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Higher Dimensional Self-Similar Actions

Amelie Skelton

Supervised by Nathan Brownlowe

University of Sydney

AMSI AUSTRALIAN MATHEMATICAL SCIENCES INSTITUTE

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Amelie Skelton, University of Sydney

Supervised by Nathan Brownlowe

Abstract

We introduce the notion of self-similar groups through the language of automata. We see how these self-similar actions can be generalised to actions on the path space of directed graphs, and further to actions of higher dimensional graphs called k-graphs. For each, we introduce a generalised notion of an automata. In particular we introduce the notion of a colour-permuting automata for k-graphs, and give an example of an automata that seems to suggest colour-splitting automata are well defined.

Contents

1 Introduction

The notion of self-similarity is a familiar geometric property—the idea that an object, like a fractal, can repeat itself at different length scales. Here we see how we can extend this notion to other mathematical objects, groups and groupoids, and their actions, in a way that does not immediately match our intuitive understanding of self-similarity. Historically, self-similar groups have made for rather interesting examples of groups. Grigorchuk's self-entitled, Grigorchuk group, for example, was the first known finitely generated group that had *intermediate* growth—growth which is faster than polynomial growth but is sub-exponential; for a precise definition of growth see [Eld12]. From the outset, it is not so obvious intuitively as to why self-similar groups have such desire-able properties.

Here, we are motivated by an entirely different field of Mathematics known as C^*-algebras. C^*-algebras are a set of operator algebras between Hilbert spaces; one easy example are the $n \times n$ matrices. It is natural to associate to objects, such as groups and groupoids, a particular C^*-algebra. Self-similar actions provide interesting material to construct C^*-algebras, where this self-similarity somehow seems to translate into the self-similarity of the C^*-algebra.

We first describe self-similar actions on trees, and their generalisations for directed graphs and the category theoretic, higher dimensional, k-graphs. The language of *automata* allow us to encode these otherwise cumbersome self-similar actions, and we introduce various definitions of them as we generalise our actions. We begin with a discursive introduction to self-similar groups.

1.1 Statement of Authorship

The work here is based on a number of sources. The historical background, details and definitions in §2 and §3 are largely based on standard texts [Eld12; Lac+18; Nek05]. The theory and extension of automata to k-graphs is based on work by [Afs+19] and [Rae05]. I was led naturally by my supervisor Nathan Brownlowe to ask whether it is possible to extend the definition of the automata in [Afs+19] to one that is colour-permuting, or indeed colour-splitting. The example of the colour-splitting automata is our own.

2 Classical Self-Similar Actions

2.1 Self Similar Actions on Rooted Trees

Let X be a finite set and let X^* be the set of all finite words in the alphabet X. That is, X^* is the free monoid containing all words $x_1x_2 \dots x_n$ for $x_i \in X$ including the empty word $\emptyset$. We say the action of a group G on X^* is *self-similar* if for any $g \in G$, and all $x \in X$ and words $w \in X^*$, there exist $y \in X$ and $h \in G$ such that

$$g \cdot (xw) = y(h \cdot w). \tag{1}$$

At first it seems odd to generalise what is quite a geometric picture we have of self-similarity to a property of elements in a group, but this idea of self-similarity becomes natural when we view our action and condition (1) in a geometric light. In fact, X^* is the vertex set of an infinite *regular rooted tree* T in disguise. Any group G, thus, acts on X^* via *automorphisms* of this tree. These trees and their generalisations, objects which themselves have a familiar sense of self-similarity, are the backdrop for all of our self-similar actions. Before we move on, then, let us introduce this geometric picture.

Definition 1 (Regular Rooted Tree)**.** A regular rooted tree T consists of vertices $T^0 = X^*$ generated by X, joined by edges T^1 in the natural way: $(w_1, w_2) \in T^1$ if and only if $w_2 = w_1x$ for some $x \in X$. The empty word $\emptyset$ is the root of the tree. The k^{th} *level of the tree* X^k is the set of words in X^* of length k.

Definition 2 (Automorphism of a Tree)**.** An automorphism ϕ of T is a series of bijections $X^k \to X^k$ that preserves adjacencies of vertices. That is, $(w_1, w_2) \in T^1$ if and only if $(\phi(w_1), \phi(w_2)) \in T^1$. Let $\mathrm{Aut}(T)$ be the set of all such automorphisms.

The *action* of G on T and therefore X^* is a homomorphism $\varphi : G \to \mathrm{Aut}(T)$ which associates to each $g \in G$ an automorphism of the tree φ_g. As we have already done, we write the action of any g on $w \in X^*$ as $g \cdot w$. For simplicity we assume the action is *faithful*; that is φ is an injective map. With this in mind, it is easier to think of g and φ_g as interchangeable, and we can assume $G \leq \mathrm{Aut}(T)$.

Example 1. A helpful picture to have in mind is a *binary tree* — all words in $\{0, 1\}$, as in Figure 1.

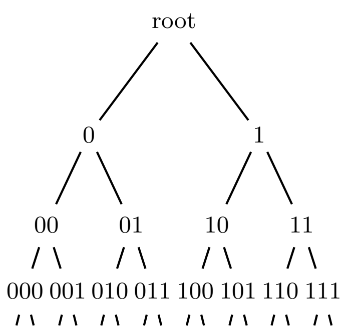

Figure 1: The infinite rooted binary tree.

It is useful to describe an automorphism by its action on the subtrees of T. We write wX^* for the subtree rooted at w, that is, all words beginning with w, $wX^* := \{w_1 \in X^* \mid w_1 = ww_2,\ \text{for } w_2 \in X^*\}$.

Example 2 (Portraits)**.** The most simple automorphism of the binary tree is the automorphism that swaps the subtrees $0X^*$ and $1X^*$ by sending words $0w$ to $1w$ and conversely, shown by its *portrait* in Figure 2. Let this automorphism be s.

Intuitively, the portrait of an automorphism of a binary tree is a decoration of the vertices by transposes $*$. A $*$ at w means swap trees at $w0$ and $w1$, by replacing words $w0w_1$ with $w1w_1$, and conversely. For us they serve as a picture. We will not define portraits precisely as we will see soon there is a more convenient way to represent automorphisms.

Example 3 (Odometer)**.** Let a be the automorphism described by the portrait in Figure 3, with decorations that continue infinitely down the right hand side of the tree. This automorphism, made of a infinite sequence of mechanical swappings, is known as the *odometer*—why? We will see soon.

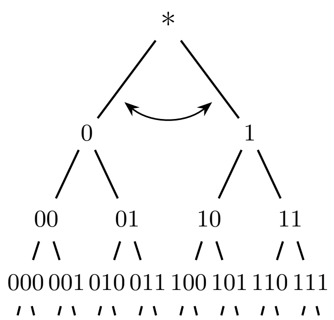

Figure 2: The portrait of automorphism s.

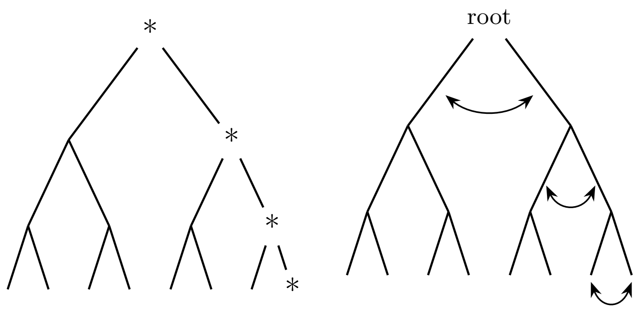

Figure 3: The portrait of the odometer a.

2.1.1 Self-similarity from the Perspective of Trees

In general, the adjacency property of Definition 2 ensures that any automorphism must map subtrees to other subtrees on the rooted on the same level of the tree. If $g \in \mathrm{Aut}(T)$, then, we have a map $g : wX^* \to g(w)X^*$ for any $w \in X^*$. Both subtrees are isomorphic to the original tree so it makes sense to consider the action on wX^* as if it were an automorphism on X^* itself. With this in mind, we define $g|_w : X^* \to X^*$ as the automorphism on the tree rooted at w translated to an action on the larger tree, so that it satisfies

$$g(ww_1) = g(w)g|_w(w_1) \tag{2}$$

for all $w_1 \in X^*$. We call this map the *restriction of g to w*. Recalling s from Example 2, both $g|_0$ and $g|_1$ are the identity map id, and Figure 4 shows this restriction map, for example on $1X^*$. With a bit of thought it is easy to see that $g|_w$ satisfies:

$$g|_{w_1w_2} = g|_{w_1}|_{w_2} \tag{3}$$

$$(g_1 \cdot g_2)|_w = g_1|_{g_2(w)} \cdot g_2|_w \tag{4}$$

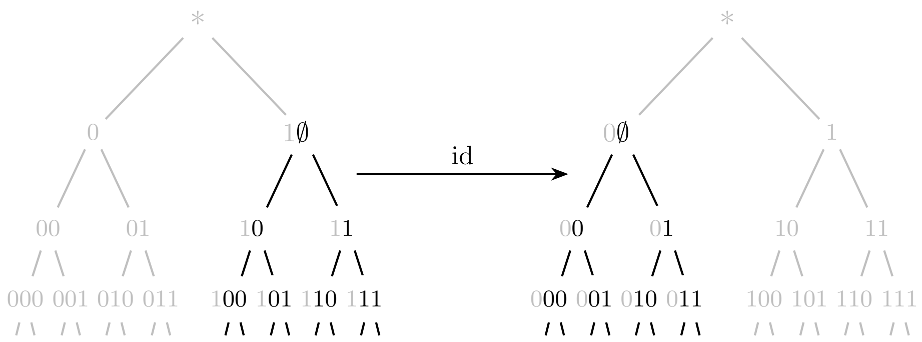

Figure 4: The restriction of s to the tree $1X^*$ is the identity.

In this light, G is self-similar only if for all $g \in G$ we can restrict the action of g to any subtree wX^* and there is another element $g|_w \in G$ which exactly describes this restricted action. Any self-similar action must therefore agree with the self-similarity of the tree itself. In principle this gives our condition on individual words of the tree in (1).

Example 4 (Odometer Revisited)**.** Restricted to the tree $0X^*$ the automorphism a acts as the identity. Restricted to $1X^*$, however, the restriction is again a. To make a self-similar group, then, we require at least the identity id and a, and therefore that the group contains at least the cyclic group $\langle a \rangle$. Property 4 gives that $(a^2)|_w = a|_{a(w)} \cdot a|_w$ and thus $(a^2)|_w \in \langle a \rangle$ for any $w \in X^*$. Continuing this, it is not hard to see that indeed $\langle a \rangle$ is a self-similar group.

Example 5 (Grigorchuk's Group)**.** For completeness let us also introduce Grigorchuk's group from §1. It is generated by five elements: a and id of the odometer, and automorphisms b, c and d whose portraits are shown in Figure 5.

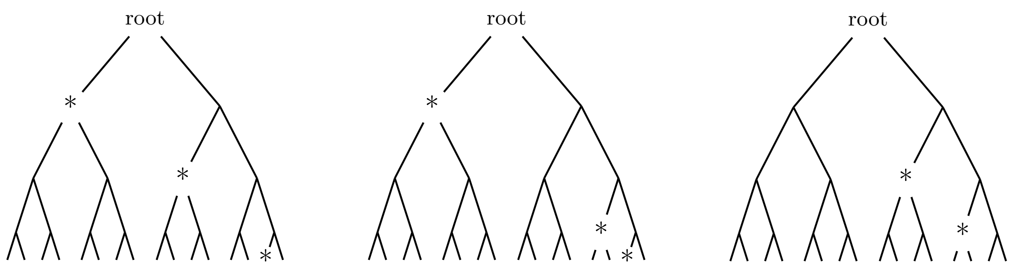

Figure 5: The portraits of b, c and d.

2.1.2 From Actions to Automata, and Back

For any $w \in X^*$, the restriction maps allow us to transit through T and make changes to w letter by letter, depending *only* on our current restriction map, some $g|_v$, before moving on to the next restriction at letter x, $g|_{vx}$. We can consider the set of these restrictions as the 'internal states' of a computer, reminiscent of a Turing machine reading a section of tape, known as an *automaton*.

Definition 3 (Automata on Trees, Laca et al. [Lac+18])**.** An *automaton* over X is a finite set of states A and a function $\tau : A \times X \to X \times A$, that sends (a, x) to $(a \cdot x, a|_x)$, such that for all $a \in A$, $x \mapsto a \cdot x$ is a bijection. If $\tau(a, x) = (a \cdot x, a|_x)$, we write $a \cdot x = (a \cdot x)\ \cdot a|_x$.

An automaton over X^* is defined recursively by

$$a|_{xw} = a|_x|_w \qquad a \cdot (xw) = a \cdot x\ a|_x(w). \tag{5}$$

It is no suprise the notation for automata mimics that we have used for trees. Let $w = x_1 \dots x_k$ for $x_i \in X$ and define $f_{a,k} : X^k \to X^k$ using (5) by

$$f_{a,k}(w) = (a \cdot x_1)(a|_{x_1} \cdot x_2) \dots (a|_{x_1 \dots x_{k-1}} \cdot x_k). \tag{6}$$

For each $a \in A$, $\{f_{a,k}\} =: f_a$ defines an automorphism of T. To prove this it is enough to show by induction on (6) that $f_{a,k+1}(wx) \in f_{a,k}(w)X$, which is a property equivalent to Definition 2 [Lac+18].

Theorem 1 (Laca et al. [Lac+18])**.** Let G_A be the group generated by $\{f_a \mid a \in A\}$. There is a faithful self-similar action of G_A on T such that $f_a \cdot w \coloneqq f_a(w)$ for $a \in A$ and $w \in X^*$.

Rather than repeat the proof of this fact we direct the reader to [Lac+18, §2, Prop. 2.2]; in short: clearly f_a satisfies (1) and it remains to prove that f_a^{-1} as well the compositon of any two automorphisms satisfying (1) also satisfy (1), as we very meekly began to do in Example 4. Conversely, every self-similar group can be described by an automata with states being the set of all group elements. Automata here, and more generally, provide a systematic way of representing and creating self-similar groups.

Example 6 (Odometer Re-Revisisted)**.** For the odometer $A = \{a, \mathrm{id}\}$, τ is defined by:

$$a \cdot 0 = 1 \cdot \mathrm{id} \qquad\qquad a \cdot 1 = 0 \cdot a \qquad\qquad .$$

Any word in the form $\underbrace{11\dots1}_{k\times 1\text{'s}}0w$ is sent to $\underbrace{00\dots0}_{k\times 0\text{'s}}1w$.

Read as a binary string in reverse, all we are doing is adding 1! Hence the infinite cyclic group $\langle a \rangle$ isomorphic to $\mathbb{Z}^+$: it is an adding machine—an *odometer*.

3 An Extension to Directed Graphs

With this picture of self-similar group actions it is relatively straightforward to generalise the notion of self-similarity to trees that are not necessarily regular. To so, however, we must first generalise the alphabet we use. We swap X with E^1 — the set of edges in a finite directed graph E — and T with a new graph T_E, which we will define shortly.

Definition 4. A *finite directed graph* E is a 4-tuple (E^0, E^1, r_E, s_E) containing a vertex set E^0, edges E^1, and range and source maps $r_E, s_E : E^1 \to E^0$.

The *path space* of E generalises the notion of words over X.

Definition 5. Let $E^k = \{\mu = e_1 \dots e_k \mid e_i \in E^1 \text{ and } s(e_i) = r(e_{i+1})\}$ be the set of paths of length $k \geq 1$. The *path space* of a directed graph is the set $E^* = \bigsqcup_{k\geq 0} E^k$, i.e. paths of any length.

Definition 6. To a directed graph E we associate the graph T_E containing vertex set $T^0 = E^*$ and edge set

$$T^1 = \{(\mu, \mu e) \mid \mu \in E^*, e \in E^1 \text{ and } s_E(\mu) = r_E(e)\}.$$

Example 7. In fact, the binary tree is the graph $\emptyset E^* := \{\mu \in E^* \mid r(\mu) = \emptyset\}$ corresponding to the directed graph in Figure 6.

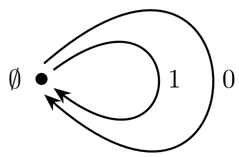

Figure 6: The directed graph which produces the binary tree.

In general, however, E has multiple vertices; instead of a single rooted tree, T_E is rather a *forest* of trees. Each subgraph $vE^* = \{\mu \in E^* \mid r(\mu) = v\}$ is a rooted tree at $v \in E^0$, and the forest T_E is the disjoint union of these trees $T_E = \bigsqcup_{v\in E^0} vE^*$. As we are acting on multiple trees and wish to consider maps not only from a tree to itself but also between trees, we now must work with *partial isomorphisms* of T_E.

Definition 7. A *partial isomorphism* of T_E is a bijection $g : vE^* \to wE^*$ between trees at $v, w \in E^0$ such that:

(P1.1) $g|_{vE^k}$ is a bijection onto wE^k for any $k \in \mathbb{N}$, and

(P1.2) $g(\mu e) \in g(\mu)E^1$ for all $\mu e \in vE^*$ and edges $e \in s_E(\mu)E^1$.

Let $\mathrm{PIso}(E^*)$ be the set of all partial isomorphisms. For each $g \in \mathrm{PIso}(E^*)$ set $d, c : \mathrm{PIso}(E^*) \to E^0$ as the domain and codomain maps of g, such that $g : d(g)E^* \mapsto c(g)E^*$, that is $d(g) := v$ and $c(g) := w$.

We also associate with each $v \in E^0$ an identity $v : vE^* \to vE^*$.

These partial isomorphisms act *partially* on T_E in the sense that we act only on one tree at a time, however it still mirrors our definition of automorphisms of trees. Condition (P1.1) ensures that each k^{th} level of tree vE^k is mapped bijectively to that of wE. Together with (P1.2), which ensures edge connections between successive levels are unchanged by g, this allows the structure of the tree to be preserved.

Example 8. Take T_E as the forest associated with the directed graph E in Figure 7. An easy example of a partial isomorphism between vE^* and wE^* occurs under the renaming of edges $E_1 : 1 \leftrightarrow 3,\ 2 \leftrightarrow 4$.

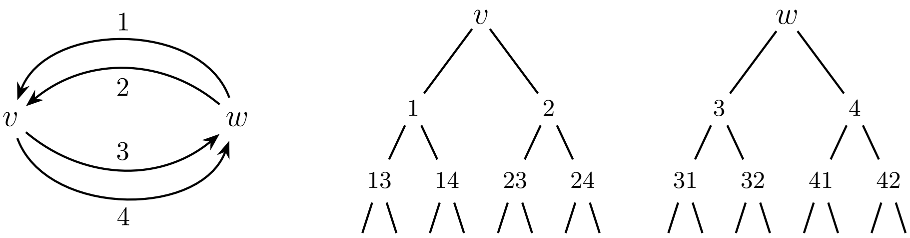

Figure 7: A directed graph E and its forest of trees T_E.

Now not all isomorphisms can be composed — only those that agree on their domain and codomain maps. Instead of a group, then, $\mathrm{PIso}(E^*)$ forms a *groupoid.* Intuitively:

Definition 8 (Groupoid)**.** A groupoid is a small category with inverses. That is, a groupoid is a set G^0 and a set of morphisms G containing inverses for each $g \in G$. The morphisms have a partially defined multiplication $(g, h) \mapsto gh$ defined by the set of composable arrows $G^2 = \{(g, h) \mid d(g) = c(h)\}$.

Lemma 2 (Laca et al. [Lac+18])**.** Let $E = (E^0, E^1, r_E, s_E)$ be a directed graph with forest T_E. Then, $(\mathrm{PIso}(E^*), E^0, c, d)$ is a groupoid with multiplication via concatenation of paths, identity arrows $v : vE^* \to vE^*$ and set-wise inverses of each $g : d(g)E^* \to c(g)E^*$.

Let G be a groupoid with unit space E^0 of a direct graph E. A *groupoid action* of G on E^* is a functor $\varphi : G \to \mathrm{PIso}(E^*)$ preserving E^0. We again assume φ is a faithful action so we can assume G is a subgroupoid of $\mathrm{PIso}(E^*)$. This action leads to an equivalent notion of self-similarity:

Definition 9 (Self-Similarity of Directed Graphs)**.** Let E be a directed graph and let G be a groupoid with unit space E^0. Suppose that G acts faithfully on the forest T_E. This action is *self-similar* if, for all $g \in G$ and $e \in d(g)E^1$, there exists $h \in G$ such that

$$g \cdot (e\mu) = (g \cdot e)(h \cdot \mu) \tag{7}$$

for all $\mu \in s(e)E^*$. We write $g|_e \coloneqq h$.

Example 9. Let E be the more complicated graph in Figure 8, with forest T_E. This example makes it clear how this definition of self-similarity matches our previous intuitive picture. A map $g : uE^* \to vE^*$ for example, must be able be restricted to an action on subtrees of uE^* and vE^* which is described completely by another element in the groupoid. For the subtrees $s(2)E^*$ and $s(4)E^*$ this is the action $g|_2$ on wE^* as in Figure 9.

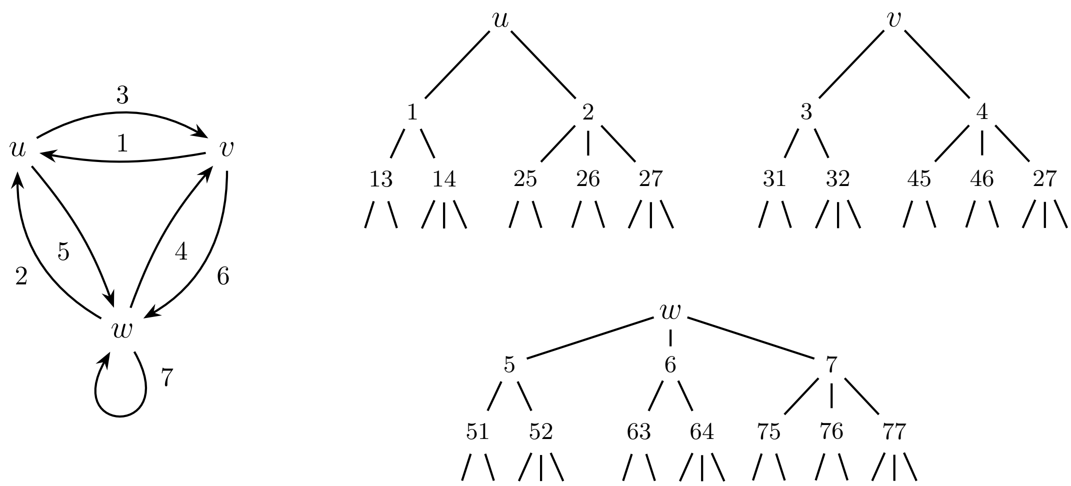

Figure 8: The graph E and corresponding rooted trees at u, v and w.

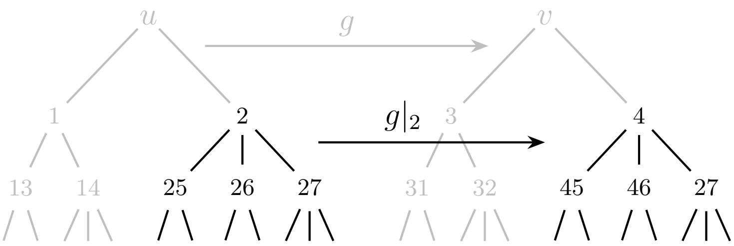

Figure 9: The restricted action of g to the subtree $g|_2$.

This self-similarity leads naturally to automata. Its definition is similar to Definition 3, however, we now have extra conditions that ensure our automaton respects the structure of the trees.

Definition 10 (Automaton, [Lac+18])**.** An automaton over a directed graph, $E = (E^0, E^1, r_E, s_E)$ is a set of states A containing E^0, range and source functions for these states: $r_A, s_A : A \to E^0$ such that $r_A(v) = v = s_A(v)$, and a function τ:

$$A_{s_A} \times_{r_E} E^1 \ni (a, e) \mapsto (a \cdot e, a|_e) \in E^1{}_{s_E} \times_{r_A} A,$$

such that

(A2.1) $e \mapsto a \cdot e$ is a bijection of trees $s_A(a)E^1 \to r_A(a)E^1$, for all $e \in E^1$ and $a \in A$;

(A2.2) For all $a \in A$ and $e \in E^1$, $s_A(a|_e) = s_E(a)$.

(A2.3) $r_E(e) \cdot e = e$, $r_E(e)|_e = s_E(e)$.

The range and source maps r_A and s_A mimic the domain and range of maps of the partial isomorphism maps in Theorem 2. By $A_{s_A}\times_{r_E} E^1$ we mean those states and edges that agree on their source and range maps respectively. This allows τ to mimic the action of a groupoid of the 1^{st} level of trees rooted at vertices in E^0. Likewise, the codomain of τ, $E^1{}_{s_E}\times_{r_A} A$, as well as condition (A2.2), ensures the restriction of a matches our intuitive picture. Condition (A2.3) ensures each $v \in E^0$ acts as the identity on vE^*.

A proof by [Lac+18], very similar to Theorem 1, shows that indeed we can construct partial isomorphisms for each $a \in A$ that generate a groupoid $G \in \mathrm{PIso}(E^*)$ acting faithfully on T_E.

Example 10. The automorphism in Example 8 is part of a self-similar action described by the states $\{a, b, v, w\}$, where v and w are identities. Using the notation of Definition 3, τ is defined by the rules:

$$a \cdot 1 = 3 \cdot b, \qquad b \cdot 3 = 1 \cdot a,$$
$$a \cdot 2 = 4 \cdot b, \qquad b \cdot 4 = 2 \cdot a.$$

4 k-Graphs and Higher-Dimensional Self-Similar Actions

We are now ready to further generalise self-similar actions to higher dimensional graphs: k-graphs. First, we will introduce k-graphs through the language of category theory, and provide some intuition that will be useful to understanding these actions. For further reading on categories see [Rae05, §10] or [Mac71].

4.1 k-Graphs and Intuition

Definition 11 (Graphs of Rank k, [Rae05])**.** A *graph of rank k*, or *k-graph*, is a countable category $\Lambda = (\Lambda^0, \Lambda^*)$ of objects Λ^0 and morphisms Λ^*, equipped with a functor $d : \Lambda \to \mathbb{N}^k$ called the *degree functor* which satisfies the factorisation property (8):

> For all morphisms $\lambda \in \Lambda$ and any decomposition of $d(\lambda)$ by $m, n \in \mathbb{N}^k$, $d(\lambda) = m + n$, there are unique elements $\mu \in d^{-1}(m)$ and $\nu \in d^{-1}(n)$ such that $\lambda = \mu\nu$. (8)

For $n \in \mathbb{N}^k$, we denote by Λ^n all morphisms $d^{-1}(n)$ of degree n. Let $\{e_i,\, 0 \le i \le k\}$ be the standard basis of $\mathbb{N}^k$. If $d(\lambda) = e_i$, we call λ an *edge*. Analogous to the case of a directed graph, we can consider a k-graph as the union of path spaces. Suppose $v \in \Lambda^0$, then $v\Lambda \coloneqq \{\lambda \in \Lambda \mid r(\lambda) = v\}$ and we write $T_\Lambda \coloneqq \bigsqcup_{v\in\Lambda^0} v\Lambda$.

In a 1-graph Λ, any morphism $\lambda \in \Lambda$ such that $d(\lambda) = p \in \mathbb{N}$ can be split uniquely into p morphisms λ_i of degree 1, as in Figure 10. The degree functor thus computes the 'length' of any morphism. A

1-graph, therefore, is simply the path space of the directed graph $(\Lambda^0, d^{-1}(1), r, s)$, where r and s are the range and source maps of morphisms in Λ.

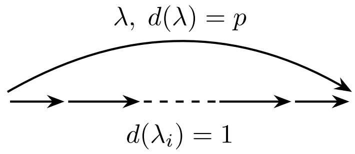

Figure 10: The decomposition of a morphism of degree $p \in \mathbb{N}$.

In two dimensions morphisms no longer have length but instead a rectangular structure.

Example 11. Set $\Omega_2 := \{(m, n) \in \mathbb{N}^2 \times \mathbb{N}^2 \mid m \leq n\}$. As morphisms, let $r(m, n) = (m, m)$ and $s(m, n) = (n, n)$ (that is, the points $m, n \in \mathbb{N}^2 = \Omega_2^0$). Define the composition of maps by $(m, n)(n, p) = (m, p)$ and set $d(m, n) = n - m$ as a component-wise difference. This makes Ω_2 a 2-graph.

In fact, Ω_2 is the 2-dimensional lattice of morphisms on $\mathbb{N}^2$ leading to $(0, 0)$, as in Figure 11. Here, the red and blue morphisms have degree $(1, 0)$ and $(0, 1)$ respectively and are therefore edges. The 2-graph contains all compositions of these edges, and we consider a morphism as equivalent to the rectangle of paths which it decomposes into.

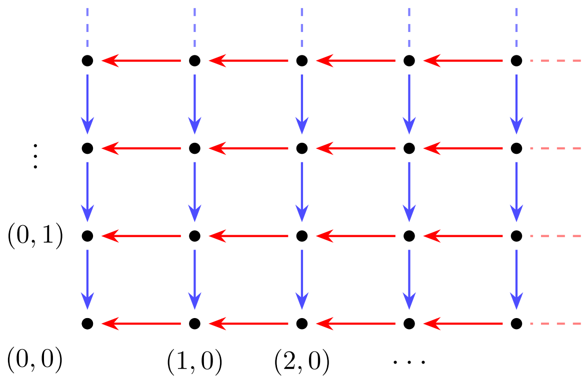

Figure 11: The 2-graph Ω_2.

This can be generalised to $\Omega_k := \{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k \mid m \leq n\}$. These 'lattice'-like categories make it clear why we like to think of a k-graph as a 'higher dimensional' graph. If Λ is k-graph, each $\lambda \in \Lambda$ must have some associated k-dimensional lattice of 'basis' edges in Ω_k (like $(1, 0)$ and $(0, 1)$) that defines its unique factorisation. It is the association between these 'basis' paths to make up these 'lattices' that completely defines a k-graph. Let us introduce this more formally. Figure 11 is an example of *k-coloured graph*. A k-coloured graph is to a k-graph what a directed graph is to its path space. They allow us to 'flatten' our k-graph.

Definition 12 (k-coloured graph)**.** A k-coloured graph of a k-graph Λ is a k-coloured directed graph $E = (\Lambda^0, \bigsqcup_{i=1}^{k} \Lambda^{e_i}, r, s)$ with each edge Λ^{e_i} assigned a unique colour c_i.

The paths of degree e_i can be thought of the 'basis' paths of our k-graph, and the directed graph gives the range and source maps of our morphisms. Here we deal mainly with 2-coloured graphs as they are the easiest non-trivial k-graphs. We use red for edges in Λ^{e_1} and blue for edges in Λ^{e_2}. Of course a k-coloured graph is not enough to uniquely describe a k-graph; we also require associations between edges, through *squares*.

Definition 13 (A Collection of Squares)**.** A *complete and associative collection of squares* $\mathcal{C}$ is a set of pairings of bi-coloured tuples of edges in which every tuple occurs exactly once. If $e_1, e_2 \in \Lambda^{e_i}$ and $f_1, f_2 \in \Lambda^{e_j}$ and $e_1 f_1$ and $f_2 e_2$ is a pairings of tuples we write $e_1 f_1 \sim f_2 e_2$. For $k \geq 3$ these squares are associative in that sense that morphisms from one side of a tri-coloured cube to the other, formed by the appropriate squares, are all equivalent, as in Figure 12.

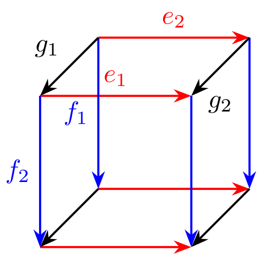

Figure 12: A cube of associative squares of red edges e_i, blue edges f_i and black edges g_i.

A theorem by [Haz+13] states that a k-graph is completely determined by a k-coloured graph and a compatible complete and associative collection of squares, and conversely.

Example 12 (An easy 2-coloured graph)**.** Figure 13 shows the simplest 2-coloured graph. There is only one square in $\mathcal{C}$ and that is $ef \sim fe$.

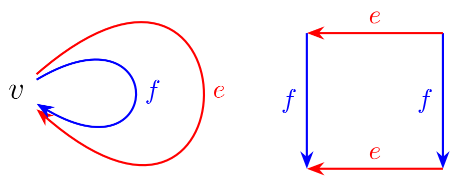

Figure 13: A 2-coloured graph and the square $ef \sim fe$.

4.2 Self-Similar Actions on k-Graphs

Most current constructions of self-similar actions on k-graphs, such as that of [Afs+19], consider only isomorphisms on T_Λ and automata on k-coloured graphs which are colour preserving. Work by [Bro+18] introduces the notion of *colour-permuting* partial isomorphisms, but there is little published work connecting these to C^*-algebras. Here we explore the idea of colour-permuting partial isomorphisms and automata. We argue that it also makes sense to define a colour-blind automata and introduce the notion of *colour-splitting.*

We begin with the definitions for the colour-permuting case. There is still a notion of partial isomorphism that generalises Definition 7, by permuting coloured edges around and between vertices.

Definition 14. Let $n = n_1 e_1 + \cdots + n_k e_k \in \mathbb{N}^k$ and let $s \in S_k$ be a permutation on k elements. A *coordinate-permuting permutation* of $\mathbb{N}^k$ is a function $\sigma(n) := n_1 e_{s(1)} + \cdots + n_k e_{s(k)}$. Let Σ be the set of all such coordinate permuting permutations.

Definition 15. Let Λ be a k-graph. A *partial isomorphism* of T_Λ is a bijection $g : v\Lambda \to w\Lambda$ between path spaces at v and $w \in \Lambda^0$ and a coordinate-permuting permutation $\sigma \in \Sigma$ such that

(P2.1) $g\,|_{v\Lambda^p}$ is a bijection onto $w\Lambda^{\sigma(p)}$ for all $p \in \mathbb{N}^k$, and

(P2.2) $g(\mu e) \in g(\mu)\Lambda$ for all $\mu \in v\Lambda$ and $e \in s(\mu)\Lambda^{e_i}$ for some e_i.

We set $\mathrm{PIso}(\Lambda)$ as the set of all such partial isomorphisms. It is no surprise that again $\mathrm{PIso}(\Lambda)$ forms a groupoid, with domain and codomain maps defined analogously to those of $\mathrm{PIso}(E^*)$. Likewise a groupoid acts faithfully on Λ if there is injective groupoid homomorphism $\phi : G \to \mathrm{PIso}(\Lambda)$. We arrive at a now familiar notion of self-similarity:

Definition 16 (Colour-permuting Self-Similarity of Directed Graphs)**.** Let Λ be a k-graph and let G be a groupoid with unit space Λ^0. Suppose that G acts faithfully on T_Λ. The action is *self-similar* is for all $g \in G$ and edges $e \in \Lambda$ such that $d(g) = r(e)$, there exists $h \in G$ such that

$$g \cdot (e\mu) = (g \cdot e)(h \cdot \mu) \tag{9}$$

for all $\mu \in s(e)\Lambda$. We again write $g|_e := h$.

Again, analogous to partial isomorphisms that act on the path spaces of directed graphs, we can define automata to represent partial isomorphisms on k-graphs by considering an automata our generalised directed graph—the corresponding k-coloured graph.

Definition 17 (Colour Permuting Automata for k-graphs)**.** A *colour permuting automata* over a k-coloured graph E and a complete and associative collection of squares $\mathcal{C}$ is a finite set A containing E^0, range and source functions $r_A, s_A : A \to E^0$ such that $r_A(v) = v = s_A(v)$ for $v \in E^0$, together with a function τ:

$$A_{s_A}\times_{r_E} E^1 \ni (a, e) \mapsto (a \cdot e, a|_e) \in E^1{}_{s_E}\times_{r_A} A,$$

such that

(A3.1) $e \mapsto a \cdot e$ is a bijection of $s_A(a)E^1 \to r_A(a)E^1$, for all $e \in E^1$ and $a \in A$;

(A3.2) $s_A(a|_e) = s_E(a)$ for all $e \in s_A(a)E^1$;

(A3.3) $r_E(e) \cdot e = e$, $r_E(e)|_e = s_E(e)$;

(A3.4) $c(a \cdot e) = \sigma(c(e))$ for all $e \in s_A(a)E^1$, where σ is a permutation that acts by a permuting $\{c_1, \dots, c_k\}$;

(A3.5) For every $\lambda \in \mathcal{C}$ such that $\lambda := e_1 f_1 \sim f_2 e_1$, we have $(a \cdot e)(a|_e \cdot f) \sim (a \cdot f')(a|_{f'} \cdot e'$;

(A3.6) For every $\lambda \in \mathcal{C}$ such that $\lambda := e_1 f_1 \sim f_2 e_1$, we have $(a|_e)|_f = (a|_{f'})|_{e'}$.

Importantly, we now have a colour permuting condition (A3.4), as well as conditions (A3.5) and (A3.6) that ensure the automata respects the square association. In particular this ensures that each partial isomorphism generated by the states in A is a colour-permuting partial isomorphism that maps equivalent morphisms the same (i.e. squares to squares), and ensures that these morphisms restrict to the same partial isomorphism in the condition for self-similarity.

All this leads to our final example—an interesting automata that raises more questions than it answers.

Example 13. Let E be the 2-coloured graph in Figure 14 and let $\mathcal{C}$ be the collection of squares:

$$e_1 f_2 \sim f_1 e_2, \;\; e_2 f_2 \sim f_2 e_2, \;\; e_3 f_4 \sim f_3 e_4, \;\; e_4 f_4 \sim f_4 e_4.$$

We let $A = \{a, b, v, u, w\}$ where v, u and w are identities, and $a : uE^* \to uE^*$ and $b : vE^* \to vE^*$ are defined by the rules:

$$\begin{array}{lll} a \cdot e_1 = e_1 \cdot w, & a \cdot f_1 = f_1 \cdot w, & b \cdot e_4 = f_4 \cdot b, \\ a \cdot e_3 = f_3 \cdot b, & a \cdot f_3 = e_3 \cdot b, & b \cdot f_4 = e_4 \cdot b. \end{array}$$

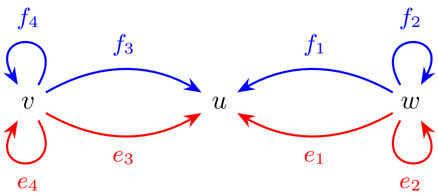

Figure 14: A 2-coloured graph with vertices v, u and w.

We claim that $A = (E, \mathcal{C})$ satisfies all conditions in Definition 17 *but* condition (A3.4). This makes A a *colour-splitting* automata.

5 Summaries and Further Questions

We have introduced the notion of self-similar actions and their recent generalisations to higher dimensional k-graphs and a provided new example of an automata that appears to allow for colour-splitting. From the C^*-algebraist's perspective, we end the report on a cliff-hanger.

A C^*-algebra can be associated with a series of self-similar colour preserving isomorphisms of k-graphs in the following way, by considering the action on paths of a single degree. For a given k-graph Λ, each Λ_p for $p \in \mathbb{N}^k$ can be a considered a directed graph. If G is a colour preserving action then (G, Λ_p) is a self-similar groupoid action on the directed graph Λ_p in the sense of Definition 9. Let $M_0 = C^*(G)$ be the C^**-algebra* associated with the groupoid G and let $M_p = M(G, \Lambda_p)$ be the *right Hilbert* $G^*(G)$*-bimodule* associated with the directed graph (G, Λ_p). We do not define these precisely, however, importantly [Afs+19, Prop. 5.1] shows that $M \coloneqq \bigsqcup_{p\in\mathbb{N}^k} M_p$ is a product system. To this product system we can associate a universal C^*-algebra.

On the other hand, it is not clear how to construct an equivalent C^*-algebra-generating formalism for colour-swapping or colour-splitting self-similar actions. Indeed, the formalism of a colour-splitting partial isomorphism in a k-graph also is not clear, which makes is difficult to obtain a groupoid from the automata. This seems the most pressing lead to follow.

5.1 Acknowledgements

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