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Dynamical system approach to modelling of electromagnetic stirring

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ABSTRACT

Analysing the behaviour of a conducting fluid under the influence of electromagnetic forces is fundamental for applications such as Electromagnetic Stirring. Robust vortical systems are observed to form on a thin annular layer of electrolytic fluid when driven by azimuthal Lorentz forces. In this project, we attempt to establish the algebraic form of the evolution of amplitudes for these vortical modes. We then analyse the resulting dynamical system for the amplitude equations by identifying the stationary points, sketching phase portraits, and interpreting them with regards to the original physical problem.

1. Introduction

1.1 Introduction to EMS and the experimental set-up

Electromagnetic Stirring (EMS) is a technique where the mixing of a fluid is caused by electromagnetic forces rather than mechanical interference. The significance of this technique is especially apparent in applications, where mechanical stirring is either inefficient or impractical, for example, when mixing molten metals. Hence analysing the behaviour of an electrolytic fluid under electromagnetic forces is of practical importance.

The set-up considered here involves an annular layer of electrolytic fluid acted upon by azimuthal Lorentz force that drives the fluid flow (Perez-Barrera et al. 2015). As seen in the figure, robust vortical system is formed close to the outer edge of the layer of fluid. These individual vortices are observed to interact with each other as time evolves. In this project, our main goal is to identify how these vortices evolve and interact.

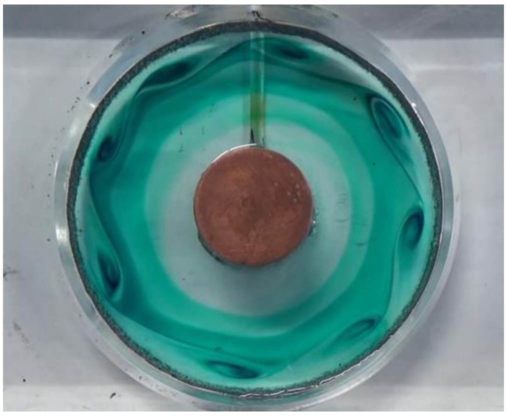


Figure 1 – Vortex patterns formed in the set-up (Perez-Barrera et al. 2015)

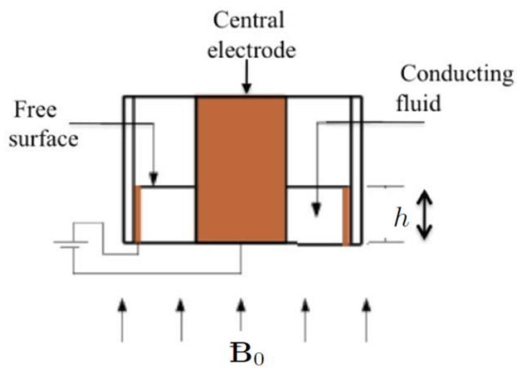


Figure 2 – Schematic diagram of cross-section (Perez-Barrera et al. 2015)

1.2 Motivation to study the evolution of amplitudes

Solving a fluid mechanics problem such as this would naturally involve the Navier-Stokes equations. Additionally, the forces that drive this motion are of electromagnetic nature. They are governed by Maxwell's equations. Hence, the full mathematical model would involve a system of coupled partial differential equations involving Navier-Stokes and Maxwell's equations, which would be very challenging to solve. However, in the work undertaken by McCloughan & Suslov (2020) it has been shown that the evolution of amplitude of a single azimuthally invariant mode (a simple solution to the original problem) can be represented by an ordinary differential equation that takes the following form:

$$\frac{dA}{dt} = \sigma A + K_1\delta + K_2A^2 + K_3\delta A + K_4A^3. \tag{1}$$

This equation is a much more straightforward to analyse compared to the original problem involving PDEs. However, the evolution of amplitude of a single mode does not necessarily describe the complete the physical problem as we clearly observe multiple vortical systems interacting with each other. Thus we attempt to establish the algebraic form of a system of multiple amplitude equations capturing more details of the original full problem starting from the simplest possible form and adding terms when necessary.

Statement of Authorship

Wickramarathne followed through existing derivations for the amplitude equations done by McCloughan & Suslov. Wolfram Mathematica code for the 1 periodic and 1 aperiodic mode case was written by Suslov. Wickramarathne worked on adding a coefficient subscript convention and trying to extend the code to include 2 periodic modes and 1 aperiodic mode. Wickramarathne worked on analysing dynamical system, deducing stationary points, and sketching phase portraits on MATLAB with help from Suslov.
This report was written by Wickramarathne and proofread by Suslov.

2. Deriving Amplitude Equations

2.1 Illustrating technique of developing Amplitude Equations

We consider the prototype equation mimicking the system of Navier-Stokes-Maxweel's equations, where the time rate of change of the quantity of interest (e.g. representing the flow velocity) *w* is given by

$$\frac{\partial w}{\partial t} = Lw + F(w.w) \tag{2}$$

where *L* is a matrix and *F* is a non-linear coupling function, namely, $F=w^2$To make a meaningful use of this relationship we would need to know the algebraic form of w. We note that the expression for quantity *w* should contain steady base w_0 and a component that depends on the (small) amplitudes *A* that evolves in time. We first assume the simplest possible relationship for the latter component for one periodic mode with a strictly linear relationship between *w* and *A*, with the complex exponential terms accounting for the periodicity. It is followed by the complex conjugate to ensure that the solution remains real as required by its physical meaning.

$$w = w_0 + [\epsilon A w_1 e^{in\theta} + \epsilon \bar{A} \bar{w}_1 e^{-in\theta}] \tag{3}$$

The epsilon terms above are used to keep track of the order of the small amplitude. Since Eq. (2) contains the time derivative term, the expression for the time derivative of the amplitude also needs to be derived. We assume the simplest possible relationship with the derivative being a product of the amplitude itself and an eigenvalue

$$\frac{dA}{dt} = \sigma_n A \tag{4}$$

Once we have the two trial expressions for perturbation and the amplitude derivative, we substitute them in Eq. (2) and fully expand both sides of the equation.

$$LHS = \frac{\partial w}{\partial t} = \epsilon\,\frac{dA}{dt} w_1 e^{in\theta} + \epsilon\,\frac{d\bar{A}}{dt}\bar{w_1} e^{-in\theta}$$

$$\begin{aligned} RHS &= Lw + w^2 \\ &= Lw_0 + \epsilon L A w_1 e^{in\theta} + \epsilon L \bar{A}\bar{w}_1 e^{-in\theta} \\ &\quad + w_0^2 + \epsilon A w_0 w_1 e^{in\theta} + \epsilon \bar{A} w_0 \bar{w}_1 e^{-in\theta} \\ &\quad + \epsilon^2 A^2 w_1^2 e^{2in\theta} + \epsilon^2 \bar{A}^2 \bar{w_1}^2 e^{-2in\theta} \\ &\quad + \epsilon^2 A\bar{A} w_1 \bar{w}_1 \end{aligned}$$

Then we compare terms on either sides of the equation for each order of the amplitude, where the order of amplitude is distinguished by the epsilon term that was introduced. The zeroth order term corresponds to the basic flow. This defines the first group of constants.

$$\epsilon^0 : Lw_0 + w_0^2 = 0 \tag{5}$$

The first order terms are similarly collected and matched with respect to functional form as follows.

$$\begin{aligned} \epsilon^1 :\ \frac{dA}{dt} w_1 e^{in\theta} + \frac{d\bar{A}}{dt}\bar{w}_1 e^{-in\theta} &= LA w_1 e^{in\theta} + L\bar{A}\bar{w}_1 e^{-in\theta} \\ &\quad + A w_0 w_1 e^{in\theta} + \bar{A} w_0 \bar{w}_1 e^{-in\theta} \end{aligned}$$

$$\sigma_n A w_1 e^{in\theta} + \bar{\sigma}_n \bar{A} \bar{w}_1 e^{-in\theta} = (L + w_0)[A w_1 e^{in\theta} + \bar{A} \bar{w}_1 e^{-in\theta}] \quad (6)$$

At this stage we should notice that the (L+w_0) part of this expression has already been solved and in the previous order, which means these constants have already been defined. Hence, we need to add some terms to the trial expressions for *w* and *A* (Eqs (3) and (4)) to ensure orthogonality of the expansion (this will be discussed further in the next section). At the next step, we collect and compare the second order terms in the full expansion

$$\epsilon^2 : 0 = A^2 w_1^2 e^{2in\theta} + \bar{A}^2 \bar{w}_1{}^2 e^{-2in\theta} + A\bar{A} w_1 \bar{w}_1 \quad (7)$$

As seen above, the solution is completely lost at the second order as there are no derivative terms to match the second order amplitude terms on the right-hand side of the equation. This implies we need to add higher order terms to our trial expressions for *w* and *A* (Eqs (3) and (4)) to produce a meaningful solution.

In summary, the procedure is to trial our current expression for *w* and *A* in Eq. (2), separate the full expansion into amplitude orders, collect terms of similar functional form, identify inconsistencies, and hence add more terms to the existing expressions for *w* and *A,* trial the new expressions in Eq. (2) and so on until a meaningful solution is obtained.

2.2 Considerations when adding terms to the perturbation and amplitude derivative expressions

We consider two main points when adding new terms to our perturbation and amplitude derivative expressions.

1. **Preserving the solution at the next order**
   As illustrated in the previous example, a higher order amplitude terms must be added to the original trial expressions to avoid losing the solution at the second (and higher) order.
2. **Maintaining orthogonality of the expansion**
   Since some of the constants in the expansion are defined in the previous order, another group of terms must be added to the expansion to determine constants that

add a degree of freedom to the expression. This is vital to ensure orthogonality of terms that are added to the expansion at each order. If orthogonality is not preserved, a part of the existing solution will be overwritten when considering the next order. This is not a particularly efficient way to form an expansion, especially if the expansion is to be truncated at a given order. When orthogonality is preserved, we will be adding a completely new part to the solution at each order.

2.3 Formulated Amplitude equations (1 Periodic mode + 1 Aperiodic mode)

We follow the above-mentioned technique until a consistent solution is obtained up to the third order. An important point to make is that the simplest case that was initially assumed (1 periodic mode) alone cannot provide a meaningful non-trivial solution to the problem. Hence while also adding new terms to the amplitude expression of the periodic mode, we also need to introduce another expression for the amplitude of an aperiodic mode, which results in a dynamical system with two amplitude equations. Hence this (1 periodic mode + 1 aperiodic mode) would be the simplest case that can meaningfully model the physical problem. Of course, adding more periodic modes would improve the model but the current case is already algebraically tedious. Hence, we make use of Wolfram Mathematica to expand the full equation, and to distinguish and collect like terms in each order, following the same logic as before when manually collecting and comparing terms. The resulting expression for perturbation w is as follows.

$$
\begin{aligned}
w = {} & w0000 + \epsilon\,(A0\,w1000 + Am\,w0101\,em + Amb\,w0101b/em) + \\
& \epsilon^2 \left(em^2\,Am^2\,w0202 + Amb^2\,w0202b\big/em^2 + Am\,Amb\,w0200 + \delta\,w0020 + \right. \\
& \qquad \left. A0^2\,w2000 + em\,A0\,Am\,w1101\right) + \\
& \epsilon^3 \left(Am^2\,Amb\,w0301\,em + Amb^2\,Am\,w0301b/em + A0^2\,Am\,w2101\,em + \right. \\
& \qquad A0^2\,Amb\,w2101b/em + em^2\,A0\,Am^2\,w1202 + A0\,Amb^2\,w1202b\big/em^2 + \\
& \qquad Am^3\,w0303\,em^3 + Amb^3\,w0303b\big/em^3 + A0\,Am\,Amb\,w1200 + \\
& \qquad \left. A0^3\,w3000 + A0\,\delta\,w1020 + \delta\,Am\,em\,w0121 + \delta\,Amb\,w0121b/em\right);
\end{aligned}
$$

More importantly, the corresponding amplitude equations for the periodic and aperiodic modes take the following form.

$$\frac{dA_m}{dt} = \sigma_m A_m + \epsilon\, K_{1101} A_m A_0 + \epsilon^2 \left(K_{0301} A_m^2 \bar{A_m} + K_{2101} A_0^2 A_m + K_{0121} \delta A_m\right)$$

$$\frac{dA_0}{dt} = \sigma_0 A_0 + \epsilon\, K_{0020}\delta + \epsilon^2 \left(K_{3000} A_0^3 + K_{1200} A_0 A_m \bar{A_m} + K_{1020} A_0 \delta\right) \tag{8}$$

As mentioned before, the existence of two amplitude modes produces a dynamical system.

3. Analysing dynamical system

3.1 Algebraic analysis of the obtained dynamical system

We firstly set both amplitude equations to zero in order to determine the stationary points. To start with, the equations are simplified setting the delta terms to 0 so that the amplitude equations can be factorised. Subsequently, the delta shift can be added separately to determine how it influences the main solution.

$$\frac{dA_m}{dt} = A_m[\sigma_m + K_{1101} A_0 + K_{0301}|A_m|^2 + K_{2101} A_0^2] = 0$$

$$\frac{dA_0}{dt} = A_0[\sigma_0 + K_{3000} A_0^2 + K_{1200}|A_m|^2] = 0 \tag{9}$$

The second equation can be expressed in terms of the periodic mode amplitude as follows

$$|A_m|_e^2 = -\left(\frac{\sigma_0 + K_{3000} A_{0e}^2}{K_{1200}}\right) \tag{10}$$

Substituting this term in the first equation results in a quadratic equation for the aperiodic mode amplitude

$$\sigma_m + K_{1101} A_0 + K_{2101} A_0^2 - \frac{K_{0301}}{K_{1200}}\left(\sigma_0 + K_{3000} A_0^2\right) = 0 \tag{11}$$

This can be solved for the stationary points of the aperiodic amplitude. It is a rather lengthy expression, however, it is fully defined by constants that depend on the physical setup

$$A_{0e} = \frac{-K_{1101} \pm \sqrt{K_{1101}^2 - 4(K_{2101} - \frac{K_{0301}K_{3000}}{K_{1200}})(\sigma_m - \frac{K_{0301}}{K_{1200}}\sigma_0)}}{2(K_{2101} - \frac{K_{0301}K_{3000}}{K_{1200}})} \tag{12}$$

This expression can be substituted back to explicitly define the stationary points for the periodic amplitude

$$|A_m|_e = \sqrt{-\frac{(\sigma_0 + K_{3000}A_{0e}^2)}{K_{1200}}} \tag{13}$$

Next, we introduce a small increment from the stationary point in an attempt to deduce the matrix of eigenvalues that define the dynamical system. Firstly, for the case of the aperiodic amplitude, an increment is added to the stationary point and the equation is expanded as follows.

$$\frac{d(A_{0e} + \epsilon A_0)}{dt} = (A_{0e}+\epsilon A_0)[\sigma_0+K_{3000}(A_{0e}+\epsilon A_0)^2+K_{1200}(|A_m|_e+\epsilon|A_m|)^2] \tag{14}$$

The portion of the solution involving the stationary point has already been found. Hence, only the incremental portion of the equation needs to be considered. We then accumulate all terms of the first order of epsilon and collect terms corresponding to the periodic and aperiodic mode amplitudes. Considering higher orders of epsilon would add concavity to the solution that is obtained here

$$\epsilon^1 : \ \frac{dA_0}{dt} = (2K_{3000}A_{0e}^2)\ A_0 + (2K_{1200}A_{0e}|A_m|_e)\ |A_m| \tag{15}$$

Exactly the same approach is used for the equation describing the Periodic amplitude:

$$\frac{d(|A_m|_e + \epsilon|A_m|)}{dt} = (|A_m|_e + \epsilon|A_m|)\left[\sigma_m + K_{1101}(A_{0e} + \epsilon A_0) + K_{2101}(A_{0e} + \epsilon A_0)^2 + K_{0301}(|A_m|_e + \epsilon|A_m|)^2\right] \tag{16}$$

$$\epsilon^1 : \frac{d|A_m|}{dt} = (|A_m|_e(K_{1101} + 2K_{2101}A_{0e}))\, A_0 + (2K_{0301}|A_m|_e^2)\, |A_m| \tag{17}$$

The above results for the first order increments to the amplitude equations can be written in a matrix form as

$$\begin{bmatrix} \frac{dA_0}{dt} \\ \frac{d|A_m|}{dt} \end{bmatrix} = \begin{bmatrix} (2K_{3000}A_{0e}^2) & (2K_{1200}A_{0e}|A_m|_e) \\ (|A_m|_e(K_{1101} + 2K_{2101}A_{0e})) & (2K_{0301}|A_m|_e^2) \end{bmatrix} \begin{bmatrix} A_0 \\ |A_m| \end{bmatrix} \tag{18}$$

where matrix

$$\chi = \begin{bmatrix} (2K_{3000}A_{0e}^2) & (2K_{1200}A_{0e}|A_m|_e) \\ (|A_m|_e(K_{1101} + 2K_{2101}A_{0e})) & (2K_{0301}|A_m|_e^2) \end{bmatrix} \tag{19}$$

depends on the constants that arise from the physical parameters of the original system.

3.2 Phase portraits of the obtained dynamical system

Since the dynamical system contains the amplitudes of aperiodic and periodic modes and of their derivatives, we can use these two equations to plot a phase portrait of the two amplitudes with amplitude derivatives shown by vectors. This is shown in the following diagrams.

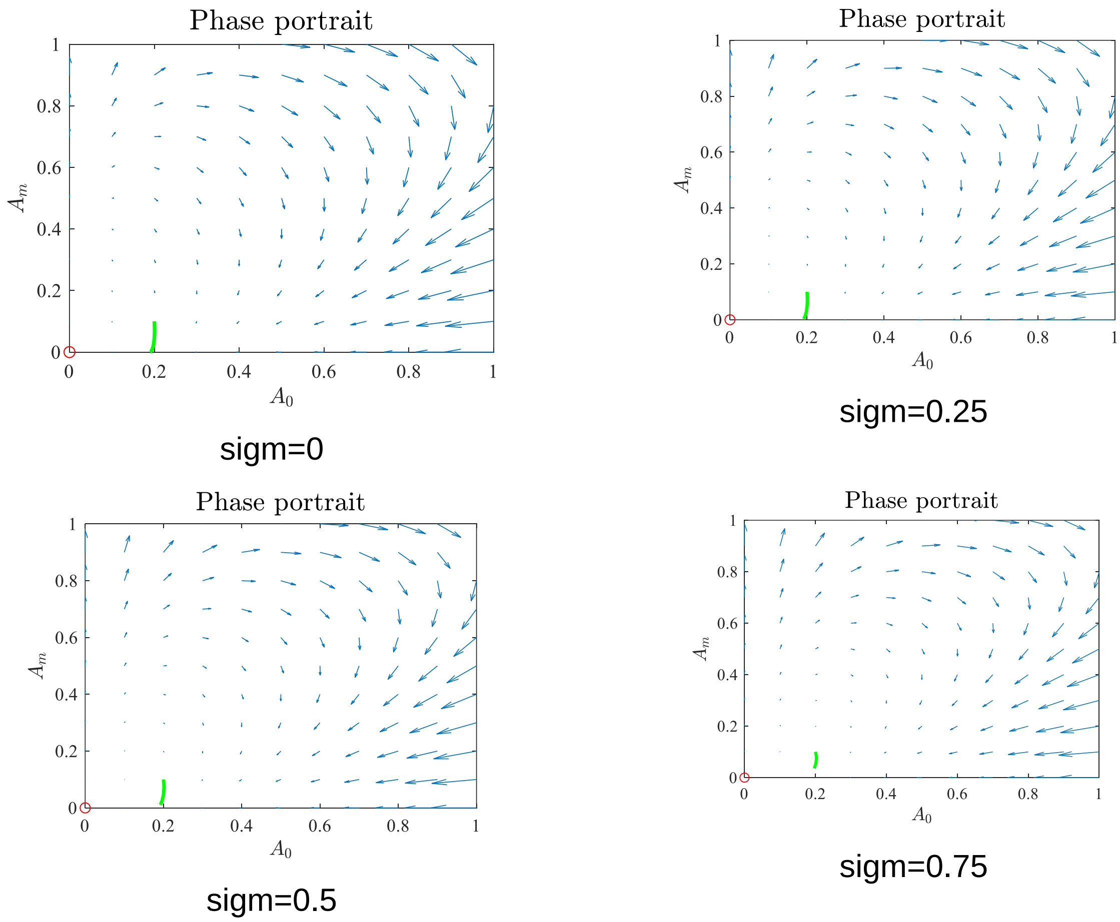
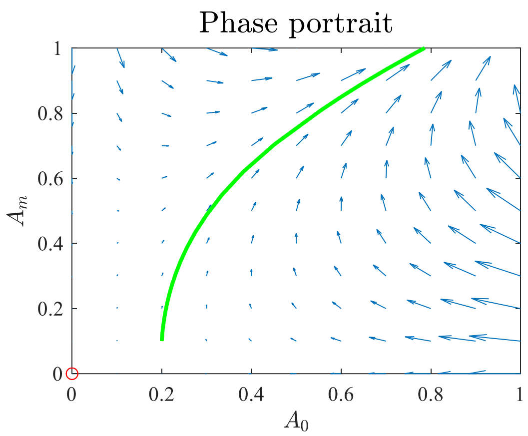


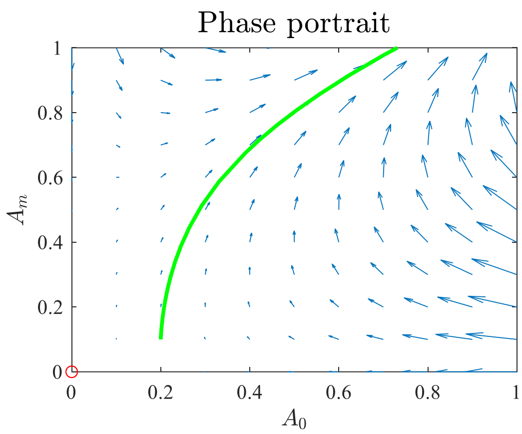
Figure 3 – Phase portraits obtained by setting K2101=1, K0301=2, K3000=-3, K1200=4, K1101=-5, K0020=-8, K1020=-9, sig0=0.1, and varying sigm

Additionally, we can also numerically solve this problem by considering it as a system of coupled ODEs. In this case the numerical solution is done using the Runge-Kutta method on MATLAB, and this solution is shown in green in the following diagrams. Naturally, this has to follow the path shown in the phase portrait as they illustrate the same system.
We note that sketching these diagrams require defining the constants that depend on the physical parameters of the set-up. However, since we are more interested in the algebraic form of the expression, we trial arbitrary values for these constants. In each of the following figures, we vary one of these constants over four values and leave the rest of the constants at fixed values.

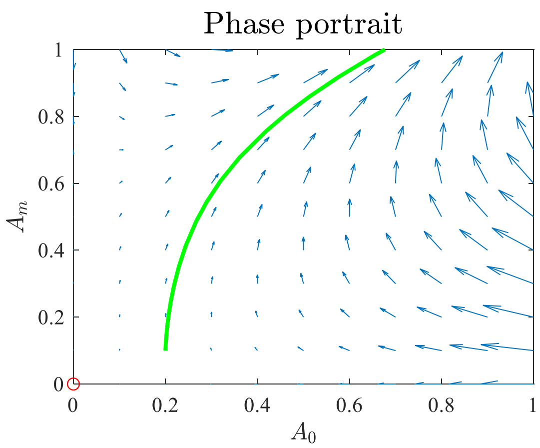
The set of phase portraits in Fig. 3 is quite interesting as it appears that a growing amplitude A_m of a periodic mode eventually leads to a decaying amplitude A_0 of an aperiodic mode, which circles back to a growing Periodic amplitude and so on. This could possibly mean a back and forth transfer of energy from one mode to another.



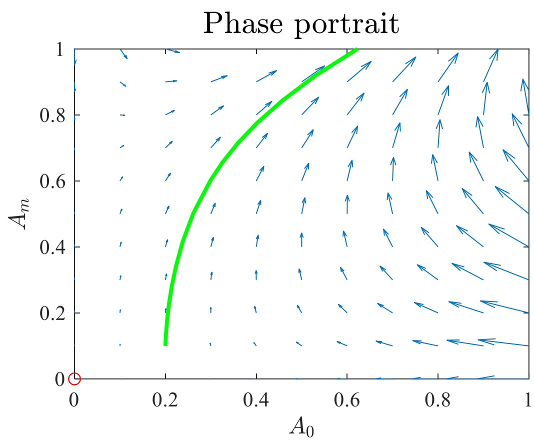
sigm=0



sigm=0.25



sigm=0.5



sigm=0.75

Figure 4 – Phase portraits obtained by setting K2101=-1, K0301=-2, K3000=-3, K1200=4, K1101=5, K0020=8, K1020=9, sig0=0.1, and varying sigm

The phase portraits in Fig. 4 seem somewhat unphysical as the decay of each of the amplitudes seems to lead back to a growth of both amplitudes, which does not make sense in terms of energy conservation.

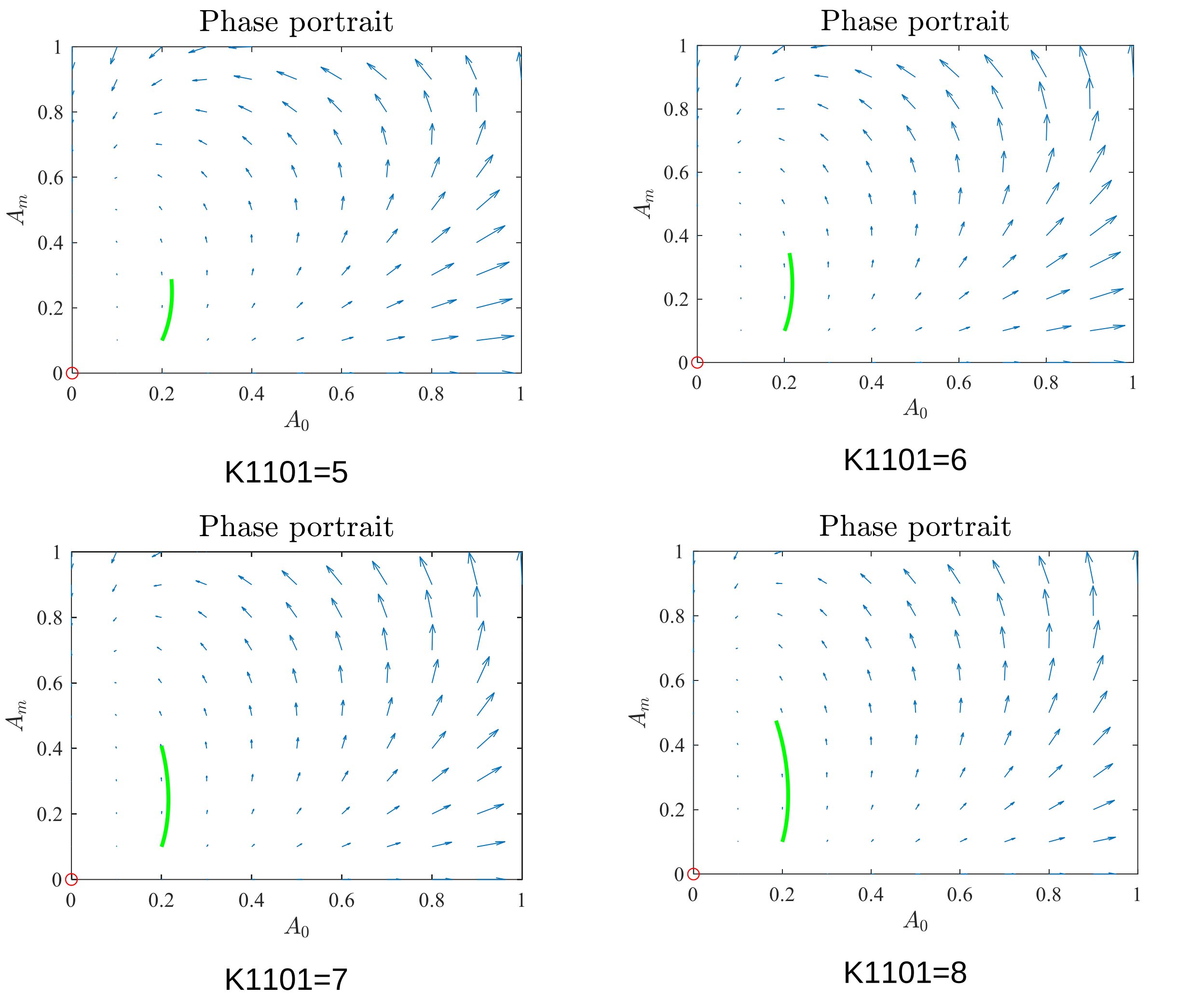


Figure 5 – Phase portraits obtained by setting K2101=-1, K0301=-2, K3000=3, K1200=-4, K0020=-8, K1020=-9, sigm=0.1, sig0=0.1, and varying K1101

The set of phase portraits in Fig. 5 seems to be very similar to what we obtained from the first selection of constants in Fig. 3, with the directions reversed. Hence this set of plots also makes sense in terms of energy conservation.

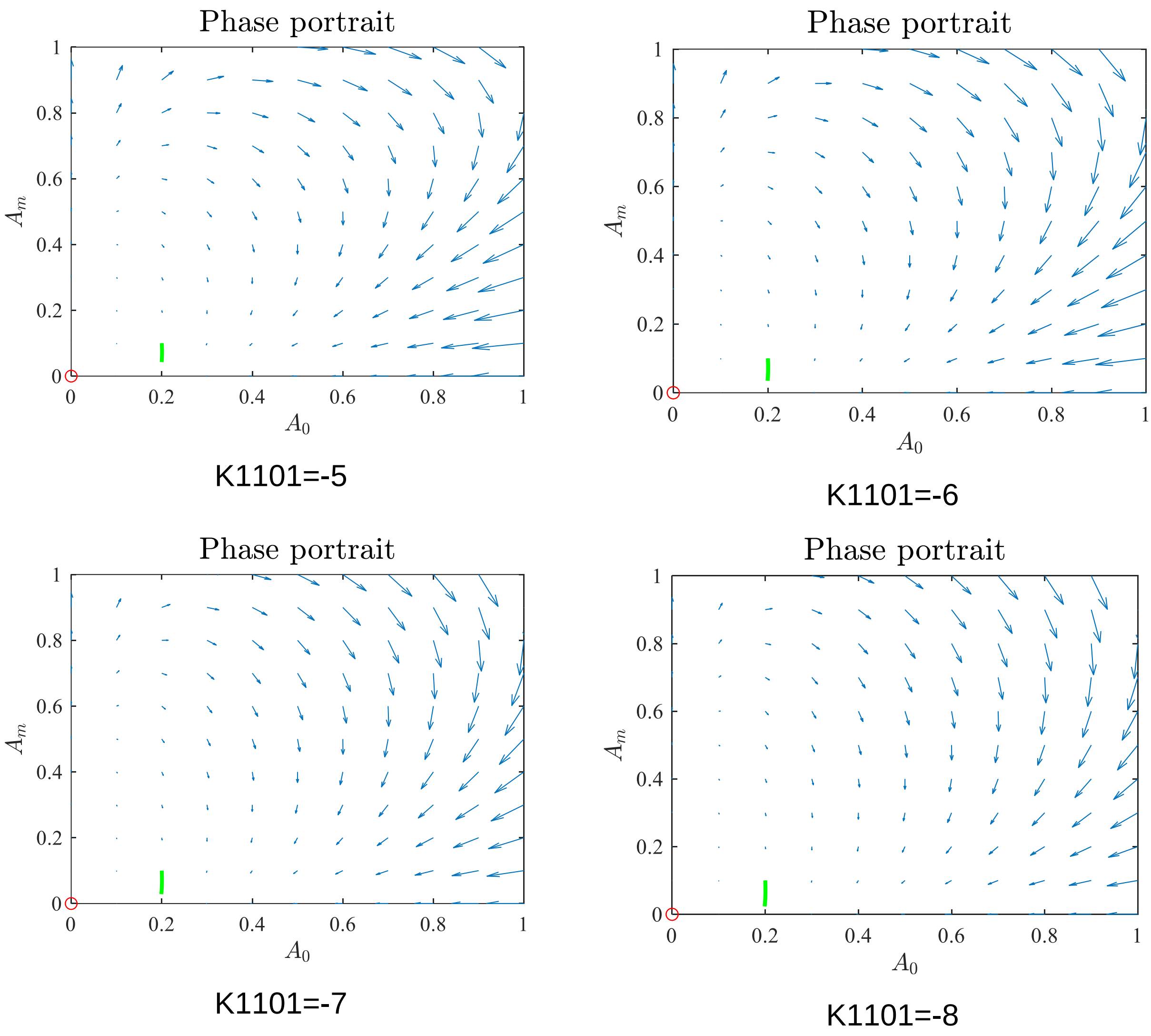


Figure 6 – Phase portraits obtained by setting K2101=1, K0301=2, K3000=-3, K1200=4, K0020=8, K1020=9, sigm=0.1, sig0=0.1, and varying K1101

The phase portraits in Fig. 6 are also very similar to the ones in Fig. 3 with one mode growing when the other decays and vice versa.

4. Discussion and Conclusion

The main aim of this project was to establish the algebraic form of the dynamical system describing the evolution of amplitudes of vortical modes, which we managed to do up to 1 periodic and 1 aperiodic mode. We then analysed the obtained dynamical system and determined an explicit formula for the stationary points of the system, and the matrix of the eigensystem in the vicinity of the stationary point. The dynamical system was graphically interpreted by sketching phase portraits and numerical solutions for arbitrary constants.

There was quite a bit of variability in the phase portraits depending on these constants. Hence, it would be of interest to do this analysis with the actual values of the constants obtained from the physical parameters of the experiment. It would also be useful to distinguish strictly real values of stationary points from the generally complex ones obtained from the derived expression for these stationary points.

Of course, the main improvement that can be done is to introduce extra modes to the system. This would be algebraically quite challenging as adding another mode adds an extra set of constants for the perturbation expression, and a third ODE to the dynamical system of amplitude equations. For reference, the full expansion at the second order for the case of 2 periodic modes + 1 aperiodic mode is as follows, which has quite a variety of terms to deal with.

Out[▫]=

$$
\begin{aligned}
&-\mathrm{L}\,\mathrm{w000200}\,\delta - 2\,\mathrm{w000000}\,\mathrm{w000200}\,\delta + \mathrm{K000200}\,\mathrm{w100000}\,\delta - \mathrm{w100000}^2\,\mathrm{a0}[t]^2 - \mathrm{L}\,\mathrm{w200000}\,\mathrm{a0}[t]^2 - 2\,\mathrm{w000000}\,\mathrm{w200000}\,\mathrm{a0}[t]^2 + 2\,\mathrm{w200000}\,\sigma 0\,\mathrm{a0}[t]^2 + \\
&\mathrm{em}\,\mathrm{K110010}\,\mathrm{w010010}\,\mathrm{a0}[t]\times\mathrm{am}[t] - 2\,\mathrm{em}\,\mathrm{w010010}\,\mathrm{w100000}\,\mathrm{a0}[t]\times\mathrm{am}[t] - \mathrm{em}\,\mathrm{L}\,\mathrm{w110010}\,\mathrm{a0}[t]\times\mathrm{am}[t] - 2\,\mathrm{em}\,\mathrm{w000000}\,\mathrm{w110010}\,\mathrm{a0}[t]\times\mathrm{am}[t] + \\
&\mathrm{em}\,\mathrm{w110010}\,\sigma 0\,\mathrm{a0}[t]\times\mathrm{am}[t] + \mathrm{em}\,\mathrm{w110010}\,\sigma\mathrm{m}\,\mathrm{a0}[t]\times\mathrm{am}[t] - \mathrm{em}^2\,\mathrm{w010010}^2\,\mathrm{am}[t]^2 - \mathrm{em}^2\,\mathrm{L}\,\mathrm{w020020}\,\mathrm{am}[t]^2 - 2\,\mathrm{em}^2\,\mathrm{w000000}\,\mathrm{w020020}\,\mathrm{am}[t]^2 + \\
&2\,\mathrm{em}^2\,\mathrm{w020020}\,\sigma\mathrm{m}\,\mathrm{am}[t]^2 + \frac{\mathrm{K11001b0}\,\mathrm{w01001b0}\,\mathrm{a0}[t]\times\mathrm{amb}[t]}{\mathrm{em}} - \frac{2\,\mathrm{w01001b0}\,\mathrm{w100000}\,\mathrm{a0}[t]\times\mathrm{amb}[t]}{\mathrm{em}} - 2\,\mathrm{w010010}\,\mathrm{w01001b0}\,\mathrm{am}[t]\times\mathrm{amb}[t] - \\
&\mathrm{L}\,\mathrm{w020000}\,\mathrm{am}[t]\times\mathrm{amb}[t] - 2\,\mathrm{w000000}\,\mathrm{w020000}\,\mathrm{am}[t]\times\mathrm{amb}[t] + \mathrm{w020000}\,\sigma\mathrm{m}\,\mathrm{am}[t]\times\mathrm{amb}[t] + \mathrm{w020000}\,\sigma\mathrm{mb}\,\mathrm{am}[t]\times\mathrm{amb}[t] - \frac{\mathrm{w01001b0}^2\,\mathrm{amb}[t]^2}{\mathrm{em}^2} - \\
&\frac{\mathrm{L}\,\mathrm{w02002b0}\,\mathrm{amb}[t]^2}{\mathrm{em}^2} - \frac{2\,\mathrm{w000000}\,\mathrm{w02002b0}\,\mathrm{amb}[t]^2}{\mathrm{em}^2} + \frac{2\,\mathrm{w02002b0}\,\sigma\mathrm{mb}\,\mathrm{amb}[t]^2}{\mathrm{em}^2} + \mathrm{en}\,\mathrm{K101001}\,\mathrm{w001001}\,\mathrm{a0}[t]\times\mathrm{an}[t] - 2\,\mathrm{en}\,\mathrm{w001001}\,\mathrm{w100000}\,\mathrm{a0}[t]\times\mathrm{an}[t] - \\
&\mathrm{en}\,\mathrm{L}\,\mathrm{w101001}\,\mathrm{a0}[t]\times\mathrm{an}[t] - 2\,\mathrm{en}\,\mathrm{w000000}\,\mathrm{w101001}\,\mathrm{a0}[t]\times\mathrm{an}[t] + \mathrm{en}\,\mathrm{w101001}\,\sigma 0\,\mathrm{a0}[t]\times\mathrm{an}[t] + \mathrm{en}\,\mathrm{w101001}\,\sigma\mathrm{n}\,\mathrm{a0}[t]\times\mathrm{an}[t] - \\
&2\,\mathrm{em}\,\mathrm{en}\,\mathrm{w001001}\,\mathrm{w010010}\,\mathrm{am}[t]\times\mathrm{an}[t] - \frac{2\,\mathrm{en}\,\mathrm{w001001}\,\mathrm{w01001b0}\,\mathrm{amb}[t]\times\mathrm{an}[t]}{\mathrm{em}} - \mathrm{en}^2\,\mathrm{w001001}^2\,\mathrm{an}[t]^2 - \mathrm{en}^2\,\mathrm{L}\,\mathrm{w002002}\,\mathrm{an}[t]^2 - 2\,\mathrm{en}^2\,\mathrm{w000000}\,\mathrm{w002002}\,\mathrm{an}[t]^2 + \\
&2\,\mathrm{en}^2\,\mathrm{w002002}\,\sigma\mathrm{n}\,\mathrm{an}[t]^2 + \frac{\mathrm{K101001b}\,\mathrm{w001001b}\,\mathrm{a0}[t]\times\mathrm{anb}[t]}{\mathrm{en}} - \frac{2\,\mathrm{w001001b}\,\mathrm{w100000}\,\mathrm{a0}[t]\times\mathrm{anb}[t]}{\mathrm{en}} - \frac{2\,\mathrm{em}\,\mathrm{w001001b}\,\mathrm{w010010}\,\mathrm{am}[t]\times\mathrm{anb}[t]}{\mathrm{en}} - \\
&\frac{2\,\mathrm{w001001b}\,\mathrm{w01001b0}\,\mathrm{amb}[t]\times\mathrm{anb}[t]}{\mathrm{em}\,\mathrm{en}} - 2\,\mathrm{w001001}\,\mathrm{w001001b}\,\mathrm{an}[t]\times\mathrm{anb}[t] - \mathrm{L}\,\mathrm{w002000}\,\mathrm{an}[t]\times\mathrm{anb}[t] - 2\,\mathrm{w000000}\,\mathrm{w002000}\,\mathrm{an}[t]\times\mathrm{anb}[t] + \\
&\mathrm{w002000}\,\sigma\mathrm{n}\,\mathrm{an}[t]\times\mathrm{anb}[t] + \mathrm{w002000}\,\sigma\mathrm{nb}\,\mathrm{an}[t]\times\mathrm{anb}[t] - \frac{\mathrm{w001001b}^2\,\mathrm{anb}[t]^2}{\mathrm{en}^2} - \frac{\mathrm{L}\,\mathrm{w002002b}\,\mathrm{anb}[t]^2}{\mathrm{en}^2} - \frac{2\,\mathrm{w000000}\,\mathrm{w002002b}\,\mathrm{anb}[t]^2}{\mathrm{en}^2} + \frac{2\,\mathrm{w002002b}\,\sigma\mathrm{nb}\,\mathrm{anb}[t]^2}{\mathrm{en}^2}
\end{aligned}
$$

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