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Modelling the dynamic response of a rail track with rubber inclusion

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1 Abstract

The development of novel technologies in the construction of railways, motivates the solution of certain mathematical models. The particular model considered in this report is the Euler-Bernoulli beam equation on a visco-elastic foundation. This model is relevant to Recycled Rubber Energy Absorbing Grids (REAG), which are being developed as a solution to the degredation of rail infrastructure from vibrational energy. The beam equation is first non-dimensionalised for two external force models - stationary and moving point forces - and then partially solved using integral transforms, paving the way for numerical solutions in further research.

2 Introduction

In this report, we model train tracks resting on a novel Recycled Rubber Energy Absorbing Grid (REAG), using the Bernoulli-Euler beam equation with a viscoelastic foundation

$$EI\frac{\partial^4 y}{\partial x^4} + ky + C\frac{\partial y}{\partial t} + m\frac{\partial^2 y}{\partial t^2} = F(x,t) \tag{1}$$

Here $E > 0$ is Young's modulus of elasticity (in $\mathrm{N/m^2}$), $I > 0$ is the moment of inertia of the beam (in $\mathrm{m^4}$), $k > 0$ is a constant of elasticity per unit length in ($\mathrm{N/m^2}$), $C > 0$ is a constant of damping per unit length (in $\mathrm{Ns/m^2}$), and F is an external force (in N/m). The equation is considered for two force terms: a stationary point force $F(x,t) = f_p\delta(x)$ and a moving point force $F(x,t) = f_p\delta(x - vt)$, where δ is the Dirac delta, f_p is the strength of the point force and v is the velocity of the moving point force.

After non-dimensionalising the equations, we solve by applying Fourier and Laplace transforms. The Fourier, Laplace, and inverse Laplace transforms are calculated analytically, leaving the inverse Fourier transform to be calculated numerically, in further research.

2.1 Background

In recent years, alongside the need for faster and heavier trains, the demand put upon rail systems has gradually increased. This has, in turn, accelerated the rate of degradation for infrastructure and necessitated preventative measures, such as strict speed limits [2]. Damage to ballasted rail tracks constitute a significant portion of maintenance costs, as they are a major infrastructure for both freight and passenger transport [1]. Previous research has indicated that the use of either rubber mats or geogrids - a type of plastic lattice - can aid in reducing wear on ballasted rail tracks [3, 5]. However, both have notable drawbacks. Rubber mats are ineffective on softer foundations and have a tendency to impede drainage [9]. Geogrids, on the other hand, while performing well on softer foundations and encouraging drainage, are ineffective at reducing wear on a rigid foundation [4]. Recycled rubber energy absorbing grids (REAG), manufactured from recycled rubber - e.g. tyres and conveyor belts - combine the benefits of rubber mats and geogrids: interlocking with the ballast to increase drainage whilst also absorbing energy from passing trains [6]. To inform the implementation of REAG to ballast tracks, mathematical modelling is required to predict the ground vibrations induced by the moving wheel load and to capture the enhanced energy absorbing capacity attributed to REAG.

2.2 Statement of Authorship

Unless otherwise referenced, all content of this report is the work of the author - Josiah Murray - done under the supervision of, and in discussion with, Proffessor Natalie Thamwattana and Professor Mike Meylan.

3 Preliminary Notes

Before beginning, there are some definitions and results that the reader may find useful to have listed. This includes a brief look at the Dirac delta, a statement of the particular forms of the Fourier and Laplace transforms used in the report and a substitution result that the author thought to be non-trivial.

3.1 The Dirac Delta

There are several ways of defining the Dirac delta, however, in this report we are mostly concerned with one property in particular. As such, we will use it as our definition. This is the 'Sifting property' [8]

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)dx = f(x_0)$$

This property is particularly relevant when using integral transforms. To properly non-dimensionalise the equation, however, we need a second property [7] - which, though we do not prove it here, can be derived from the sifting property.

$$\delta(f(x)) = \sum_i \frac{1}{|f'(x_i)|}\delta(x - x_i), \ \forall x_i \text{ such that } f(x_i) = 0$$

In particular, we then have

$$\delta(ax) = \frac{1}{|a|}\delta(x)$$

$$\delta(ax - b) = \frac{1}{|a|}\delta(x - \frac{b}{a})$$

3.2 Integral Transforms

Fourier transforms in this report are of the form

$$\mathcal{F}\{f(x)\} = \hat{f}(\Omega) = \int_{-\infty}^{\infty} f(x)e^{i\Omega x}dx$$

and inverse Fourier transforms are of the form

$$\mathcal{F}^{-1}\{\hat{f}(\Omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\Omega)e^{-i\Omega x}d\Omega$$

Laplace transforms are of the form

$$\mathcal{L}\{f(t)\} = \bar{f}(s) = \int_{0}^{\infty} f(t)e^{-st}dt$$

and the inverse Laplace transforms are of the form

$$\mathcal{L}^{-1}\{\bar{f}(s)\} = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} \bar{f}(s)e^{st}ds$$

3.3 Substituting into derivatives

In the process of non-dimensionalising, we make substitutions of the form $\tilde{u} = au, \tilde{v} = bv$ into terms of the form $\frac{\partial^n u}{\partial v^n}$. This gives

$$\frac{\partial^n u}{\partial v^n} = \frac{a}{b^n}\frac{\partial^n \tilde{u}}{\partial \tilde{v}^n}$$

What follows is a demonstration of how we can derive this result

$$\begin{aligned}
\frac{\partial^k u}{\partial v^k} &= \frac{\partial^k (a\tilde{u})}{\partial (b\tilde{v})^k} \\
&= a\frac{\partial^k \tilde{u}}{\partial (b\tilde{v})^k} \\
&= a\frac{\partial}{\partial (b\tilde{v})}\left(\frac{\partial^{k-1}\tilde{u}}{\partial (b\tilde{v})^{k-1}}\right) \\
&= a\frac{\partial g}{\partial (b\tilde{v})} \qquad \text{, where g} = \frac{\partial^{k-1}\tilde{u}}{\partial (b\tilde{v})^{k-1}} \\
&= a\left(\frac{\partial g}{\partial \tilde{v}} \cdot \frac{\partial \tilde{v}}{\partial (b\tilde{v})}\right) \\
&= a\left(\frac{\partial g}{\partial \tilde{v}} \cdot \frac{\partial (b\tilde{v}\frac{1}{xs})}{\partial (b\tilde{v})}\right) \\
&= \frac{a}{b}\left(\frac{\partial g}{\partial \tilde{v}} \cdot \frac{\partial (x_s\tilde{v})}{\partial (b\tilde{v})}\right) \\
&= \frac{a}{b}\left(\frac{\partial g}{\partial \tilde{v}}\right) \\
&= \frac{a}{b}\frac{\partial}{\partial \tilde{v}}\left(\frac{\partial^{k-1}\tilde{u}}{\partial (b\tilde{v})^{k-1}}\right) \\
&\dots \\
&= \frac{a}{b^k}\frac{\partial^k \tilde{u}}{\partial \tilde{v}^k}
\end{aligned}$$

4 Non-dimensionalisation

To simplify the calculations in this report, we non-dimensionalise our equations. Making appropriate substitutions and relabelling parameters allows us to find a form of the PDE in which all of the variables and terms are unitless and which has minimal parameters. Below are the standard and non-dimensionalised forms of the equations considered in this report, as derived in sections 4.1 and 4.2:

$$EI\frac{\partial^4 y}{\partial x^4} + ky + C\frac{\partial y}{\partial t} + m\frac{\partial^2 y}{\partial t^2} = f_p\delta(x) \implies \frac{\partial^4 \gamma}{\partial \chi^4} + \gamma + A\frac{\partial \gamma}{\partial \tau} + \frac{\partial^2 \gamma}{\partial \tau^2} = B\delta(\chi) \tag{2}$$

$$EI\frac{\partial^4 y}{\partial x^4} + ky + C\frac{\partial y}{\partial t} + m\frac{\partial^2 y}{\partial t^2} = f_p\delta(x - vt) \implies \frac{\partial^4 \gamma}{\partial \chi^4} + \gamma + A\frac{\partial \gamma}{\partial \tau} + \frac{\partial^2 \gamma}{\partial \tau^2} = B\delta(\chi - V\tau) \tag{3}$$

4.1 Non-dimensionalising with a stationary point force

To begin our non-dimensionalisation, we express the dimensional variables as a non-dimensional variable multiplied by a scaling factor - the particular value of these factors will be defined later. Specifically, we take

$$y = y_s\gamma \qquad\qquad x = x_S\chi \qquad\qquad t = t_s\tau$$

where y, x, t have units - say metres, metres, and seconds respectively - γ, χ, τ are dimensionless, and y_s, x_s, t_s have the same units as their associated variables.

Substituting into our PDE we get

$$\begin{aligned} EI\frac{\partial^4 y}{\partial x^4} + ky + C\frac{\partial y}{\partial t} + m\frac{\partial^2 y}{\partial t^2} = f_p\delta(x) &\implies EI\frac{\partial^4 (y_s\gamma)}{\partial (x_s\chi)^4} + k(y_s\gamma) + C\frac{\partial (y_s\gamma)}{\partial (t_s\tau)} + m\frac{\partial^2 (y_s\gamma)}{\partial (t_s\tau)^2} = f_p\delta(x_s\chi) \\ &\implies \frac{y_s}{x_s^4}EI\frac{\partial^4 \gamma}{\partial \chi^4} + y_sk\gamma + \frac{y_s}{t_s}C\frac{\partial \gamma}{\partial \tau} + \frac{y_s}{t_s^2}m\frac{\partial^2 \gamma}{\partial \tau^2} = f_p\delta(x_s\chi) \\ &\implies \frac{\partial^4 \gamma}{\partial \chi^4} + \frac{x_s^4}{EI}k\gamma + \frac{x_s^4 C}{EIt_s}\frac{\partial \gamma}{\partial \tau} + \frac{x_s^4 m}{EIt_s^2}\frac{\partial^2 \gamma}{\partial \tau^2} = \frac{x_s^4 f_p}{EI}\delta(x_s\chi) \\ &\implies \frac{\partial^4 \gamma}{\partial \chi^4} + \frac{x_s^4}{EI}k\gamma + \frac{x_s^4 C}{EIt_s}\frac{\partial \gamma}{\partial \tau} + \frac{x_s^4 m}{EIt_s^2}\frac{\partial^2 \gamma}{\partial \tau^2} = \frac{x_s^4 f_p}{EI|x_s|}\delta(\chi) \end{aligned}$$

We can continue to simplify, if we now specify $x_s = \sqrt[4]{\frac{EI}{k}}$ and $t_s = \sqrt{\frac{mx_s^4}{EI}}$, then let $A := \frac{x_s^4 C}{EIt_s}$ and $B := \frac{x_s^4 f_p}{EI|x_s|} = \frac{x_s^3 f_p}{EI}$ - noting that $EI > 0 \;\; k > 0 \implies x_s \geq 0$.

$$\frac{\partial^4 \gamma}{\partial \chi^4} + \frac{x_s^4}{EI}k\gamma + \frac{x_s^4 C}{EIt_s}\frac{\partial \gamma}{\partial \tau} + \frac{x_s^4 m}{EIt_s^2}\frac{\partial^2 \gamma}{\partial \tau^2} = \frac{x_s^4 f_p}{EI}\delta(x_s\chi) \implies \frac{\partial^4 \gamma}{\partial \chi^4} + \gamma + A\frac{\partial \gamma}{\partial \tau} + \frac{\partial^2 \gamma}{\partial \tau^2} = B\delta(\chi) \tag{4}$$

4.2 Non-dimensionalising with a moving point force

For a moving point force, we repeat the same process as in the preceding section 4.1 with the addition of letting $V := \frac{vt_s}{x_s}$. This gives

$$EI\frac{\partial^4 y}{\partial x^4} + ky + C\frac{\partial y}{\partial t} + m\frac{\partial^2 y}{\partial t^2} = f_p\delta(x - vt) \implies \frac{\partial^4 \gamma}{\partial \chi^4} + \gamma + A\frac{\partial \gamma}{\partial \tau} + \frac{\partial^2 \gamma}{\partial \tau^2} = B\delta(\chi - V\tau) \tag{5}$$

5 Solution with Stationary Point Force

We begin with the non-dimensional form of the Euler-Bernoulli beam equation on a viscoelastic foundation with a stationary point force at $x = 0$, as in (4)

$$\frac{\partial^4 \gamma}{\partial \chi^4} + \gamma + A\frac{\partial \gamma}{\partial \tau} + \frac{\partial^2 \gamma}{\partial \tau^2} = B\delta(\chi)$$

For initial conditions, we assume that the beam is undisturbed, so $\gamma(x, 0) = 0 \ \forall x \in \mathbb{R}$. We now aim to solve this equation using applications of both the Fourier and Laplace transforms.

5.1 Fourier Transform

Applying the Fourier transform to both sides of our equation, we get

$$\int_{-\infty}^{\infty} \left(\frac{\partial^4 \gamma}{\partial \chi^4} dt + \gamma + A\frac{\partial \gamma}{\partial \tau} + \frac{\partial^2 \gamma}{\partial \tau^2}\right) e^{i\Omega\chi} dt = \int_{-\infty}^{\infty} B\delta(x_s\chi) e^{i\Omega\chi} dt$$
$$\Omega^4 \hat{\gamma} + \hat{\gamma} + A\frac{\partial \hat{\gamma}}{\partial \tau} + \frac{\partial^2 \hat{\gamma}}{\partial \tau^2} = B$$

5.2 Laplace Transform

Now applying the Laplace transform, we get

$$\int_{0}^{\infty} \left(\Omega^4 \hat{\gamma} + \hat{\gamma} + A\frac{\partial \hat{\gamma}}{\partial \tau} + \frac{\partial^2 \hat{\gamma}}{\partial \tau^2}\right) e^{-st} dt = \int_{0}^{\infty} Be^{-st} dt$$
$$\Omega^4 \bar{\hat{\gamma}} + \bar{\hat{\gamma}} + sA\bar{\hat{\gamma}} + s^2\bar{\hat{\gamma}} = B\frac{1}{s}$$
$$\bar{\hat{\gamma}} = B\frac{1}{s}\left(\frac{1}{s^2 + As + \Omega^4 + 1}\right)$$

5.3 Inverse Laplace Transform

Denoting convolution with $*$, we apply the inverse Laplace transform and use the convolution theorem to achieve

$$
\begin{aligned}
\mathcal{L}^{-1}\left\{\bar{\hat{\gamma}}\right\} = \hat{\gamma} &= \int_{\lambda-i\infty}^{\lambda+i\infty} B\frac{1}{s}\left(\frac{1}{s^2+As+\Omega^4+1}\right)e^{st}ds \\
&= B\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+As+\Omega^4+1}\right\} \\
&= B\mathcal{L}^{-1}\left\{\frac{1}{s^2+As+\Omega^4+1}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} \\
&= \frac{B}{\sqrt{\Omega^4+1-\frac{A^2}{4}}}\mathcal{L}^{-1}\left\{\frac{\sqrt{\Omega^4+1-\frac{A^2}{4}}}{(s+\frac{A}{2})^2+\Omega^4+1-\frac{A^2}{4}}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} \\
&= \frac{B}{\sqrt{\Omega^4+1-\frac{A^2}{4}}}\left(e^{-A\tau/2}\sin\left((\Omega^4+1-\frac{A^2}{4})\tau\right)\right) * 1 \\
&= \frac{B}{\sqrt{\Omega^4+1-\frac{A^2}{4}}}\int_0^{\tau} e^{-A\xi/2}\sin\left(\sqrt{\Omega^4+1-\frac{A^2}{4}}\,\xi\right)d\xi \\
&= \frac{B}{\sqrt{\Omega^4+1-\frac{A^2}{4}}}\left[\frac{e^{-A\tau/2}\sin\left(\sqrt{\Omega^4+1-\frac{A^2}{4}}\,\tau\right)}{\frac{-A}{2}\left(1+\frac{4(\Omega^4+1-\frac{A^2}{4})}{A^2}\right)}\right. \\
&\quad \left. - \frac{\sqrt{\Omega^4+1-\frac{A^2}{4}}\,e^{-A\tau/2}\cos\left(\sqrt{\Omega^4+1-\frac{A^2}{4}}\,\tau\right) - \sqrt{\Omega^4+1-\frac{A^2}{4}}}{\frac{A^2}{4}\left(1+\frac{4(\Omega^4+1-\frac{A^2}{4})}{A^2}\right)}\right] \\
&= \frac{B}{\sqrt{\Omega^4+1-\frac{A^2}{4}}}\left[\frac{e^{-A\tau/2}\sin\left(\sqrt{\Omega^4+1-\frac{A^2}{4}}\,\tau\right)}{\frac{-A}{2}-\frac{2(\Omega^4+1-\frac{A^2}{4})}{A}}\right. \\
&\quad \left. - \frac{\sqrt{\Omega^4+1-\frac{A^2}{4}}\,e^{-A\tau/2}\cos\left(\sqrt{\Omega^4+1-\frac{A^2}{4}}\,\tau\right) - \sqrt{\Omega^4+1-\frac{A^2}{4}}}{\frac{A^2}{4}+(\Omega^4+1-\frac{A^2}{4})}\right] \\
&= \frac{B}{\sqrt{\Omega^4+1-\frac{A^2}{4}}}\left[\frac{e^{-A\tau/2}\sin\left(\sqrt{\Omega^4+1-\frac{A^2}{4}}\,\tau\right)}{\frac{2}{A}(\Omega^4+1)}\right. \\
&\quad \left. - \frac{\sqrt{\Omega^4+1-\frac{A^2}{4}}\,e^{-A\tau/2}\cos\left(\sqrt{\Omega^4+1-\frac{A^2}{4}}\,\tau\right)}{\Omega^4+1}\right] + \frac{B}{\Omega^4+1}
\end{aligned}
$$

5.4 Inverse Fourier Transform

We can now recover γ by applying the inverse Fourier transform

$$\mathcal{F}^{-1}\{\hat{\gamma}\} = \gamma = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{B}{\sqrt{\Omega^4+1-\frac{A^2}{4}}}\left[\frac{e^{-A\tau/2}\sin\left(\sqrt{\Omega^4+1-\frac{A^2}{4}}\,\tau\right)}{\frac{2}{A}(\Omega^4+1)}\right.\right.$$
$$\left.\left. - \frac{\sqrt{\Omega^4+1-\frac{A^2}{4}}\,e^{-A\tau/2}\cos\left(\sqrt{\Omega^4+1-\frac{A^2}{4}}\,\tau\right)}{\Omega^4+1}\right] + \frac{B}{\Omega^4+1}\right)e^{-i\Omega x}d\Omega$$
$$= \frac{B}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{\Omega^4+1-\frac{A^2}{4}}}\left[\frac{e^{-A\tau/2}\sin\left(\sqrt{\Omega^4+1-\frac{A^2}{4}}\,\tau\right)}{\frac{2}{A}(\Omega^4+1)}\right.\right.$$
$$\left.\left. - \frac{\sqrt{\Omega^4+1-\frac{A^2}{4}}\,e^{-A\tau/2}\cos\left(\sqrt{\Omega^4+1-\frac{A^2}{4}}\,\tau\right)}{\Omega^4+1}\right] + \frac{1}{\Omega^4+1}\right)e^{-i\Omega x}d\Omega$$

6 Solution with Moving Point Force

In this section, we consider the second external force model, a moving point force. We begin with equation (5)

$$\frac{\partial^4 \gamma}{\partial \chi^4} + \gamma + A\frac{\partial \gamma}{\partial \tau} + \frac{\partial^2 \gamma}{\partial \tau^2} = B\delta(\chi - V\tau)$$

We, again, aim to solve this equation using applications of both the Fourier and Laplace transforms.

6.1 Fourier Transform

Applying the Fourier transform to both sides of our equation, we get

$$\int_{-\infty}^{\infty} \left(\frac{\partial^4 \gamma}{\partial \chi^4} dt + \gamma + A\frac{\partial \gamma}{\partial \tau} + \frac{\partial^2 \gamma}{\partial \tau^2} \right) e^{i\Omega\chi} dt = \int_{-\infty}^{\infty} B\delta(\chi - V\tau) e^{i\Omega\chi} dt$$
$$\Omega^4 \hat{\gamma} + \hat{\gamma} + A\frac{\partial \hat{\gamma}}{\partial \tau} + \frac{\partial^2 \hat{\gamma}}{\partial \tau^2} = Be^{i\Omega V\tau}$$

6.2 Laplace Transform

Now applying the Laplace transform, we get

$$\int_{0}^{\infty} \left(\Omega^4 \hat{\gamma} + \hat{\gamma} + A\frac{\partial \hat{\gamma}}{\partial \tau} + \frac{\partial^2 \hat{\gamma}}{\partial \tau^2} \right) e^{-st} dt = \int_{0}^{\infty} Be^{i\Omega V\tau} e^{-st} dt$$
$$\Omega^4 \bar{\hat{\gamma}} + \bar{\hat{\gamma}} + sA\bar{\hat{\gamma}} + s^2\bar{\hat{\gamma}} = \frac{B}{s - i\Omega V}$$
$$\bar{\hat{\gamma}} = \frac{B}{s - i\Omega V} \left(\frac{1}{s^2 + As + \Omega^4 + 1} \right)$$

6.3 Inverse Laplace Transform

Applying the inverse Laplace transform and again leveraging the convolution theorem, we get

$$\begin{aligned}
\hat{\gamma} &= \int_{\lambda-i\infty}^{\lambda+i\infty} \frac{1}{s-i\Omega v t_s}\left(\frac{1}{s^2+As+\Omega^4+1}\right)e^{st}ds \\
&= \mathcal{L}^{-1}\left\{\frac{B}{s-i\Omega V}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+As+\Omega^4+1}\right\} \\
&= B\mathcal{L}^{-1}\left\{\frac{1}{s^2+As+\Omega^4+1}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s-i\Omega V}\right\} \\
&= \frac{B}{\sqrt{\Omega^4+1-\frac{A^2}{4}}}\mathcal{L}^{-1}\left\{\frac{\sqrt{\Omega^4+1-\frac{A^2}{4}}}{(s+\frac{A}{2})^2+\Omega^4+1-\frac{A^2}{4}}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s-i\Omega V}\right\} \\
&= \frac{B}{\sqrt{\Omega^4+1-\frac{A^2}{4}}}\left(e^{-A\tau/2}\sin\left(\sqrt{\Omega^4+1-\frac{A^2}{4}}\,\tau\right)\right) * e^{i\Omega V\tau} \\
&= \frac{B}{\sqrt{\Omega^4+1-\frac{A^2}{4}}}\int_0^{\tau} e^{-A\xi/2+i\Omega V(\tau-\xi)}\sin\left(\sqrt{\Omega^4+1-\frac{A^2}{4}}\,\xi\right)d\xi \\
&= \frac{B}{\sqrt{\Omega^4+1-\frac{A^2}{4}}}\left[\frac{e^{-A\tau/2}\sin\left(\sqrt{\Omega^4+1-\frac{A^2}{4}}\,\tau\right)}{(-\frac{A}{2}-i\Omega V)\left(1+\frac{(\Omega^4+1-\frac{A^2}{4})}{(-\frac{A}{2}-i\Omega V)^2}\right)}\right. \\
&\qquad \left. - \frac{\sqrt{\Omega^4+1-\frac{A^2}{4}}e^{-A\tau/2}\cos\left(\sqrt{\Omega^4+1-\frac{A^2}{4}}\,\tau\right) - \sqrt{\Omega^4+1-\frac{A^2}{4}}e^{i\Omega V\tau}}{(-\frac{A}{2}-i\Omega V)^2\left(1+\frac{(\Omega^4+1-\frac{A^2}{4})}{(-\frac{A}{2}-i\Omega V)^2}\right)}\right] \\
&= \frac{B}{\sqrt{\Omega^4+1-\frac{A^2}{4}}}\left[\frac{(-\frac{A}{2}-i\Omega V)e^{-A\tau/2}\sin\left(\sqrt{\Omega^4+1-\frac{A^2}{4}}\,\tau\right)}{(-\frac{A}{2}-i\Omega V)^2+\left(\Omega^4+1-\frac{A^2}{4}\right)}\right. \\
&\qquad \left. - \frac{\sqrt{\Omega^4+1-\frac{A^2}{4}}e^{-A\tau/2}\cos\left(\sqrt{\Omega^4+1-\frac{A^2}{4}}\,\tau\right) - \sqrt{\Omega^4+1-\frac{A^2}{4}}e^{i\Omega V\tau}}{(-\frac{A}{2}-i\Omega V)^2+\left(\Omega^4+1-\frac{A^2}{4}\right)}\right]
\end{aligned}$$

6.4 Inverse Fourier Transform

We can now recover γ by applying the inverse Fourier transform

$$\mathcal{F}^{-1}\{\hat{\gamma}\} = \gamma = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{B}{\sqrt{\Omega^4+1-\frac{A^2}{4}}}\left[\frac{(-\frac{A}{2}-i\Omega V)e^{-A\tau/2}\sin\left(\sqrt{\Omega^4+1-\frac{A^2}{4}}\,\tau\right)}{(-\frac{A}{2}-i\Omega V)^2+\left(\Omega^4+1-\frac{A^2}{4}\right)} - \frac{\sqrt{\Omega^4+1-\frac{A^2}{4}}e^{-A\tau/2}\cos\left(\sqrt{\Omega^4+1-\frac{A^2}{4}}\,\tau\right)-\sqrt{\Omega^4+1-\frac{A^2}{4}}e^{i\Omega V\tau}}{(-\frac{A}{2}-i\Omega V)^2+\left(\Omega^4+1-\frac{A^2}{4}\right)}\right]\right)e^{-i\Omega x}d\Omega$$

$$= \frac{B}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{\Omega^4+1-\frac{A^2}{4}}}\left[\frac{(-\frac{A}{2}-i\Omega V)e^{-A\tau/2}\sin\left(\sqrt{\Omega^4+1-\frac{A^2}{4}}\,\tau\right)}{(-\frac{A}{2}-i\Omega V)^2+\left(\Omega^4+1-\frac{A^2}{4}\right)} - \frac{\sqrt{\Omega^4+1-\frac{A^2}{4}}e^{-A\tau/2}\cos\left(\sqrt{\Omega^4+1-\frac{A^2}{4}}\,\tau\right)-\sqrt{\Omega^4+1-\frac{A^2}{4}}e^{i\Omega V\tau}}{(-\frac{A}{2}-i\Omega V)^2+\left(\Omega^4+1-\frac{A^2}{4}\right)}\right]\right)e^{-i\Omega x}d\Omega$$

7 Conclusion

Motivated by a relevance to the development of novel railway technologies, this report developed partial solutions for the Bernoulli-Euler beam equation, on a visco-elastic foundation, with stationary and moving point forces. Fourier, Laplace and inverse Laplace transforms were completed analytically, with the inverse Fourier transform being applied but left in integral form, paving the way for numerical techniques to be applied in further research.

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