11 Discussion and Conclusion

This brings us up to the end of the first chapter of Dynkin and Yushkevich. We have introduced the subject of random walks, and proved in reasonable depth some important properties. The book contains another three chapters, all at least as challenging as the first. It would be a enormous task to summarise the rest of the book in the same fashion as I have done here, but I expect that doing so would create a fantastic resource for future students.

References

[1] Evengii B. Dynkin and Aleksandr A. Yushkevich. *Markov Processes Theorems and Problems*. Plenum Press, Moscow, 1969.