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Random Games and Independent Bond Percolation of the Hypercube

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Contents

1 Prelude

1.1 Abstract

We look at properties of pure Nash equilibrium in finite games with random payoff functions. In particular we look at games in which each player can choose between one of two strategies and the payoffs are independent and identically distributed. We look at asymptotic results about the expected number of pure equilibrium in such a game along with the existence and accessibility of equilibria. It can be shown that these properties of games depend on the probability of having a tie. We attempt (with some difficulty) to answer the question of how long it will take to absorb into an equilibrium in a random game.

1.2 Acknowledgements

I would like to acknowledge my supervisors Andrea and Kais for supervising my project providing support and guidance for over the summer.

I would also like to thank Minh Nguyen for attending meetings with my supervisors and helping me understand some of the concepts.

Finally I would like to thank my older brother Josh for helping me when I was stuck with coding and for providing feedback on my report and talk.

1.3 Statement of Authorship

This report was written by me (Jack Mills). This project was mostly an exercise in learning high level maths from a source that is not a text book or lectures, as such the theorems and proofs provided in this report are from Amiet B., Collevecchio A., Scarsini M., and Zhong Z(2021) unless otherwise stated. Commentary, examples and the attempt to answer the question of how long it will take to absorb into an equilibrium are written by me(Jack Mills).

2 Introduction

Game Theory is the study of models of interactions between agents or players in a game. A game is a set of players, a set of strategies each player can take and a set of payoffs for each strategy. In games where players try to maximise their payoff functions, a *pure Nash equilibrium*(PNE) may arise. A PNE is a strategy profile(one strategy for each player) such that no player will benefit from changing their strategies. The idea of a probability distribution over strategy profiles is known as a mixed strategy, and a mixed equilibrium is a profile of mixed strategies that no player will deviate from. Modern game theory began with John von Neumann's study of mixed equilibria in two person zero sum games. John Nash showed that mixed equilibria always exist in games with a finite number of players and strategies.

While mixed equilibria always exist, games may not contain PNE and trying to find them is hard. In the paper that this report will focus on, Amiet B., Collevecchio A., Scarsini M., and Zhong Z take a stochastic approach to the question of the existence of PNE. "If payoffs are drawn at random what is the distribution of the number of PNE in a game. To answer this question for a fixed number of players and strategies is computationally expensive, so the authors of the paper look at the limit distribution of the number of PNE as the number of players goes to infinity.

The paper demonstrates a link between random games where players have two strategies and bond percolation(the study of graphs when edges are removed) of the hypercube. This link allows for a analytic results about the behaviour of PNE in a random game, rather than the use of simulations.

3 Background Information

The following sections will contain some information on game theory and bond percolation theory to provide some context to the paper.

3.1 Game Theory

Game theory is the study of interactions between agents or players. A game Γ is a set of players, a set of strategies for each player and a set of payoff functions for each player. Let:

$$\Gamma = ([N], (S_i)_{i \in [N]}, (g_i)_{i \in [N]})$$

Where $[N] := \{1, \ldots, N\}$ is the set of players and S_i is the set of strategies for each player $i \in [N]$. A strategy profile $\mathbf{s} = (s_1, \ldots, s_N) \in S$ is a set of strategies taken by each player. g_i maps each strategy profile to a payoff function. In all games we consider players will have two strategies to choose from. So there will be 2^n strategy profiles.

Bellow is an example of a three player game we will use this game as an example for the properties of games that the paper considers.

Player 3	Player 2 \ Player 1	0	1
Player 3 (0)	0	(-1,-1,1)	(-1,2,0)
Player 3 (0)	1	(-1,2,0)	(2,1,0)
Player 3 (1)	0	(1,1,-2)	(0,0,-1)
Player 3 (1)	1	(1,3,1)	(2,1,1)

Table 1: In this game, the set of players is $1, 2, 3$. The set of strategy profiles is each three digit long binary number. The columns and rows represent player 1 and 2's strategies respectively and the table represents player 3's strategies. So each cell corresponds to a strategy profile which is then mapped to a payoff for each player. For example $g_1(1, 0, 0) = -1, g_2(1, 0, 0) = 2, g_3(1, 0, 0) = 0$.

In the games we consider, the payoffs will be randomly generated from a set distribution.

In the games we will consider, players will have opportunities to change their strategies sequentially. When picking their strategy, the players will know what other players have chosen and then pick the strategy which maximises their payoff given the other players choices. Note that players will only be able to choose between any two strategy profiles at any given time. In the case of a draw between the two strategy profiles available, the player will choose not to change their strategy.

As described in the introduction, if no player will change their strategy to deviate from a current strategy profile, this profile is called a pure Nash equilibrium(PNE). If there are no draws then this strategy profile is a strict pure Nash equilibrium(SPNE). In our example the strategy profile (1,1,1) is a PNE because no player will deviate from this position.

In the game in table 1, there are two PNE, one of which is strict, and PNE are accessible from every strategy profile. It is also not hard to imagine games where there no PNE or games in which PNE exist but there is a set of vertices from which none of the PNE are accessible. This poses some questions about random games. What is the probability of having a PNE? How many PNE do we expect a random game to have? What is the probability of absorbing into a PNE?

3.2 Bond Percolation

A useful tool to answer the questions posed at the end of the previous section is percolation theory. Percolation theory is the study of graphs when vertices("sites") or edges("bonds") are added or removed. Bond percolation, first introduced by Broadbent and Hammersly(1957), can be linked to random games. In this section we will look at some theorems about bond percolation of the hypercube this is because later in the paper we will show a link between bond percolation of the hypercube and the behaviour of random games. To give you an idea of what a bond percolation may look like, figure 1 shows a bond percolation of a cube.

Figure 1: A percolated cube, where each edge was removed or kept independently with probability 1/2 (image of coffee percolator from Public Domain Vectors).

The question we will ask about percolation of a large hypercube is: If edges of a hypercube are removed independently at random, what is the probability that the hypercube remains connected? i.e. there is a path between any two vertices on the random sub-graph generated by the percolation. The answer to this question depends on the probability that an edge is removed. Let p be the probability that an edge is removed in a bond percolation.

Let $\mathcal{Q}_n$, denote the n-hypercube. Let $V(\mathcal{Q}_n) = (\epsilon_1, ...\epsilon_n), \epsilon_i = 0, 1$ be the vertex set of the hypercube. An edge exists between two vertices if and only if the vertices differ by exactly one co-ordinate.

Let $\mathcal{G}_{n,p}$ be the random sub graph of $\mathcal{Q}_n$ defined by letting $\mathbf{P}(\{i,j\} \in E(\mathcal{G}_{n,p})) = p$ for all $\{i,j\} \in E(\mathcal{Q}_n)$ and letting all these probabilities be mutually independent and let:

$$f_n(p) = \mathbf{P}(\mathcal{G}_{n,p} \text{is connected})$$

We want to understand what happens to $f_n(p)$ as n approaches infinity.

The following theorem is from Erdős and Spencer's(1979) paper Evolution of the n-Cube.

$$\lim_{n\to\infty} f_n(p) = \left\{ \begin{array}{ll} 0, & \text{if } p < 0.5 \\ e^{-1}, & \text{if } p = 0.5 \\ 1, & \text{if } p > 0.5 \end{array} \right\} \tag{1}$$

Proofs are for all three parts are shown in Evolution of the n-Cube. The following is a proof for the first part of the equation.

Let $h_n(p)$ be the probability that $\mathcal{G}_{n,p}$ contains vertices with no incident edges. Clearly $h_n(p) \leq 1 - f_n(p)$ because if $\mathcal{G}_{n,p}$ it will always be disconnected, however if $\mathcal{G}_{n,p}$ is disconnected, all vertices may still have at least one incident edge.

For each $i \in V(\mathcal{G}_{n,p})$ we define:

$$\mathbf{X}_i = \begin{array}{ll} 0, & \text{if i has at least one incident edge} \\ 1, & \text{if not} \end{array} \tag{2}$$

Set:

$$\mathbf{X} = \sum i \in V(\mathcal{G}_{n,p}) X_i$$

as the number of vertices with no incident edges.

Since each vertex in $\mathcal{G}_{n,p}$ has degree n and each edge is removed with probability 1-p

$$\mathbf{E}(\mathbf{X}_i) = (1-p)^n$$

Through linearity of expectation we get

$$\mathbf{E}(\mathbf{X}) = 2^n (1-p)^n$$

.

Let $\mu = \mathbf{E}(\mathbf{X})$. By using the formula:

$$\text{Var}(\mathbf{X}) = \sum_i \text{Var}(\mathbf{X}_i) + \sum_{i \neq j} \text{Cov}(\mathbf{X}_i, \mathbf{X}_j)$$

we can find the variance of $\mathbf{X}$.

We get:

$$\text{Var}(\mathbf{X}) = \mu + \mu(1-p)^n [np/(1-p)) - 1)]$$

.

By applying Kolmogorov's inequality we get:

$$1 - h_n(p) = \mathbf{P}(X = 0) \leq \text{Var}(\mathbf{X})/\mu^2$$

From our calculation of the variance we use

$$\lim_{n\to\infty} \text{Var}(\mathbf{X})/\mu^2 = 0$$

.

Since $h_n(p) \leq 1 - f_n(p)$

$$\lim_{n\to\infty} f_n(p) \leq \lim_{n\to\infty} 1 - h_n(p) \leq 0$$

and since $f_n(p)$ is a probability

$$f_n(p) = 0$$

.

Erdős and Spencer also provide proofs for the other two parts of equation 1. McDiarmid, Scott and Withers(2018)'s work on the structure of random subgraphs of the hypercube is also used in answering questions about random games, however their work has mostly been omitted from the report due to space constraints.

4 Properties of Random Games

4.1 Linking Random Games and Percolation Theory

To be able to apply the theory of bond percolation to random games. We must first establish a link between the two. First we will start by representing games as graphs. To do this we can assign strategy profile a vertex and put an edge between two vertices if they differ by exactly one strategy. This way the edges represent possible changes in strategy from strategy profile. We then proceed by directing the edges in graph in the following way. If two vertices u and v differ at point i, then direct the edge from u to v if $g_i(u) > g_i(u)$ and direct the edge from v to u if $g_i(u) > g_i(u)$. Notice while this representation of a game keeps all the information about PNE, it looses information about the magnitude of the difference between two payoffs. For us this isn't a worry since there are only two strategy's for any player to choose from at any given time.

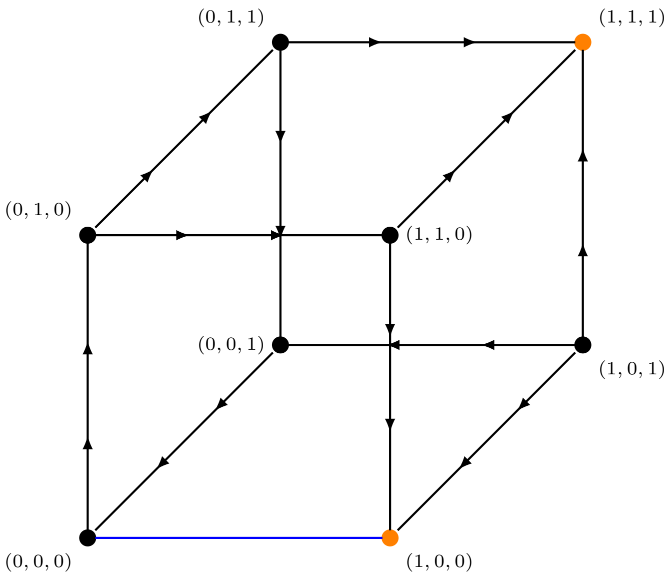


Figure 2: A directed graph of the game in table 1

The graph that represents a game where each player has two strategies to choose from will be a hypercube and the geometry of this graph will depend only on the probability of a draw between two payoffs. Let α be the probability of a draw. Instead of considering games with random payoffs, we can consider randomly directed cubes that represent the games.

Let

$$\beta := (1 - \alpha)/2$$

. Here β is the probability that one payoff is greater that another.

Theorem 4.1 from Amiet B., Collevecchio A., Scarsini M., and Zhong Z.(2021) links partially directed hypercubes and bond percolations. The theorem states: for any $\beta \in [0, 1/2]$ there exists an independent bond percolation $\mathcal{B}^\beta$ with the following property: the set of vertices of that contains the vertex $\mathbf{0}$ is equal in distribution to the set of vertices accessible from $\mathbf{0}$ in a random game.

The proof for the theorem is quite long so it is not included here but here is a sketch of it. The proof is a constructive proof which provides a way to construct the bond percolation $\mathcal{B}^{\beta}$ with an appropriate choice of the percolation parameter p.

Begin by starting at vertex $\mathbf{0}$ and setting $p := (1-\alpha)/2 = \beta$. The probability that a single edge incident to $\mathbf{0}$ is in the bond percolation is equal to the probability that the other vertex incident to that edge is accessible in a random game from $\mathbf{0}$ and each edge is independent in both the bond percolation and the random game. So the set of vertices adjacent to $\mathbf{0}$ in $\mathcal{B}^{\beta}$ is the same as the set of vertices accessible from $\mathbf{0}$ in one step of a the random game. The proof then continues recursively on all the vertices now in the bond percolation.

4.2 Number of PNE

Now that bond percolation theory and random games are linked, we can begin to answer some of the questions about random games that we had. Namely, what is the probability of a PNE or SPNE existing and how many PNE do we expect.

To find the probability of a PNE existing we can proceed as follows:

Let Z be the random variable which $g_i(s)$ is a realisation of.

As before define

$$\alpha := P(Z_1 = Z_2),$$
$$\beta := P(Z_1 < Z_2) = \frac{1-\alpha}{2}$$

Note that when $\alpha = 0$, a PNE is also always a SPNE since the probability of a draw is 0.

Since the probability that a strategy profile $\mathbf{s}$ is a PNE is $(1-\beta)^N$ and the probability that it is an SPNE is β^N and there are 2^N strategies, we can use linearity of expectation to find the expected number of PNE and SPNE in a random game.

$$E[\text{number of PNE}] = 2^N(1-\beta)^N,$$
$$E[\text{number of SPNE}] = 2^N(\beta)^N$$

.

The paper also present the following theorem. It can be interpreted as follows. In the case where there are no draws, the number of SPNE follows a Poisson distribution. In the case where draws are possible the number of SPNE is almost surely 0 for large enough games.

(a) if $\alpha = 0$, then

$$\lim_{N\to\infty} P(\text{card}(\text{SPNE}(\Xi_N) = k)) = \frac{e^{-1}}{k!}$$

(b) and if $\alpha > 0$, then

$$P(\lim_{N\to\infty} \text{card}(\text{SPNE}(\Xi_N) = 0) = 1$$

Part, a was proved in a previous paper. A proof to theorem 2.2,b is presented in the paper it uses the fact that $E[card(\text{SPNE}(\Xi_N)] = 2^N(\beta)^N$ and Markov's inequality to show that $\sum_{N=1}^{\infty} P(\text{card}(\text{SPNE}(\Xi_N) \geq 0) \leq \infty$. The theorem follows from the Borel-Cantelli Lemma.

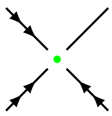
Notice here that when $\alpha > 0$ the behavior of PNE and SPNE is radically different.

4.3 Accessibility of PNE

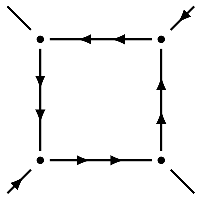
Another questions about the behavior of random games is what is the probability of absorbing into a PNE from a given strategy?

To answer this question we must make the key observation that a game will not reach a PNE if and only if the game is stuck in a "trap". A trap is a set of vertices whose induced sub-graph form a cycle, and every other edge incident to those vertices is not pointing away from the cycle.

Bellow is what a trap of size 4 looks like.



a PNE



a trap of size 4

With this knowledge, the only way for a PNE not to be accessible is for the game to either start in a trap, or for all arcs out of the starting vertices to be leading to traps. The way the game behaves based on α is as follows:

- (a) If $0 \leq \alpha < 1/2$, then in a sequence of games increasing in size there exists a finite random number N^* of players such that in each game with N^* or more players all PNE are accessible.
- (b) If $\alpha = 1/2$, there exist PNE that are not accessible with positive probability.
- (c) If $\alpha > 1/2$, as the number of players increases, the number of inaccessible PNE approaches ∞ with probability approaching 1.

While PNE may be accessible from a given vertex, there is also a chance of reaching a trap before reaching a PNE. If $0 < \alpha < 2^{3/4} - 1$ then, then in a sequence of games increasing in size there exists a finite random number N^* of players such that in each game with N^* or more players, the game will converge to a PNE.

5 Further Questions and Remarks

Some further questions asked about the behavior of random games are:

- How long does it take for a game to reach a PNE?
- What is the geometry of PNE when more strategies are available.

I attempted to answer both these questions very unsuccessfully.

It is easy to answer the first question for single instances of random games, a system of equations can be set up for the probability of absorption for each vertex and this system can be solved. However when a trap exists in the game, there may be a non 0 chance for the game to never absorb into a PNE. This means that the expected time to absorb into a PNE will be ∞. To get around this I tried to find the expected time to absorb into a PNE given the game absorbs into one.

For small games (one or two players) we can just look at every possible game and find the expected time to absorb. For example, for a one player game, the probability of starting in a PNE is clearly $1-\beta$ so the expected number of steps to absorb into a PNE is just β.

For larger games the number of possible games gets very big very fast, so this strategy isn't very good. I also tried to model the problem with simulations, the only problem is that this was the first time in my life writing a program.

To answer the second question we must consider more than just the direction of the inequality between two random games. For example in games where players have three available strategies and both strategies that a player can move to better than their current strategy where do they go? They can either pick the best of their two available options or they can pick one of the two strategies at random. Amiet, Collevecchio and Hamza (2019) discuss this in the paper "When better is better than best".

Once we decide on a process for picking a strategy we must also consider the shape of the graph the corresponds to the game. In games where players may pick from two strategies, the graph corresponding to the game is a hypercube. Luckily for us bond percolation of hypercube has been well studied. A hypercube is also the repeated Cartesian product of the complete graph on two vertices. For games with n strategies for each player, all of which are available from each other strategy, we get a graph which is the repeated Cartesian product of the complete graph on n vertices. Erde et. al. (2022) recently published work on the percolation of higher dimensional product graphs. This may be useful in understanding the geometry of games with more strategies available.

6 Conclusion

Random games can be linked to bond percolation. Large random games have many PNE when the probability of ties is greater than 0. When the probability of a draw is small enough, random games will converge to a PNE. Otherwise they may not. Despite random games having a large amount of PNE when the probability of a draw is greater than 0, many of the PNE will not be accessible from any given vertex when the probability of a draw is greater than 1/2. When the chance of a draw is exactly 1/2 there may be some PNE which are inaccessible, this is an interesting transition point.

Questions about how long it takes to absorb into a PNE and what games look like when more than two strategies are available remain open and are interesting.

7 References

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Amiet B., Collevecchio A., Scarsini M., and Zhong Z. (2021) *Pure nash equilibria and best- response dynamics in random games.* Mathematics of Operations Research, 46(4):1552– 1572, 2021. 8

Amiet, B., Collevecchio, A., Hamza, K. (2021). *When "Better" is better than "Best".* Operations Research Letters, 49(2), 260-264. https://doi.org/10.1016/j.orl.2021.01.009

Broadbent, S. R. and Hammersley, J. M. (1957). *Percolation processes. I. Crystals and mazes.* Proc. Cambridge Philos. Soc., 53:629–641.

Erde, J., Kang, M., Krivelevich, M., and Diskin, S. (2022). *Percolation on High-dimensional Product Graphs.* https://arxiv.org/abs/2209.03722

Erdős, P. and Spencer, J. (1979). *Evolution of the n-cube.* Comput. Math. Appl., 5(1):33–39.

McDiarmid, C., Scott, A., and Withers, P. (2020). *The component structure of dense random subgraphs of the hypercube.* Technical report, arXiv:1806.06433.

8 Apendix