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The Category of Dessins d'Enfant The Existence of a Topos

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1 Prelude

1.1 Acknowledgements

I would like to acknowledge Professor Finnur Larusson and Dr Daniel Stevenson for all their guidance during my research project. I would also like to acknowledge Thomas Dee and Paawan Jethva, whom I worked with very closely during my research. Finally, I would like to thank Dr Tom Leinster for his invaluable book, titled *Basic Category Theory*, which I used to learn category theory.

1.2 Abstract

Dessin d'enfants are connected graphs that are bipartite and have a cyclic order of edges on each node. These graphs can be used to study topics in complex analysis, number theory, topology, and the absolute Galois group of rational numbers. Hence studying dessin d'enfants provides a unique vantage point to study these topics from – making certain results more apparent. This report looked at the existence of a topos in the category of dessin d'enfants by studying dessin d'enfants through the lens of combinatorics. While we found that the category of dessin d'enfants is not a topos, if we relax the definition of a dessin to include disconnected graphs, we can show that a topos does indeed exist.

1.3 Statement of Authorship

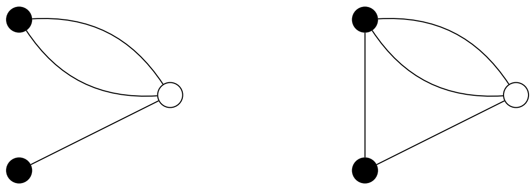
Lachlan Schilling researched the mathematics within this report as well as writing this paper. Professor Finnur Larusson, Dr Daniel Stevenson, Thomas Dee, and Paawan Jethva assisted in research as well as providing report feedback. Professor Finnur Larusson also formulated the project plan.

2 Introduction

2.1 What is a Dessin?

Dessin d'enfants (*singular. dessin*) are **connected** undirected graphs consisting of nodes connected via edges. However, unlike regular undirected graphs, dessins are **bipartite** and have a **cyclic ordering of edges on each node**. This means that the nodes are classified into two groups – black and white – and an edge can only be between a black and a white node. Because of the bipartite condition, the graph on the left of fig. 1 the following example is possible, while the graph on the right is not.

Furthermore, by the cyclic ordering of edges property, the following dessin in fig. 2 are distinct despite having the same number of edges and nodes.



(a) A valid dessin

(b) A invalid dessin

Figure 1: An example of a valid and invalid dessin by the bipartite condition.

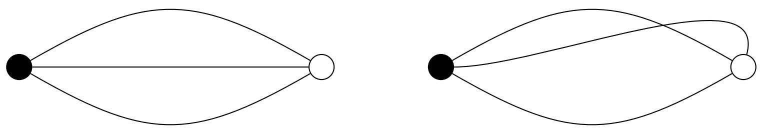


Figure 2: Two distinct dessins despite both having the same number of edges and nodes.

2.2 History of Dessin D'enfant

While the history of dessin d'enfants can be traced back to the 19th century, the study of dessin d'enfants flourished in the late 20th century by Alexander Grothendieck. Grothendieck became fascinated by dessin d'enfant as a means to study other fields including complex analysis, topology, number theory and combinatorics. The most notable of these was the study of the absolute Galois group of rational numbers which is connected to an unsolved problem in mathematics to this day.

Despite the widespread reach of dessin d'enfants, we will be limiting this paper to a combinatorial study of finite dessin d'enfants. More specifically, this report will be discussing the existence of an elementary topos in the category of dessin d'enfants.

2.3 What is a Topos?

The notion of a topos (*plural. topoi*) was first introduced by Alexander Grothendieck to describe features of topological spaces. Grothendieck's original definition of a topos (known as the Grothendieck topos) was later extended to what are known as elementary topoi. These are categories with additional structure that encapsulate some of the key properties of the category of sets – allowing us to do mathematics within these categories.

3 Dessins d'Enfants and Their Categorical Construction

Naturally, we first start by defining a dessin as;

Definition 1 (**Dessin d'Enfants**). *A* ***dessin*** *is a non-empty undirected graph consisting of edges and nodes that satisfy the following axioms:*

D1. *The graph is bipartite;*

D2. *There is a cyclic ordering on the edges connected to each node;*

D3. *The graph is connected.*

Furthermore, we will say a dessin is finite if and only if the dessin's corresponding edge set is finite. With this, we will be focusing our study to the category of finite dessin d'enfants. But before we can begin studying the category of finite dessin d'enfants, we need to establish the category itself. To do this, let us first notice that due to axioms D1 and D3, there is a canonical partitioning of a dessin's edge set.

Corollary 3.0.1. *For any dessin, all nodes of the same colour partition the set of edges.*

Proof. Let F be a dessin and fix a colour c. We will show that there exists an equivalence relation given by $e_1 \sim e_2$ if and only if e_1 and e_2 are connected to the same node of colour c.

Reflexivity. From D1, an edge can only be connected to one node of colouring c. As such, it is clear that $e \sim e$ for any edge e.

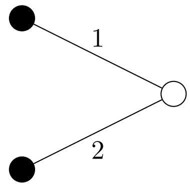
Symmetry. For any two edges e_1 and e_2, if $e_1 \sim e_2$, then they must share the same node n of colouring c. However from D1, e_1 and e_2 can only connect to node n of colouring c, and so $e_2 \sim e_1$.

Transitivity. Suppose $e_1 \sim e_2$ and $e_2 \sim e_3$ for edges e_1, e_2, e_3. Then let e_1 and e_2 connect to the node n_1 of colour c, and e_2 and e_3 both connect to the node n_2 of colour c. However from D1, e_2 can only connect to one node of colour c, and so $n_1 = n_2$. Thus e_1 and e_3 connect to the same node and $e_1 \sim e_3$.

Thus there exists an equivalence relation on the set of edges. As such, the equivalence classes partition the set of edges. ☐

3.1 The Combinatorial Construction of Dessins d'Enfant

From corollary 3.0.1, it is sufficient to define a dessin by its edges and the cyclic ordering of edges for each node – with information about the nodes implicitly given through the disjoint cycles of edges. In the following example, the black and white cycles encapsulate the structure of the dessin;

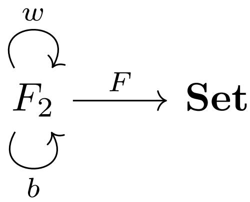


has black cycles of $(1), (2)$
has white cycles of $(1\ 2)$.

However, because these cycles are disjoint for each colour (corollary 3.0.1), we can take the product of cycles for all the same coloured nodes without losing information. In the case of the example above[1];

$$\text{black permutation} = (1)(2)$$
$$\text{white permutation} = (1\ 2).$$

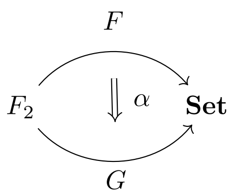
So we now have a description of a dessin F; an edge set E equipped with two functions, $b_F, w_F : E \to E$, that describe the black and white permutation of edges at each node. This is precisely a G-Set for a specific group G! More specifically, if we have the group F_2 – which is the free group on two generators b, w which correspond to the black and white permutations – the dessin F is a F_2-Set that maps b and w to the black and white permutations b_F and w_F respectively. This can be represented by the following diagram, where F is the corresponding dessin;



However up until this point we have neglected the connectedness property of dessin d'enfant. However, this can easily be rectified by restricting our construction to a transitive F_2-Set. With this, we arrive at our final construction of a dessin, and we can begin to discuss morphisms of dessin.

4 Morphisms of Dessin d'Enfant

Given that a dessin corresponds to a functor from the category F_2 with generators b and w, to the category of sets, it follows naturally that a morphism between dessins corresponds to a natural transformation. Formally, in the following diagram, α is a morphism from dessin F to dessin G.



Being a natural transformation, morphisms between dessin must satisfy the following commutative square:

[1]Displaying single cycles removes the need to explicitly specify the edge set.

$$\begin{array}{ccc} F(A) & \xrightarrow{F(c)} & F(A') \\ \big\downarrow \widehat{\alpha_A} & & \big\downarrow \widehat{\alpha_{A'}} \\ G(B) & \xrightarrow{G(c)} & G(B') \end{array}$$

where c is either b or w. However as the F_2 category has just one object, this commutative diagram becomes:

$$\begin{array}{ccc} E_1 & \xrightarrow{c_F} & E_1 \\ \big\downarrow \hat{\alpha} & & \big\downarrow \hat{\alpha} \\ E_2 & \xrightarrow{c_G} & E_2 \end{array}$$

where c_F and c_G are the c-coloured permutation on edges for F and G respectively[2] and E_1 and E_2 is the edge set of F and G respectively. These conditions can be nicely summarised by the equation

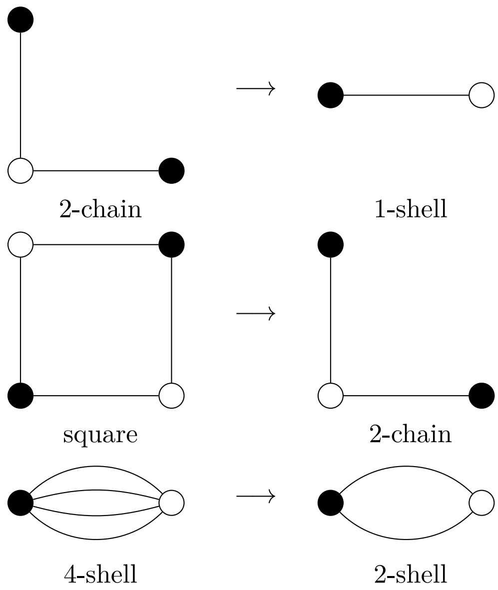
$$\hat{\alpha} c_F = c_G \hat{\alpha}. \tag{1}$$

It is important to note here that a morphism between dessins and a function between their edge sets are not the same thing. However, a function between their edge sets satisfying eq. (1) is a condition for a morphism between dessins to exist. As such, it might feel natural to think of these are the same thing. This is an idea we will formalise shortly.

4.1 Examples of Morphisms

The following are examples to illustrate some morphisms of dessins, along with the names of some common types of dessin.

[2] We will be using this notation hereafter.



4.2 The Category of Dessin d'Enfant and Concreteness

With morphisms defined, we can define the category of dessin d'enfant as the following:

Theorem 4.1 (Category of Dessin d'Enfants). *The category of dessin d'enfants, denoted by **Des**, is defined by;*

- $\text{ob}(\mathbf{Des}) = \{$*finite dessin d'enfants*$\}$,
- *for each* $F, G \in \text{ob}(\mathbf{Des})$, $\mathbf{Des}(F,G) = \{$*morphisms from* F *to* G *as defined above*$\}$,
- *for each* $F, G, H \in \text{ob}(\mathbf{Des})$, $\circ(f,g) = g \circ f$ *where the corresponding function between edge sets for* $g \circ f$ *is simply composition of the individual functions between edge sets.*

Proof. To show that **Des** is indeed a category, we need to show that **Des** has identity morphisms for each dessin and that the composition map is associative.

Associativity. Let F, G, H, D be dessins. Then we need to show that for $f \in \mathbf{Des}(F,G)$, $g \in \mathbf{Des}(G,H)$ and $h \in \mathbf{Des}(H,I)$, we have

$$(h \circ g) \circ f = h \circ (g \circ f).$$

However because composition is defined by composition of the corresponding functions between edge sets for f, g and h, their composition is associative, and thus so is the composition of morphisms.

Identities Let F, G be dessins. Then for $f \in \mathbf{Des}(F,G)$, we clearly have morphisms $1_F : F \to F$ and $1_G : G \to G$ such that

$$f \circ 1_F = f = 1_G \circ f$$

namely the morphisms with corresponding identity function between edge sets.

Thus as identity morphisms exist and as composition is associative, **Des** is a category. □

Up until now we have been discussing morphisms of dessins as functions in the underlying edge sets – despite not being the same thing. However, as we saw when defining the notion of a morphism, the only condition of a morphism to exist between two dessins is for the existence of a function between their edge sets such that eq. (1) is satisfied. As such, it seems that a morphism between dessins is really a morphism between their edge set. We can make this idea precise by showing that there exists a functor $U : \mathbf{Des} \to \mathbf{Set}$ such that a morphism from dessins F to G give a unique morphism from $U(F)$ to $U(G)$. This is the idea of a concrete category, and it is shown that **Des** is concretisable;

Theorem 4.2 (**Concreteness**)**.** *The category of dessin d'enfants equipped with the functor that returns a dessin's edge set is concrete.*

Proof. Let $U : \mathbf{Des} \to \mathbf{Set}$ be a functor that returns the edge set of dessins.

To prove that **Des** is concrete, we need to show that U is faithful. This amounts to showing the function

$$f_{F,G} : \mathbf{Des}(F, G) \to \mathbf{Set}(U(F), U(G)).$$

is injective (the existence of such a function is trivial as we naturally need a function between edge sets to describe a morphism between dessin).

Injectivity. Let $F, G \in \mathbf{Des}$ and suppose $\alpha_1, \alpha_2 : F \to G$. Then If $f_{F,G}(\alpha_1) = f_{F,G}(\alpha_2)$, then we have $\hat{\alpha_1} = \hat{\alpha_2}$, and hence $\alpha_1 = \alpha_2$ as eq. (1) is the only defining feature for a morphism between F and G. Thus $f_{F,G}$ is injective.

Hence U is faithful, and thus **Des** equipped with U is a concrete category. □

With theorem 4.2, we can not only say that **Des** behaves like **Set**, only with some additional structure, but we can naturally talk about morphisms of two dessin as a morphism in their underlying edge set. As such, if $\alpha : F \to G$ is a morphism, we will often refer to the induced function between their edge sets as $\hat{\alpha} : U(F) \to U(G)$. Furthermore, we will refer to **Des** as the category equipped with the forgetful functor U.

Additionally, while concreteness does not necessarily restrict the nature of objects in **Des** (such as limits, exponential, and subobject classifiers), it often provides a reasonable initial guess as to the structure of these objects. Finally, the concreteness of **Des** allows us to conclude that the functor $U : \mathbf{Des} \to \mathbf{Set}$ that sends a dessin to its underlying edge set is indeed a forgetful functor.

4.3 Surjections and Isomorphisms

Having now spoken about morphisms of dessins, it is natural to now investigate different kinds of morphisms. We will first start by looking at surjections.

Lemma 4.3 (**Surjections**)**.** *A morphism from F to G induces a surjective function from $U(F)$ to $U(G)$.*

Proof. Let $F, G \in \mathbf{Des}$ and suppose there exists a morphism $\alpha : F \to G$ such that the induced morphism $\hat{\alpha} : U(F) \to U(G)$ is not surjective: i.e. there exists $e \in U(G)$ such that $\alpha^{-1}(e) = \emptyset$. However because α is a morphism, we require

$$\hat{\alpha} c_F = c_G \hat{\alpha}.$$

This is a contradiction because G is connected and c_G is a bijection, and thus there exists $e' \in U(F)$ such that $c_G \alpha = e$, however this cannot be reached αc_F. Thus by contradiction, $\hat{\alpha}$ is surjective. □

As a direct result of lemma 4.3, we can conclude the following:

Corollary 4.3.1. *For dessin $F, G \in$ **Des**, a morphism from F to G implies that $|U(F)| \geq |U(G)|$.*

Proof. This result directly follows from lemma 4.3 and the fact that for surjections $A \to B$, $|A| \geq |B|$. □

It is obvious that morphism-induced functions do not have to be injective (consider the examples in section 4.1). However, when they are injective, we get an isomorphism;

Lemma 4.4 (**Isomorphisms**)**.** *A morphism $\alpha : F \to G$ as an isomorphism if and only if the induced morphism $\hat{\alpha} : U(F) \to U(G)$ is an isomorphism.*

Proof. This result follows directly from the fact that isomorphisms $\alpha : F \to G$ of concrete categories must correspond to an isomorphism from $U(F)$ to $U(G)$. □

With lemma 4.4, we can see that the isomorphism class of a dessin is just the same dessin with all possible edge sets.

Another interesting type of morphism to consider are inclusion maps. We will define an inclusion map as:

Definition 2 (**Inclusions**)**.** *An inclusion map from dessins F to G, denoted by $\alpha : F \hookrightarrow G$, has a corresponding inclusion map from $U(F)$ to $U(G)$.*

With this, we can show that inclusion maps in **Des** are simply isomorphisms;

Lemma 4.5. *An inclusion map in **Des** is an isomorphism.*

Proof. Let $F, G \in \mathbf{Des}$ with $\alpha : F \hookrightarrow G$, and $\hat{\alpha} : U(F) \to U(G)$ be the α induced function.

From definition 2, $\hat{\alpha}$ must be an inclusion map, and thus injective. Furthermore, $\hat{\alpha}$ is surjective from lemma 4.3. Thus as $\hat{\alpha}$ is bijective, α is an isomorphism from lemma 4.4. □

Notice that not all isomorphisms are inclusion maps as isomorphisms can be between different edge sets, while inclusion maps are between the same edge set – to this end, inclusion maps in **Des** are **automorphisms**.

5 Existence of a Topos

5.1 Elementary Topoi

With the category of dessin d'enfants now defined, we can begin to investigate the existence of a topos. Formally, a topos is defined as:

Definition 3 (**Elementary Topoi**). *An **elementary topos** is a category equipped with the following objects:*

- *Finite limits*
- *Exponentials*
- *A subobject classifier*

As such, this section will be dedicated to proving the existence of these objects in the category of dessin d'enfants.

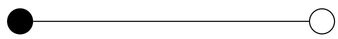
5.2 Finite Limits

A category has finite limits if it has terminal objects, binary products and equalisers. However, in this section, we will only show that **Des** has a terminal object and binary products as the other conditions for a topos prove the existence of equalisers.

5.2.1 Terminal Object

As was shown in corollary 4.3.1, if there exists a morphism from any dessin F to the terminal dessin $\mathbf{1}$, then $\mathbf{1}$ must have the least number of edges. This means that $\mathbf{1}$ must have a single edge, which leaves a single isomorphism class for the terminal object;

Theorem 5.1 (**Terminal object**). *The **terminal object** of **Des** is:*

$\boldsymbol{1} =$ 

Proof. Suppose the single edge of $\mathbf{1}$ is e, and let F be any dessin. We need to show that there exists a unique morphism from F to $\mathbf{1}$:

Existence. The morphism $\alpha : F \to \mathbf{1}$ that maps all edges in F to e is a valid morphism.

Uniqueness. Suppose that $\alpha_1, \alpha_2 : F \to \mathbf{1}$ are both morphisms. Thus as the codomain of α_1 and α_2 is just e, $\alpha_1(i) = \alpha_2(i) = e$ for all edges i in F. Thus $\alpha_1 = \alpha_2$: there is a unique morphism from F to $\mathbf{1}$.

Hence as there is a unique morphism from F to $\mathbf{1}$, $\mathbf{1}$ is indeed the terminal object. □

5.2.2 Binary Products

Intuitively, the binary product of two dessins F and G is simply another dessin, $F \times G$, such that there exists morphisms from $F \times G$ to F and G. Despite this, defining $F \times G$ can be tricky. However, from theorem 4.2,

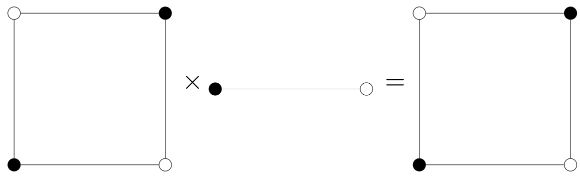
Des is concrete, and thus we will take inspiration from **Set** to define binary products in **Des**. Namely;

$$U(F \times G) = U(F) \times U(G)$$
$$p_i((e_1, e_2)) = e_i.$$

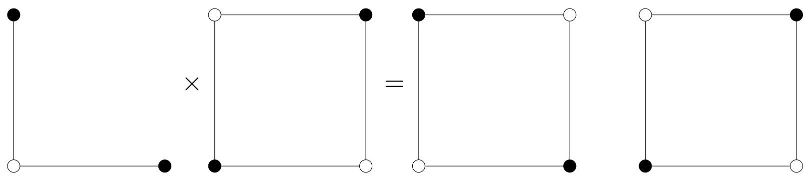
With this choice, the most natural guess for the black and white permutations are

$$c_{F\times G}((e_1, e_2)) = (b_F(e_1), b_G(e_2)).$$

With this definition of binary products, let us consider two examples of products:



This is clearly correct as there is a morphism from the square to the square (identity) and a morphism from the square to the terminal (theorem 5.1).



While this seems mostly correct – there is a morphism[3] from the double square to the single square, and likewise for the 2-chain, the product dessin is not connected. This is very problematic as axiom D3 requires dessins to be connected. As such, we will adapt our notion of a dessin to accommodate for this.

5.3 Generalised Dessin d'Enfant

As we saw in our examples of products, our definition of binary products seems correct except for the issue of connectedness. As such, we will introduce the notion of a generalised dessin to be a dessin without the connectedness condition:

Definition 4 (Generalised Dessin d'Enfants). *A **generalised dessin** is an undirected graph consisting of edges and nodes that satisfy the following axioms:*

GD1. *The graph is bipartite;*

GD2. *There is a cyclic ordering on the edges connected to each node;*

[3] We will come to define what a morphism of disconnected dessin means shortly, but for now, we will proceed with intuition

As our combinatorial construction did not rely on the connectedness of dessin, the construction of generalised dessin remains the same, and thus our notion (and conditions) for morphisms between generalised dessin are identical to that of connected dessin. With objects and morphisms defined, we can define the category of generalised dessin like above – the proof of theorem 4.1 applies to the category of generalised dessin d'enfants – we will denote the category of generalised dessin d'enfants as **GDes**.

Furthermore, as the proof of corollary 3.0.1, theorem 4.2, lemma 4.4, and theorem 5.1 did not rely on the connectedness of **Des**, these results all apply in **GDes**. The most notable of these is that **GDes** is concrete and that **GDes** has a terminal object – namely the same as that in **Des**. However, because lemma 4.3 and lemma 4.5 did rely on the connected nature of dessin, it is not true that morphisms between generalised dessin are surjective, and inclusion maps are no longer isomorphisms.

Finally, in addition to removing the connectedness condition, we will permit the existence of an **initial** dessin in **GDes**; denoted by $\boldsymbol{\emptyset}$. This dessin has no edges and no nodes – thus there is a unique morphism from $\boldsymbol{\emptyset}$ to F for all dessins F. However, from the fact that $U(\boldsymbol{\emptyset}) = \emptyset$, there exists no morphisms to $\boldsymbol{\emptyset}$ (theorem 4.2).

5.4 Inclusions and Monomorphisms of Generalised Dessin

It is often useful to think of a generalised dessin consisting of many connected dessin. As such, we will denote the generalised dessin made up of connected dessins $D_1, ..., D_n$ as $\langle D_1, ..., D_n \rangle$[4]. Furthermore, if there exists an inclusion map α from F to G, we will say that $F \subseteq G$. If F is connected and $F \subseteq G$, we say that $F \in G$. With this, we can show that $D \in \langle D_1, ..., D_n \rangle$ if and only if $D \in \{\boldsymbol{\emptyset}, D_1, ..., D_n\}$.

With this new notation, a natural question to ask is whether we can think of a morphism between two generalised dessin as a set of morphisms between their connected dessin. This leads us to the following theorem;

Theorem 5.2. *A morphism α from generalised dessin F to G can be expressed as a set of morphisms α_i from F_i to G_i where $F_i \in F$ and $G_i \in G$.*

Proof. Let F and G be dessins, and let $\alpha : F \to G$ with induced function $\widehat{\alpha} : U(F) \to U(G)$. Then for each $F_i \in F$, let G_i be the image of F_i under α, along with the map $\alpha_i : F_i \to G_i$ such that the induced function $\widehat{\alpha_i} : U(F_i) \to U(G_i)$ sends $e \mapsto \widehat{\alpha}(e)$. $G_i \in G$ as there is obviously an inclusion from G_i to G (by definition of G_i), and G_i is also connected as if it was not, then there could not be a morphism from F to G.

Thus we now have a collection of morphisms $\alpha_i : F_i \to G_i$ for each $F_i \in F$. With this, we can construct a new morphism $\alpha' : F \to G$ such that the induced function $\widehat{\alpha'} : U(F) \to U(G)$ such that for all $F_i \in F$, and $e \in U(F_i)$;

$$\widehat{\alpha'}(\iota_{F_i}(e)) = (\iota_{G_i}\widehat{\alpha_i})(e)$$

[4]This notation is making the assumption that all of the edge sets of $D_1, ..., D_n$ are disjoint.

where ι_{F_i} is the inclusion map from $U(F_i) \to U(F)$. However, by the construction of α', it is clear that $\hat{\alpha} = \widehat{\alpha'}$. Thus we have shown that each morphism from $F \to G$ can be expressed as morphisms from $F_i \to G_i$ for $F_i \in F$ and $G_i \in G$. □

With theorem 5.2, we can talk about morphisms of dessin as being a set of morphisms from connected dessins to connected dessins.

We will now briefly turn our attention to monomorphisms in **GDes**.

Lemma 5.3. *A morphism in **GDes** is a monomorphism if and only if the induced function is injective.*

Proof. This result follows directly from the concreteness of **GDes**. □

From lemma 5.3 and definition 2, we can think of a monomorphism as an inclusion map composed with an isomorphism. This is formalised as so;

Corollary 5.3.1. *A morphism in **GDes** is a monomorphism if and only if it is isomorphic to an inclusion map.*

Proof. Let $F, G \in$ **GDes** with $\alpha : F \to G$ and the induced function $\hat{\alpha} : U(F) \to U(G)$.

- ($\implies$). Suppose α is a monomorphism. Then from lemma 5.3, $\hat{\alpha}$ is injective. Thus $\hat{\alpha}$ is isomorphic to an inclusion map, and thus from definition 2, α is an inclusion map.
- ($\impliedby$). Suppose α is an inclusion map. Then from definition 2, $\hat{\alpha}$ is injective. Thus from lemma 5.3, α is a monic.

Hence a morphism is a monomorphism if and only if it is an inclusion map. □

From corollary 5.3.1, we can conclude that a morphism $\alpha : F \to G$ is a monomorphism if and only if F is isomorphic to F' such that $F' \subseteq G$. As a result, the only possible monics to a connected dessin F are those from $\emptyset$ and isomorphisms of F.

5.5 Binary Products of Generalised Dessin

With generalised dessin now defined, we can finally define binary products of dessin.

Theorem 5.4 (Binary Products). *The binary product of $F, G \in$ **GDes** is defined by:*

$$U(F \times G) = U(F) \times U(G)$$

$$c_{F \times G}((e_1, e_2)) = (c_F(e_1), c_G(e_2))$$

$$p_i((e_1, e_2)) = e_i.$$

Proof. Consider a dessin F with morphisms $\alpha_1 : F \to X$ and $\alpha_2 : F \to Y$. Then we can construct $X \times Y$ as per above. We need to show that there exists a unique morphism from F to $X \times Y$ along with morphisms from $X \times Y$ to X and Y called projections.

Existence. Consider the morphism $\alpha : F \to X \times Y$ with induced function $\widehat{\alpha} : U(F) \to U(X \times Y)$ in which $e \mapsto (\alpha_1(e), \alpha_2(e))$. This map is a morphism from F to $X \times Y$ as it satisfies eq. (1):

$$\begin{aligned} \alpha c_F &= (\widehat{\alpha_1} c_F, \widehat{\alpha_2} c_F) \\ &= (c_X \widehat{\alpha_1}, c_Y \widehat{\alpha_2}) \\ &= (c_X, c_Y)\widehat{\alpha} \\ &= c_{X \times Y}\widehat{\alpha}. \end{aligned}$$

Uniqueness. Suppose there exists another morphism $\alpha' : F \to X \times Y$. Let $\hat{\alpha}'(e) = (x, y)$ for all edges e in F. Then because products must commute, we have:

$$\widehat{\alpha_1}(e) = p_1(\hat{\alpha}'(e)) = p_1(x, y) = x$$

likewise with $\widehat{\alpha_2}(e) = y$. Thus we have $\hat{\alpha} = (\widehat{\alpha_1}, \widehat{\alpha_2}) = \widehat{\alpha}'$, and hence $\alpha = \alpha'$.

□

With theorem 5.1 and theorem 5.4, we have shown that **GDes** has a terminal object and binary products.

5.6 Exponential Objects

Much like with binary products, we can use the concreteness of **GDes** to make a reasonable guess towards the edge set of the exponential. Namely, for two dessins F and G, we have:

$$U\left(F^G\right) = U(F)^{U(G)}.$$

This leaves us to determine the black and white permutations of the exponential objects. Which, from aligning the domains and codomains of the permutations in F and G, one of the most natural guess for the black and white permutations are

$$\begin{aligned} b_{F^G}(\alpha) &= b_F \alpha b_G^{-1} \\ w_{F^G}(\alpha) &= w_F \alpha w_G^{-1}. \end{aligned}$$

These guesses lead us to the following.

Theorem 5.5 (Exponential Objects). *The exponential object of dessins $F, G \in$ **GDes** is defined by:*

$$\begin{aligned} U\left(F^G\right) &= U(F)^{U(G)} \\ c_{F^G}(\alpha) &= c_F \alpha c_G^{-1}. \end{aligned}$$

Proof. Let F, G, and H be dessins. We need to show that

$$\mathbf{Des}(F \times G, H) \cong \mathbf{Des}(F, H^G)$$

naturally in F and G. Thus let us consider the map $f : \mathbf{Des}(F \times G, H) \to \mathbf{Des}(F, H^G)$ with

$$f : [(e_1, e_2) \mapsto e_3] \mapsto \left[e_1 \mapsto \left[c_G^{-1}(e_2) \mapsto e_3\right]\right].$$

Let $\alpha \in \mathbf{Des}(F \times G, H)$ and suppose $\alpha((e_1, e_2)) = e_3$. Then we will show that $f\alpha$ is actually a morphism from F to H^G:

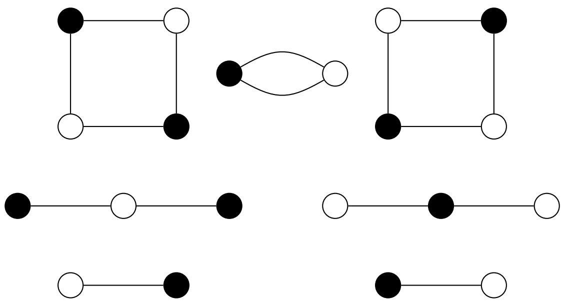
$$\alpha c_{F \times G}((e_1, e_2)) = c_H \alpha((e_1, e_2)) = \alpha(c_F(e_1), c_G(e_2))$$

Thus sending $\alpha((e_1, e_2))$ to $f\alpha(e_1, e_2) = \alpha'(e_1)(c_G^{-1}(e_2))$ as per the proposed isomorphism above gets us

$$\alpha'(c_F(e_1))(c_G^{-1}(c_G(e_2))) = (c_H \alpha'(e_1))(c_G^{-1} e_2)$$
$$\alpha'(c_F(e_1))(e_2) = C_{H^G}(\alpha'(e_1))(e_2).$$

Thus α' satisfies eq. (1), and thus α' is indeed a morphism from F to H^G – i.e. $f\alpha \in \mathbf{Des}(F, H^G)$. Furthermore, as c_G is a bijection, it is obvious that f is injective. For brevity, we will not rigorously show that f is surjective, but it is clear to see that the proof above can be simply reversed to show that $f^{-1}\alpha \in \mathbf{Des}(F \times G, H)$ for all $\alpha \in \mathbf{Des}(F, H^G)$. Thus f is surjective, and thus f is an isomorphism between $\mathbf{Des}(F \times G, H)$ and $\mathbf{Des}(F, H^G)$. □

The following is an example of the *2-chain* exponentiated with the *square*:



5.7 Subobject Classifier of Generalised Dessin d'Enfants

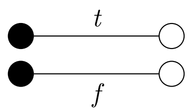
The final condition for a topos to exist is the existence of a subobject classifier. However, before investigating the existence of a subobject classifier, we ought to understand the subobjects of dessins.

Theorem 5.6 (Subobjects). *The subobjects of the dessin $D = \langle D_1, ..., D_n \rangle$ are the isomorphism classes of dessin generated by the powerset of $\{D_1, ..., D_n\}$.*

Proof. Let $F = \langle D_1, ..., D_n \rangle$, and suppose there exist monics $u : S \rightarrowtail F$ and $v : T \rightarrowtail F$. Then from corollary 5.3.1, u and v must be isomorphic to inclusion maps. Hence $S \cong S'$ and $T \cong T'$ with $S' \subseteq F$ and $T' \subseteq F$. Hence the subobjects classes are the isomorphism classes of all combinations of $D_1, ..., D_n$ – i.e. the powerset of $\{D_1, ..., D_n\}$. □

With all of the above work, we can finally investigate the existence of a subobject classifier.

Theorem 5.7 (**Subobject Classifier**). *The subobject classifier of **GDes** is;*

$\mathbf{\Omega} =$ 

Proof. Let $F \in GDes$. We need to show that

$$\mathbf{GDes}(F, \mathbf{\Omega}) \cong \text{sub(F)}$$

naturally in F. Thus let us consider the map $f : \text{sub(F)} \to \mathbf{GDes}(F, \mathbf{\Omega})$ where $G \mapsto f_G : F \to \mathbf{\Omega}$ where the morphism f_G has the induced map $\widehat{f_G} : U(F) \to U(\mathbf{\Omega})$ where

$$\widehat{f_G}(e) = \begin{cases} t & e \in U(G) \\ f & \text{otherwise.} \end{cases}$$

First we note that $f(G)$ is certainly a morphism from F to $\mathbf{\Omega}$ as $\widehat{f(G)}$ is simply mapping the connected dessins of F to either one of the two terminal dessins in $\mathbf{\Omega}$, and thus $f(G)$ is a morphism by theorem 5.2. With this, we will now show that this map is a bijection:

Injective. Let $G_1, G_2 \in \text{sub(F)}$ and suppose $f(G_1) = f(G_2)$. Then $\widehat{f_{G_1}} = \widehat{f_{G_2}}$. Hence $U(G_1) = U(G_2)$, and thus $G_1 = G_2$ and f is injective

Surjective. Let $\alpha \in \mathbf{GDes}(F, \mathbf{\Omega})$. Then the dessin G generated by the connected dessins that map to t (the preimage of t) is a subobject of F as G consists of the connected dessin in F, which is a subobject of F by theorem 5.2 and definition 2. Thus f is surjective.

As f is bijective, $\mathbf{GDes}(F, \mathbf{\Omega}) \cong \text{sub(F)}$, and thus the proposed subobject classifier satisfied the universal property for a subobject classifier. □

Thus from from definition 3, theorem 5.1, theorem 5.4, theorem 5.5, and theorem 5.7, the category of generalised dessin d'enfants is a topos.

6 Conclusion

In this research, it was shown that while the category of dessin d'enfants is not a topos, the category of generalised dessin is indeed a topos. Interestingly, because the F_2 nature of dessins was never explicitly used, this result can be generalised to the category of G-Sets. Given additional time, I would research the implications of the topos – e.g. can we use the fact that **GDes** is a topos to redefine connectedness? How can we use the topos to do other mathematics? I also think it could be interesting to look closer into the nature of different structures in **GDes**; such as exponentials, as these seemed to encapsulate some interesting symmetries.

7 Bibliography

Leinster, T 2014, *Basic Category Theory, Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, UK.

Sergei L Alexander Z 2013, *Graphs on Surfaces and Their Applications*, *Encyclopaedia of Mathematical Sciences*, Springer Berlin, Heidelberg.