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Metrical Properties of Continued Fractions

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Abstract

The aim of this work is to investigate Dirichlet's approximation theorem established in 1842, its improvements, and its connections with the growth of product of partial quotients in their continued fractions. Furthermore, we investigate the metrical theory for the sets of real numbers such that when they are expanded in their continued fractions have digits restricted to be prime numbers. We prove that the corresponding set of real numbers obey a zero-full law if a certain series converges or diverges depending upon the growth rate of the partial quotients.

1 Introduction

Through this report we will build upon the works of Kleinblock and Wadleigh [7] and Schindler and Zweimüller [3]. This area of study is centred around approximation of irrational numbers by rational numbers. Whilst every irrational number can be approximated, certain numbers containing certain properties have better approximations. Every number is approximated by a continued fraction and the *partial quotients* of the continued fractions can be considered in sets. The partial quotients of a continued fraction are the varible values of the diagonal and are expressed in the list notation of the continued fraction. Through restrictions to the partial quotients of the continued fractions of irrational numbers the way these partial quotients grow can be modelled by functions $\phi(n)$. Through restricting these partial quotients to being prime we can find the set of functions $\phi(n)$ for which the product of two consecutive partial quotients of the irrational number is less than the value of $\phi(n)$.

2 Statement Of Authorship

This research was completed as a collaboration between Lauren White and Dr Mumtaz Hussain. The theorem developed was originate by Dr Hussain and the proof written by White, as based on proofs by Kleinbock and Wadleigh and the Prime Number Theorem.

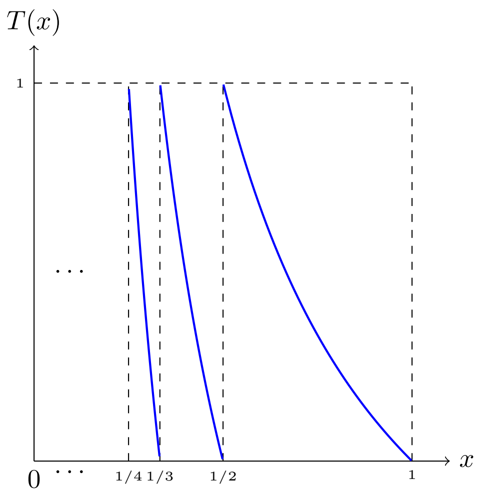
3 Continued Fractions and Diophantine approximation

Any real number x can be written as a sum of the integer part and the fractional part, denoted $x = \{x\} + [x]$ where $\{x\}$ is the fractional part and $[x]$ as the integer part. Now, let $\left[\frac{1}{x}\right] = n$, then

$$n \leq \frac{1}{x} < n+1 \Longleftrightarrow \frac{1}{n+1} < x \leq \frac{1}{n}.$$

The Gauss map $T : [0,1) \to [0,1)$ is a relation defined by

$$T(x) = \begin{cases} 0 & \text{if} \quad x = 0, \\ \frac{1}{x} - n & \text{if} \quad \frac{1}{n+1} < x \leq \frac{1}{n}. \end{cases}$$



Every iteration of this graph is restricted to the interval $\left(\frac{1}{n+1}, \frac{1}{n}\right]$ which is called the *branch*. These branches are invertible due to their properties.

For any $x \in (0,1)$ define the sequence of natural numbers $(a_n)_{n=1}^{\infty} = (a_n(x))_{n=1}^{\infty}$ by

$$\frac{1}{1+a_n} < T^{n-1}(x) < \frac{1}{a_n}. \tag{1}$$

Equivalently,

$$a_n(x) = \left\lfloor \frac{1}{T^{n-1}(x)} \right\rfloor \quad \text{for each} \quad n \geq 1. \tag{2}$$

By induction, it is straightforward to see that, for any $x \in (0,1)$ the sequence $(a_n(x))_{n=1}^{\infty}$ defined in (2) gives the entries of the continued fraction expansion to x that is

$$x := \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cdots \cfrac{1}{a_n(x) + T^n(x).}}} \tag{3}$$

$$x = [a_1(x), a_2(x), \ldots] = [a_1(x), a_2(x), \ldots, a_n(x) + T^n(x)].$$

If a continued fraction contains a restricted number of partial quotients the number is rational, these continued fractions are not unique, for example,

$$\frac{67}{29} = 2 + \cfrac{1}{3 + \frac{1}{4+\frac{1}{2}}} = [2,3,4,2]$$

If $n \to \infty$, x is an irrational number and the continued fraction expansion is unique, for example the continued fraction expansion of $\sqrt{3}$ is given uniquely as

$$\sqrt{3} = 1 + \cfrac{1}{1 + \frac{1}{2+\frac{1}{1+\frac{1}{2+\ldots}}}} = [1,1,2,1,2,1,2,1,2\ldots] = [1, \overline{1,2}]$$

Consider the *convergents* "$\frac{p_n(x)}{q_n(x)}$" which are obtained by terminating the continued fraction expansion of x at level n, thus calling them the n^{th} convergents. Let $p_{-1} := 1,\ q_{-1} := 0,\ p_0 := 0,\ q_0 := 1$ (sometimes we write p_n, q_n for $p_n(x), q_n(x)$). Define the recursive formula

$$p_n = a_n(x)p_{n-1} + p_{n-2}, \quad \text{and} \quad q_n = a_n(x)q_{n-1} + q_{n-2} \quad \text{for} \quad n \in \mathbb{N}. \tag{4}$$

Also,

$$p_{n-1}q_n - p_n q_{n-1} = (-1)^n, \text{ for all } n \geq 1. \tag{5}$$

Theorem 3.1 (Lagrange)**.** *The convergents of x are optimal rational approximations of x in the sense that*

$$\left| x - \frac{p_n(x)}{q_n(x)} \right| \leq \inf_{p \in \mathbb{Z}, q \leq q_n(x)} \left| x - \frac{p}{q} \right|.$$

Theorem 3.2 (Legendre)**.** *If p/q satisfies $\left| x - \frac{p}{q} \right| < \frac{1}{2q^2}$ then*

$$\frac{p}{q} = \frac{p_n(x)}{q_n(x)} \quad \text{for some } n \geq 1. \tag{6}$$

Hence the n^{th} convergents are the best approximates of any irrational number x. In fact, we have

$$\frac{1}{(a_{n+1}+2)q_n^2} < \left|x - \frac{p_n}{q_n}\right| = \frac{1}{q_n(q_n T^{n+1}x + q_{n+1})} < \frac{1}{a_{n+1}q_n^2}. \tag{7}$$

Thus it follows that for an irrational number x there exits infinitely many rational numbers $\frac{p_n}{q_n}$ such that

$$\left|x - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2}. \tag{8}$$

By choosing $q_n \geq q$, we conclude that for any irrational numbers x there are infinitely many rational numbers p/q such that $|x - \frac{p}{q}| < \frac{1}{q^2}$. This fact is also a consequence of the well-known Dirichlet's approximation theorem.

Theorem 3.3 (Dirichlet, 1842)**.** *For any real number x and $N \in \mathbb{N}$, there exists integers p, q with $1 \leq q \leq N$ such that*

$$\left|x - \frac{p}{q}\right| \leq \frac{1}{qN}.$$

Proof. Let $x \in \mathbb{R}$, with $[x]$ being the integer part of x and $\{x\}$ being the fractional part such that $x = [x] + \{x\}$. The unit interval (0,1) can be divided into N smaller intervals of size 1/N. There are N + 1 numbers within this interval, we will call them $a_0, a_1, ..., a_n$. By the pigeonhole principle, we know that there must be two numbers within the same smaller interval. We denote these two values a_i and a_j where $0 < i < j < N$. Now:

$$|\{a_j x\} - \{a_i x\}| < 1/N$$
$$|(a_j x - [a_j x]) - (a_i x - [a_i x])| = |(a_j - a_i)x - ([a_j x] - [a_i x])| < 1/N.$$

Let $q = a_i - a_j$ and $p = [a_j x] - [a_i x]$ where $0 < q < N$ then

$$|qx - p| < 1/N.$$

□

Corollary 3.4. *For any irrational $x \in \mathbb{R}$, there exists infinitely many rational numbers p/q such that*

$$\left|x - \frac{p}{q}\right| < \frac{1}{q^2}. \tag{9}$$

Can we replace the right-hand side of (9) with a quantity tending to zero faster as the denominator size increases? The following result answers this question.

Theorem 3.5 (Hurwitz, 1895)**.** *For any irrational real number x there exist infinitely many integers p and $q > 0$ such that*

$$\left|x - \frac{p}{q}\right| \leq \frac{1}{\sqrt{5}q^2}. \tag{10}$$

It has been found that $\sqrt{5}$ is the largest constant possible for this inequality to hold.

Going back to the Legendre's theorem and properties of continued fractions (7) we note that the set can be written in terms of continued fractions as

$$\Lambda(\psi) = \left\{ x \in [0,1) : a_n(x) \geq \frac{1}{q_n^2(x)\psi(q_n(x))} \text{ for infinitely many } n \in \mathbb{N} \right\}. \tag{11}$$

Since the partial quotients can be obtained through Gauss map, the theory has close connections with dynamical systems, ergodic theory, and Diophantine approximation. To be precise, the ergodic theory of Gauss iterated functions systems and Diophantine approximations have been intertwined through the metrical properties of the following set

$$\mathcal{E}(\Psi) := \{x \in [0,1) : a_n(x) \geq \Psi(n) \text{ for infinitely many } n \in \mathbb{N}\}.$$

where $\Psi : \mathbb{N} \to \mathbb{R}^+$ is a positive function. One of the fundamental results in the metrical theory of continued fraction is Borel-Bernstein's theorem (1911, 1912) which gives the Lebesgue measure of the set $\mathcal{E}(\Psi)$.

Theorem 3.6 ([1, 2], Borel-Bernstein)**.** *Let* Ψ *be a positive function. Then*

$$\mathcal{L}\left(\mathcal{E}(\Psi)\right) = \begin{cases} 0 & \text{if} \quad \sum_{n=1}^{\infty} \frac{1}{\Psi(n)} < \infty, \\ 1 & \text{if} \quad \sum_{n=1}^{\infty} \frac{1}{\Psi(n)} = \infty. \end{cases}$$

"What if the partial quotients are all prime?" is a natural question that leads to the following study of the metrical properties of the set $\mathcal{E}(\Psi)$ by restricting the partial quotients to be prime. Let $\mathbb{P}$ denote the set of primes. To single them out as per Schindler-Zweimüller [3] we will define x in the unit interval and $n \geq 1$,

$$a_n'(x) := 1_P(a_n(x)) \cdot a_n(x) = \begin{cases} a_n(x) & \text{if} \quad a_n(x) \in \mathbb{P}, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

With the aid of the prime number theorem and recent developments in this area we can derive information about prime digits in continued fractions. We can use some of the important behaviours of prime digits, such as the proportion of those $k \in 1, ..., n$ for which $a_k(x)$ is prime converges:

Proposition 3.7 ([3], Asymptotic frequency of prime digits)**.** *We have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} 1_P \circ a_k = \frac{1}{\log(2)} \log \prod_{p\in\mathcal{P}} (1 + \frac{1}{p(p+2)}) \quad a.e. \tag{13}$$

We consider the set $\widehat{\mathcal{E}}(\Phi)$ as a variation of the set $\mathcal{E}(\Phi)$:

$$\widehat{\mathcal{E}}(\Psi) := \{x \in [0,1) : a_n'(x) \geq \Psi(n) \text{ for infinitely many } n \in \mathbb{N}\}.$$

The Lebesgue measure of this set was given by Schindler and Zweimüller in 2022

Theorem 3.8 ([3], Schindler-Zweimüller, 2022)**.** *Let* Ψ *be a positive function. Then*

$$\mathcal{L}\left(\widehat{\mathcal{E}}(\Psi)\right) = \begin{cases} 0 & \text{if} \quad \sum_{n=1}^{\infty} \frac{1}{\Psi(n)\log(\Psi(n))} < \infty, \\ 1 & \text{if} \quad \sum_{n=1}^{\infty} \frac{1}{\Psi(n)\log(\Psi(n))} = \infty. \end{cases}$$

The above results express properties derived from dynamical systems and the Gauss map, which is known to preserve the probability density

$$\mu(x) := \frac{1}{\log(2)} \frac{1}{1+x}, \quad x \in I$$

The invariant *Gauss measure* μ_φ on $\mathcal{B}_I$ defined by the latter, $\mu_\varphi(B) := \int_B \mu(x)dx$, is exact and ergodic. As previously defined the continued fraction digits can be revealed by the Gauss map, T, in that for $x \in (0,1)$

$$x = [a_1(x), a_2(x), ...] \quad with \quad a_n(x) = a \circ T^{(n-1)}(x), n \geq 1$$

where $a : I \to \mathbb{N}$ is the *digit function* corresponding to the partition $\epsilon := \{I_k : k \geq 1\}$, that is, $a(x) := [\frac{1}{x}] = k$ for $x \in I_k$. The obtained sequence $(a \circ T^n)_{n \geq 0}$ on the probability space (I, B_I, μ_φ) exhibits interesting properties since a has infinite expectation

$$\int_I a d\mu_\varphi = \sum_{k \geq} k \mu_\varphi(I_k) = \infty$$

as

$$\mu_\varphi(I_k) = \log(\frac{(k+1)^2}{k(k+2)}) / \log(2) \sim \frac{1}{(\log 2\dot{k}^2)} \quad \text{for} \quad k \to \infty.$$

As in classical probability theory, the tail behaviour of the distribution, given by

$$\mu_\varphi\{x \in I : a(x) \geq K\} = \frac{1}{\log(2)} \cdot \log \frac{K+1}{K} \sim \frac{1}{\log(2)} \cdot \frac{1}{K} \quad \text{as} \quad K \to \infty$$

(which entails $L(N) := \int_I (a \wedge N) d\mu_\varphi = \sum_{K=1}^{N} \mu_\varphi(a \geq K) \sim \log N / \log 2$ as $N \to \infty$) is the key to fine asymptotic results. The study of continued fractions digit sequence goes beyond standard results, since the random variables $a \circ T^n$ are not independent. Yet, it is well known that they still satisfy a strong form of *asymptotic independence* or *mixing.*

Lemma 3.9 ([3], Tail behaviour and truncated expectation of a')**.** *The distribution of* a' *(with respect to the Gauss measure) satisfies*

$$\mu_\varphi\{x \in I : a'(x) \geq K\} \sim \frac{1}{\log 2} \cdot \frac{1}{K \log(K)} \quad as \quad K \to \infty. \tag{14}$$

Proof. First, the Prime Number Theorem is easily seen (cf. [4], Theorem 1.8.8) to imply that

$$p_n \sim n \log n \quad as \quad n \to \infty \tag{15}$$

where p_n denotes the nth prime number. Therefore

$$\sum_{n \geq N} \frac{1}{p_n^2} \sim \sum_{n \geq N} \frac{1}{n^2 (\log n)^2} \sim \frac{1}{N(\log N)^2} \quad as \quad N \to \infty.$$

Letting N(K) denote the least n with $p_n \geq K$, we have, as $K \to \infty$,

$$\mu_\varphi\{x \in I : a'(x) \geq K\} = \sum_{p \geq K, p \in \mathcal{P}} \mu_\varphi(I_p) \sim \frac{1}{\log 2} \sum_{p \geq K, p \in \mathcal{P}} \frac{1}{p^2} = \frac{1}{\log 2} \sum_{n \geq N(K)} \frac{1}{p_n^2}$$

and by the Prime Number Theorem, $N(K) \sim K/\log K$. Combining these observations yields the above lemma.

□

Proof of Theorem 1.8, *[3]*. Note that $\{x \in I : a'_j > c\} = \{x \in I : T^{-(j-1)}a' > c\}$ with $\{x \in I : a' > c\}$ measurable with respect to ξ. As a consequence of the continued fraction-mixing property, we see that Renyi's Borel-Cantelli Lemma applies to show that

$$\mu_\varphi\{x \in I : a'_n(x) > \phi(n)\} > 0 \quad \Longleftrightarrow \quad \sum_{n \geq 1} \mu_\varphi\{x \in I : a'_n(x) > \phi(n)\} = \infty \tag{16}$$

□

By T-invariance of μ_φ and Lemma 1.9, we have $\mu_\varphi(a'_n \geq \phi(n)) = \mu_\varphi(a' \geq \phi(n)) \sim \frac{1}{b_n \log(b_n)}$, so that divergence on the right-hand side series in Lemma (1.10) is equivalent to that of $\sum_{n \geq 1}(b_n \log(b_n))^{-1}$. Finally, again because of $a'_j > c = T^{-(j-1)}a' > c$, the set $a'_n > b_n i.o$ is easily seen to belong to the tail-σ-field $\bigcap_{n \geq 0} T^{-n}B_I$ of T. The system (I, B_I, μ_φ, T) begin exact, the latter is trivial mod μ_φ. Hence $\mu_\varphi(a'_n > \phi(n)i.o) > 0$ implies $\mu_\varphi(a'_n > \phi(n)) = 1$.

4 Product of partial quotients and uniform Diophantine approximation

Recall that the Gauss map $T : [0,1) \to [0,1)$ is defined by

$$T(0) = 0, \; T(x) = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor \text{ for } x \in (0,1),$$

where $\lfloor x \rfloor$ denotes the integer part of x. We write $x := [a_1(x), a_2(x), a_3(x), \ldots]$ for the continued fraction of x where $a_1(x) = \lfloor 1/x \rfloor$, $a_n(x) = a_1(T^{n-1}(x))$ for $n \geq 2$ are called the partial quotients of x. The sequences $p_n = p_n(x)$, $q_n = q_n(x)$, referred to as n^{th} convergents, has the recursive relation

$$p_{n+1} = a_{n+1}(x)p_n + p_{n-1}, \quad q_{n+1} = a_{n+1}(x)q_n + q_{n-1}, \quad n \geq 0.$$

Thus $p_n = p_n(x), q_n = q_n(x)$ are determined by the partial quotients $a_1, \ldots, a_n$, so we may write $p_n = p_n(a_1, \ldots, a_n), q_n = q_n(a_1, \ldots, a_n)$. When it is clear which partial quotients are involved, we denote them by p_n, q_n for simplicity.

Building on a work of Davenport-Schmidt [5], Kleinbock-Wadleigh [7] we consider the set

$$D(\psi) := \left\{ x \in \mathbb{R} : \begin{array}{l} \exists\, N \text{ such that the system } |qx - p| < \psi(t), \; |q| < t \\ \text{has a nontrivial integer solution for all } t > N \end{array} \right\}.$$

calling it a set of ψ-Dirichlet improvable numbers. A simple calculation shows the following simple yet extremely important criterion.

$$x \in D(\psi) \iff |q_{n-1}x - p_{n-1}| \,<\, \psi(q_n) \text{ for all } n \gg 1$$
$$\iff [a_{n+1}, a_{n+2}, \dots] \cdot [a_n, a_{n-1}, \dots, a_1] \,<\, \frac{1}{\Phi(q_n)} \text{ for all } n \gg 1,$$

where

$$\Phi(t) := \frac{t\psi(t)}{1 - t\psi(t)} = \frac{1}{1 - t\psi(t)} - 1. \tag{17}$$

In what follows, ψ and Φ will always be related by (17). This leads to the following criterian for Dirichlet improvability.

Lemma 4.1 ([7], Lemma 2.2)). *Let $x \in [0,1) \setminus \mathbb{Q}$. Then*

(i) *$x \in D(\psi)$ if $a_{n+1}(x)a_n(x) \,\leq\, \Phi(q_n)/4$ for all sufficiently large n.*

(ii) *$x \in D^c(\psi)$ if $a_{n+1}(x)a_n(x) \,>\, \Phi(q_n)$ for infinitely many n.*

As a consequence of this lemma, we have

$$\mathcal{G}(\Phi) \subset D^c(\psi) \subset G(\Phi/4), \tag{18}$$

where

$$G(\Phi) := \Big\{x \in [0,1) : a_n(x)a_{n+1}(x) \,>\, \Phi\big(q_n(x)\big) \text{ for infinitely many } n \in \mathbb{N}\Big\}.$$

The Lebesgue measure and Hausdorff measure of the set $D^c(\psi)$ was proved by Kleinbock-Wadleigh in [7, Theorem 1.8] and Hussain-Kleinbock-Wadleigh-Wang [8] respectively.

Theorem 4.2 ([7, Theorem 1.8]). *Let ψ be any non-increasing positive function and Φ as in* (17) *non-decreasing such that $t\psi(t) < 1$ for all t large enough. Then*

$$\mathcal{L}(D^c(\psi)) = \begin{cases} 0 & \text{if} \quad \sum_t \frac{\log \Phi(t)}{t\Phi(t)} \,<\, \infty, \\ 1 & \text{if} \quad \sum_t \frac{\log \Phi(t)}{t\Phi(t)} \,=\, \infty. \end{cases} \tag{19}$$

In this report we consider the product of two consecutive prime partial quotients. This consideration is motivated by the recent developments in the theory of uniform Diophantine approximation, specifically to the set of real numbers admitting improvements to Dirichlet's theorem. Let φ be a monotonically non-increasing function. The set $\mathcal{D}(\varphi)$ of φ-Dirichlet improvable numbers is the set of all $x \in \mathbb{R}$ such that

$$|qx - p| < \varphi(t), \;\; 1 \leq |q| < t$$

has a nonzero integer solution (p,q) for all large enough t. The set $\mathcal{D}(\varphi)$ has an elegant characterisation in terms of growth of product of consecutive partial quotients as

$$\{x\in(0,1): a_n(x)a_{n+1}(x) > \widetilde{\varphi}(q_n(x)) \text{ for i.m. } n\in\mathbb{N}\} \subset \mathcal{D}^c(\varphi)$$
$$\subset \{x\in(0,1): a_n(x)a_{n+1}(x) \geq \widetilde{\varphi}(q_n(x))/4 \text{ for i.m. } n\in\mathbb{N}\}$$

where $\widetilde{\varphi}(r)=\frac{r\varphi(r)}{1-r\varphi(r)}$, $p_n(x)/q_n(x)=[a_1(x),a_2(x),\ldots,a_n(x)]$ is the n-th convergent of the continued fraction expansion of x, and 'i.m.' stands for infinitely many. The Lebesgue measure of $\mathcal{D}^c(\varphi)$ has been determined in [7]. The study of comparisons of the set of Dirichlet non-improvable numbers with that of the set of well-approximable numbers was carried out in [11, 10], and the study of level sets about the growth rate of $\{a_n(x)a_{n+1}(x) : n\geq 1\}$ relative to that of $\{q_n(x): n\geq 1\}$ was discussed in [12]. In particular, to get the Lebesgue measure of $\mathcal{D}^c(\varphi)$, Kleinbock and Wadleigh [7] obtained the Lebesgue measure of the set

$$G(\Phi) := \{x\in(0,1): a_n(x)a_{n+1}(x) > \Phi(n) \text{ for i.m. } n\in\mathbb{N}\}$$

and then as a corollary, they deduced the Lebesgue measure of $\mathcal{D}^c(\varphi)$. See also [13] for a detailed analysis of a generalised version of the set $G(\Phi)$.

Theorem 4.3 ([7, Theorem 3.6]). *Let $\Phi:\mathbb{N}\to[1,\infty)$ be a function with $\lim_{n\to\infty}\Phi(n)=\infty$. Then the Lebesgue measure of $G(\Phi)$ is either zero or full according as the series $\sum_{n=1}^{\infty}\log\Phi(n)/\Phi(n)$ converges or diverges respectively.*

In this report, we restrict the digits of the continued fractions to be prime and consider the analogous set $G(\Phi)$. To be precise, we prove Lebesgue measure for the following set

$$G'(\Phi) := \Big\{x\in(0,1): a'_n(x)a'_{n+1}(x) > \Phi(n) \text{ for infinitely many } n\in\mathbb{N}\Big\}$$

Theorem 4.4.

$$\mathcal{L}(G'(\Phi)) = \begin{cases} 0 & \text{if} \quad \sum\limits_{t} \frac{\log\log\Phi(n)}{\Phi(n)\log\Phi(n)} < \infty, \\ 1 & \text{if} \quad \sum\limits_{n} \frac{\log\log\Phi(n)}{log\Phi(n)\Phi(n)} = \infty. \end{cases} \tag{20}$$

5 Some definitions, notions and auxiliary results

For any integer vector $(a_1,\ldots,a_n)\in\mathbb{N}^n$ with $n\geq 1$, write

$$I_n(a_1,\ldots,a_n) := \{x\in[0,1): a_1(x)=a_1,\ldots,a_n(x)=a_n\}$$

for the corresponding 'cylinder of order n', i.e. the set of all real numbers in $[0,1)$ whose continued fraction expansions begin with $(a_1,\ldots,a_n)$.

We will frequently use the following well known properties of continued fraction expansions. They are explained in the standard texts [6].

The next proposition concerns the position of a cylinder in $[0, 1)$.

Proposition 5.1 ([6], Khintchine). *Let $I_n = I_n(a_1, \ldots, a_n)$ be a cylinder of order n, which is partitioned into sub-cylinders $\{I_{n+1}(a_1, \ldots, a_n, a_{n+1}) : a_{n+1} \in \mathbb{N}\}$. When n is odd, these sub-cylinders are positioned from left to right, as a_{n+1} increases from 1 to ∞; when n is even, they are positioned from right to left.*

Theorem 5.2 ([9]). *Let E_n, $n \geq 1$, be a sequence of measurable set in a probability space (X, ν). Denote by $A(N, x)$ the number of integers $n \leq N$ such that $x \in E_n$. Put*

$$\phi(N) = \sum_{n \leq N} \nu(E_n). \tag{21}$$

Suppose that there exists a convergent series $\sum_{(j \leq 1)} C_j$ with $C_j \geq 0$ such that for all integers $m > n$ we have

$$\nu(E_n \cap E_m) \leq \nu(E_n)\nu(E_m) + \nu(E_m)C_{m-n} \tag{22}$$

There for any $\epsilon > 0$ one has

$$A(N, x) = \phi(N) + O_\epsilon(\phi^{1/2}(N) \log^{3/2+\epsilon} \phi(N)) \tag{23}$$

for almost all x

Theorem 5.3. *[[9]] There exists constants $c_0 > 0$ and $0 < \gamma < 1$ with the following property. Fix $\mathbf{r} = (r_1, ..., r_k) \in N^k$, and write*

$$E_r := x \in [0, 1] \setminus \mathbb{Q} : a_1(x) = r_1, a_2(x) = r_2, ..., a_k(x) = r_k. \tag{24}$$

Let $F \subset [0, 1]$ be any measurable set. Then for all $n \geq 0$,

$$|\mu(E_r \cap T^{-n-k}F) - \mu(E_r)\mu(F)| \leq c_0\mu(E_r)\mu(F)\gamma^{\sqrt{n}}. \tag{25}$$

This estimate admits passing to unions:

Corollary 5.4. *Let c_0 and γ be as in the previous theorem. Let $F \subset [0, 1]$ be any measurable set. Fix $k \in N$ and let $\mathcal{R} \subset N^k$. Then (25) holds for all $n \geq 0$ when E_r is replaced with $\cup_{r \in R} E_r$.*

Proof. We have

$$\begin{aligned} |\mu(\cup_{r \in \mathbb{R}} E_r \cap T^{-n-k}F) - \mu(\cup_{r \in R} E_r)\mu(F)| &= |\sum_{r \in R} \mu(E_r \cap T^{-n-k}F) - \mu(E_r)\mu(F)| \\ &\leq \sum_{r \in R} c_0\mu(E_r)\mu(F)\gamma^{\sqrt{n}} = c_0\mu(\cup_{r \in R} E_r)\mu(F)\gamma^{\sqrt{n}}. \end{aligned}$$

□

Lemma 5.5 ([7]). *Fix $k \in N$. Suppose $A_n (n \in N$ is a sequence of sets such that each A_n is a union of sets of the form $E_r, \boldsymbol{r} \in N^k$ (E_r as defined in (5.3)). If $\sum_n \mu(A_n) = \infty$ (resp. $< \infty$), then for almost every (resp. almost no) $x \in [0,1]$ on has $T^n(x) \in A_n$ for infinitely many n.*

Proof. The convergence case follows from the Borel-Cantelli Lemma and the fact that μ is T-invariant. Suppose $\sum_n \mu(A_n) = \infty$. For $m \geq n + k$ write

$$\begin{aligned} \mu(T^{-n}A_n \cap T^{-m}A_m) &= \mu(A_n \cap T^{-(m-n)}A_m) \leq \mu(A_n)\mu(A_m) + c_0\mu(A_m)\mu(A_n)\gamma^{\sqrt{m-n-k}} \\ &\leq \mu(A_n)\mu(A_m) + \mu(A_m)c_0\gamma^{\sqrt{m-n-k}} \\ &= \mu(T^{-n}A_n)\mu(T^{-m}A_m) + \mu(T^{-m}A_m)c_0\gamma^{\sqrt{m-n-k}} \end{aligned}$$

for c_0, γ as in (5.3). The sets $T^{-n}A_n$ therefore satisfy the condition of (5.3). By that theorem, $\sum_n \mu(T^{-n}A_n) = \infty$ guarantees that almost all x lie in $T^{-n}A_n$ for infinitely many n. □

6 Proof of Theorem 4.4

In this section, we prove Theorem 4.4. The above lemma can now be applied to describe real numbers which belong to infinitely many sets of the form $\{x : a'_1(x)a'_2(x) > \Phi(n)\}$:

Theorem 6.1. *Let $\phi : N \to [1, \infty)$ be any function with $\lim_{n\to\infty} \phi(n) = \infty$. If*

$$\sum_n \frac{\log\log(\phi(n))}{\phi(n)\log(\phi(n))} < \infty \quad (resp = \infty), \tag{26}$$

then almost every (resp. almost no) $x \in [0,1] \setminus \mathbb{Q}$ has

$$a'_{n+1}(x)a'_n(x) \geq \phi(n) \tag{27}$$

for sufficiently large n.

Proof. Define

$$\begin{aligned} A_n &:= \{x : a'_1(x)a'_2(x) > \phi(n)\} \\ &= \bigcup_{a' \in \mathcal{P}} \bigcup_{b=[\frac{\phi(n)}{a'}+1]}^{\infty} \left(\frac{1}{a' + \frac{1}{b}}, \frac{1}{a\prime + \frac{1}{b+1}}\right) \\ &= \bigcup_{a' \in \mathcal{P}} \left(\frac{1}{a' + \frac{1}{[\frac{\phi(n)}{a'}+1]}}, \frac{1}{a'}\right) \end{aligned}$$

Clearly $x \in [0,1] \setminus \mathbb{Q}$ has $a'_{n+1}a'_n > \phi(n)$ if and only if $T^{n-1}(x) \in A_n$ where T denotes the Gauss map (3.1). By Lemma 3.5, it is suffice to show

$$c^{-1}\mu(A_n) \leq \frac{\log(\log(\phi(n)))}{\phi(n)\log(\phi(n))} \leq c\mu(A_n)$$

We have

$$
\begin{aligned}
A_n \subset (\bigcup_{a'\leq\phi(n)} (\frac{1}{a'+\frac{a'}{\phi(n)}}, \frac{1}{a'}))\bigcup(\bigcup_{a'>\phi(n)} (\frac{1}{a'+1}, \frac{1}{a'})) \subset \\
(\bigcup_{a'\leq\phi(n)} (\frac{1}{a'+\frac{a'}{\phi(n)}}, \frac{1}{a'})\bigcup(0, \frac{1}{\phi(n)}) = \\
(\bigcup_{a'=2}^{[\phi(n)]} (\frac{1}{a'+\frac{a'}{\phi(n)}}, \frac{1}{a'})\bigcup(0, \frac{1}{\phi(n)}).
\end{aligned}
$$

So

$$
\begin{aligned}
\mathcal{L}(A_n) &\leq 1 - \frac{1}{1+\frac{1}{\phi(n)\log(\phi(n))}} + \int_2^{\phi(n)} (\frac{1}{a'} - \frac{1}{a'+\frac{a'}{\phi(n)\log(\phi(n))}})da' + \frac{1}{\phi(n)\log(\phi(n)} \\
&= 1 - \frac{1}{1+\frac{1}{\phi(n)}} + \log(\phi(n)\log(\phi(n))(1 - \frac{1}{1+\frac{1}{\phi(n)}}) + \frac{1}{\phi(n)\log(\phi(n))} \\
&= \frac{1}{\phi(n)+1} + \frac{\log(\phi(n)\log(\phi(n)))}{1+\phi(n)\log(\phi(n))} + \frac{1}{\phi(n)\log(\phi(n))} \asymp \frac{\log(\phi(n)\log(\phi(n)))}{\phi(n)\log(\phi(n))}
\end{aligned}
$$

To see the asymptotic lower bound, we start with

$$
A_n \supset \bigcup_{a'=2}^{[\phi(n)]} (\frac{1}{a'+\frac{1}{\frac{\phi(n)}{a'}+1}}, \frac{1}{a'}).
$$

Then

$$
\begin{aligned}
\mathcal{L}(A_n) &\geq \int_2^{\phi(n)\log(\phi(n))} (\frac{1}{a'} - \frac{1}{a'+\frac{1}{\frac{\phi(n)}{a'}+1}})da' = \int_2^{\phi(n)\log(\phi(n))} \frac{1}{a'(a'+\phi(n)+1)}da' \\
&= \int_2^{\phi(n)\log(\phi(n))} (\frac{(\phi(n)\log(\phi(n))+1)^{-1}}{a} - \frac{(\phi(n)\log(\phi(n))+1)^{-1}}{\phi(n)\log(\phi(n))+a'+1})da' \\
&= \frac{\log(\phi(n)\log(\phi(n))}{\phi(n)\log(\phi(n))} + \frac{\log(\frac{\phi(n)\log(\phi(n))+2}{2\phi(n)\log(\phi(n))})}{\phi(n)\log(\phi(n))} \asymp \frac{\log(\phi(n)\log(\phi(n))}{\phi(n)\log(\phi(n))} \\
&= \frac{\log(\phi(n))}{\phi(n)\log(\phi(n))} + \frac{\log(\log(\phi(n))}{\phi(n)\log(\phi(n))} \\
&\asymp \frac{\log(\log(\phi(n))}{\phi(n)\log(\phi(n))}.
\end{aligned}
$$

Hence, by appealing to Theorem 5.2 and Lemma 5.5, we conclude that

$$
\mathcal{L}(G'(\Phi)) = 1 \quad \Longleftrightarrow \quad \sum_n \mathcal{L}(A_n) = \sum_n \frac{\log(\log(\phi(n))}{\phi(n)\log(\phi(n))} = \infty.
$$

□

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