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Circles in 3-dimensional Lie Groups

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1 Prelude

1.1 Abstract

This project concerns the curves defined by one-dimensional subgroups of three-dimensional unimodular Lie groups, equipped with a left Invariant metric. It is well known when these curves are geodesics. This project studies the case where the curves are circles; i.e., they have nonzero (constant) curvature and zero torsion. We find that such circles exist precisely when the Ricci curvature has a particular signature.

1.2 Acknowledgment

I would like to express my special thanks of gratitude to my supervisors Dr Yuri Nikolayevsky and Dr Grant Cairns for spending their valuable time to guide and direct me throughout this research project. I am lucky enough to have them as supervisors.

My thanks and appreciations also go to AMSI Vacation Research teams in creating this meaningful program that allowed me to experience a life of a researcher.

I am also thankful to my family that always encourage and support me to fulfill this project.

1.3 Statement of Authorship

The main idea of this project was created by Dr Yuri Nikolayevsky regarding the theorems and the findings from Dr John Milnor (1976). Calculations and interpretation of the findings were done and checked by Dr Yuri Nikolayevsky, Dr Grant Cairns and Mr Soprom Meng. We have sourced the theorems and the formulas from various sources, namely textbook and online. AMSI funded the whole research project.

2 Introduction

Basically, the classical differential geometry of a curve is the study of local property of the curve to determine the characteristic related to their neighbourhood. The Frenet formulas is one of the most essential equations that use the curvature (κ) and the torsion (τ) to show the relationship between the vector fields $\{T, N, B\}$ and their derivatives.

Particularly, when the curvature (κ) is non-zero constant and the torsion (τ) is zero, the curve is a (portion of a) circle (Nikolayevsky 2018). In this project, we will use the covariant derivative ∇_X by the Koszul formula and the Frenet formulas for a left-invariant curve in a 3-dimensional metric Lie group to classify left-invariant circles which are not geodesics and prove the following result.

Theorem 1. *A three dimensional, unimodular, metric Lie group admits left-invariant circles which are not geodesics if and only if the signature of its Ricci form is either* $(+,-,-)$ *or* $(0,0,-)$.

3 Preliminaries

3.1 Lie Group and Lie Algebra

A Lie group is a set $\mathfrak{g}$ with two structures: $\mathfrak{g}$ is a group and $\mathfrak{g}$ is a smooth and real manifold. These structures agree in the following sense: multiplication and inversion are smooth map (Kirillov 2008).

A Lie algebra over a field of the real numbers P is a vector space $\mathfrak{g}$ with a bi-linear operation

$$[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

The operation $[.,.]$ is called the Lie bracket for $\mathfrak{g}$ and satisfies the following properties:

- $[.,.]$ is bi-linear.
  $[aX + bY, Z] = a[X, Z] + b[Y, Z]$
  $[Z, aX + bY] = a[Z, X] + b[Z, Y]$, for all elements $X, Y, Z \in \mathfrak{g}$ and for all $a, b \in P$.

- Skew-symmetry
  $[X, Y] = -[Y, X]$ for all $X, Y \in \mathfrak{g}$.

- Jacobi identity
  $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$ for all $X,Y,Z\in\mathfrak{g}$ (Bordag 2015).

A Lie algebra $\mathfrak{g}$ is called Abelian if $[X,Y]=0$,for all $X,Y\in\mathfrak{g}$ (Bordag 2015).

Let $(G,\langle,\rangle)$ be a Lie group with a left-invariant metric. Then the diffeomorphisms $L_g : G\to G$ are isometries (Tumarkin n.d.).

The covariant derivative ∇_X is defined by the Koszul formula:

$$\langle\nabla_X Y,Z\rangle=\frac{1}{2}(\langle[X,Y],Z\rangle-\langle[Y,Z],X\rangle+\langle[Z,X],Y\rangle). \tag{1}$$

for all $X,Y,Z\in\mathfrak{g}$ (Milnor 1976).

3.2 Three-dimensional Unimodular Lie groups

For every element X of a Lie algebra $\mathfrak{g}$, we define the operator ad_X on $\mathfrak{g}$ by $ad_XY=[X,Y]$. A Lie algebra $\mathfrak{g}$ is called unimodular if $tr(ad_X)=0$, for all $X\in\mathfrak{g}$ (Milnor 1976).

Let $\mathfrak{g}$ be a 3-dimensional, metric, unimodular Lie algebra. Then there is an orthonormal basis $\{e_1,e_2,e_3\}$ for $\mathfrak{g}$ such that

$$[e_1,e_2]=\lambda_3e_3,\qquad [e_3,e_1]=\lambda_2e_2,\qquad [e_2,e_3]=\lambda_1e_1,$$

where $\lambda_1,\lambda_2,\lambda_3\in\mathbb{R}$ (Milnor 1976, Lemma 4.1).
It is convenient to define numbers μ_1,μ_2,μ_3 by the formula (Milnor 1976) :

$$\mu_i=\frac{1}{2}(\lambda_1+\lambda_2+\lambda_3)-\lambda_1.$$

By a theorem of Milnor, the orthonormal basis e_1,e_2,e_3 diagonalized the Ricci quadratic form, the principle Ricci curvatures being given by:

$$r(e_1)=2\mu_2\mu_3,\quad r(e_2)=2\mu_1\mu_3,\quad r(e_3)=2\mu_1\mu_2.$$

A three dimensional, unimodular, metric Lie group admits left-invariant circles which are not geodesics if and only if the signature of its Ricci form is either $(+,-,-)$ or $(0,0,-)$ (Milnor 1976).

Signs of λ_1 , λ_2 , λ_3	Associated Lie group	Description
$+, +, +$	$SU(2)$ or $SO(3)$	compact, simple
$+, +, -$	$SL(2, \mathbf{R})$ or $O(1, 2)$	noncompact, simple
$+, +, 0$	$E(2)$	solvable
$+, -, 0$	$E(1, 1)$	solvable
$+, 0, 0$	Heisenberg group	nilpotent
$0, 0, 0$	$\mathbf{R} \oplus \mathbf{R} \oplus \mathbf{R}$	commutative

Figure 1: The signs of λ_i's and Associated Lie group, (Milnor 1976)

3.3 Frenet formulas for Left-invariant curves in 3-dim Lie groups

For each point on a curve of 3 dimensional space, the Frenet frame is formed by three orthogonal unit vectors $\{T, N, B\}$. For a regular curve $\gamma(t)$ in $\mathbb{R}^3$. Then for a smooth parametrisation, we get

$$T = \frac{\gamma'}{||\gamma'||}, \quad N = \frac{(\gamma' \times \gamma'') \times \gamma'}{||\gamma' \times \gamma''||||\gamma'||}, \quad B = \frac{\gamma' \times \gamma''}{||\gamma' \times \gamma''||}$$
$$\kappa = \frac{||\gamma' \times \gamma''||}{||\gamma'||^3}, \quad \tau = \frac{det(\gamma', \gamma'', \gamma''')}{||\gamma' \times \gamma''||^2}$$

where

- T is the unit tangent vector.
- N is the unit normal vector.
- B is the principle unit binormal vector.
- κ is the curvature.
- τ is the torsion.

Moreover, the Frenet frame satisfies the following Frenet formulas:

$$T' = kN.$$

$$N' = -kT + \tau B.$$

$$B' = -\tau N.$$

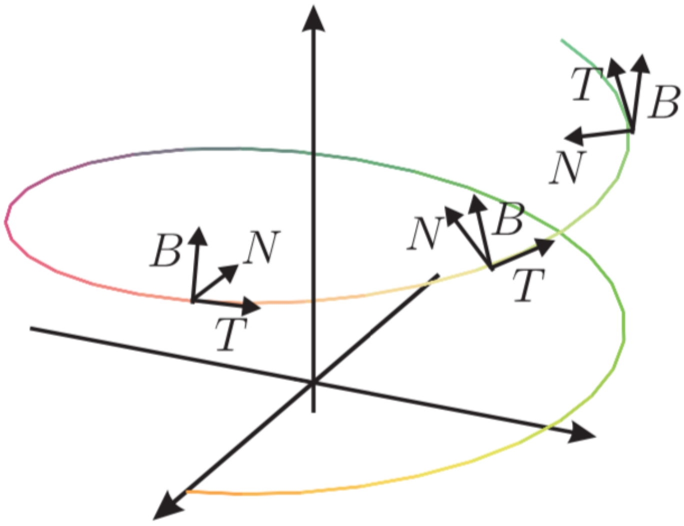

Figure 2: Frenet frame of a helix, (Nikolayevsky 2018)

- If $\kappa = 0$, the curve is a (portion of a) straight line (a geodesic).
- If $\kappa \neq 0$, but $\tau = 0$, the curve is a (portion of a) circle.

Frenet formulas for a left-invariant curve in a 3-dimensional metric Lie group can be translated to the level of Lie algebra as follows:

$$\nabla_X X = kN. \tag{2}$$

$$\nabla_X N = -kT + \tau B.$$

$$\nabla_X B = -\tau N.$$

where $X \neq 0$, $X \in \mathfrak{g}$ is the tangent vector to the curve, κ is the curvature and τ is the torsion (defined when $\kappa \neq 0$).

From (2), we get:

$$\begin{aligned}\nabla_X \nabla_X X &= \nabla_X kN \\ &= k\nabla_X N \\ &= k(-kX + \tau B) \\ &= -k^2 X + k\tau B.\end{aligned}$$

We need to calculate the cross product between $\nabla_X \nabla_X X$ and X to find τ,

We have

$$\begin{aligned}(\nabla_X \nabla_X X) \times X &= (-k^2 X + k\tau B) \times X \\ &= k\tau (B \times X) \\ &= \tau k N \\ &= \tau \nabla_X X.\end{aligned}$$

4 The Proof of the Theorem

Let $\mathfrak{g}$ be a 3-dimensional, metric, unimodular Lie algebra. Then there is an orthonormal basis $\{e_1, e_2, e_3\}$ for $\mathfrak{g}$ such that

$$[e_1, e_2] = \lambda_3 e_3, \qquad [e_3, e_1] = \lambda_2 e_2, \qquad [e_2, e_3] = \lambda_1 e_1,$$

where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ (Milnor 1976, Lemma 4.1).

To prove the theorem, we have to calculate the Torsion τ by finding $(\nabla_X \nabla_X X) \times X$ and $\nabla_X X$.

First, we need to find $\nabla_X X$.

From Koszul formular (1), we get for all $X, Z \in \mathfrak{g}$

$$\begin{aligned}\langle \nabla_X X, Z \rangle &= \frac{1}{2}(\langle [X, X], Z \rangle - \langle [X, Z], X \rangle] + \langle [Z, X], X \rangle) \\ &= -\langle [X, Z], X \rangle.\end{aligned}$$

We have

$$\begin{aligned}[X, Z] &= (x_1 z_2 - x_2 z_1)[e_1, e_2] + (x_1 z_3 - x_3 z_1)[e_1, e_3] + (x_2 z_3 - x_3 z_2)[e_2, e_3] \\ &= (x_1 z_2 - x_2 z_1)\lambda_3 e_3 - (x_1 z_3 - x_3 z_1)\lambda_2 e_2 + (x_2 z_3 - x_3 z_2)\lambda_1 e_1,\end{aligned}$$

where $\quad X = x_1e_1 + x_2e_2 + x_3e_3$ and $Z = z_1e_1 + z_2e_2 + z_3e_3$.

We obtain

$$\begin{aligned}\langle \nabla_X X, Z\rangle &= -x_1\lambda_1(x_2z_3 - x_3z_2) + x_2\lambda_2(x_1z_3 - x_3z_1) - x_3\lambda_3(x_1z_2 - x_2z_1)\\ &= -x_1x_2\lambda_1z_3 + x_1x_3\lambda_1z_2 + x_1x_2\lambda_2z_3 - x_2x_3\lambda_2z_1 - x_1x_3\lambda_3z_2 + x_2x_3\lambda_1z_1\\ &= x_2x_3(\lambda_3 - \lambda_2)z_1 + x_1x_3(\lambda_1 - \lambda_3)z_2 + x_1x_2(\lambda_2 - \lambda_1)z_3.\end{aligned}$$

Therefore

$$\nabla_X X = x_2x_3(\lambda_3 - \lambda_2)e_1 + x_1x_3(\lambda_1 - \lambda_3)e_2 + x_1x_2(\lambda_2 - \lambda_1)e_3 = \kappa N. \tag{3}$$

where κ is the curvature and N is the unit principle normal vector.

Denote $\tilde{N} = \kappa N$.

From equation (1), we have

$$\langle \nabla_X \tilde{N}, Z\rangle = \frac{1}{2}(\langle [X, \tilde{N}], Z\rangle - \langle [\tilde{N}, Z], X\rangle] + \langle [Z, X], \tilde{N}\rangle).$$

Since

$$\begin{aligned}[X, \tilde{N}] &= [x_1^2x_3(\lambda_1 - \lambda_3) - x_2^2x_3(\lambda_3 - \lambda_2)][e_1, e_2] + [x_1^2x_2(\lambda_2 - \lambda_1) - x_3^2x_2(\lambda_3 - \lambda_2)][e_1, e_3]\\ &\quad + [x_2^2x_1(\lambda_2 - \lambda_1) - x_3^2x_1(\lambda_1 - \lambda_3)][e_2, e_3]\\ &= [x_1^2x_3(\lambda_1 - \lambda_3) - x_2^2x_3(\lambda_3 - \lambda_2)]\lambda_3e_3 + [x_1^2x_2(\lambda_2 - \lambda_1) - x_3^2x_2(\lambda_3 - \lambda_2)]\lambda_2e_2\\ &\quad + [x_2^2x_1(\lambda_2 - \lambda_1) - x_3^2x_1(\lambda_1 - \lambda_3)]\lambda_1e_1.\end{aligned}$$

This implies that

$$\begin{aligned}\langle [X, \tilde{N}], Z\rangle &= [x_1^2x_3(\lambda_1 - \lambda_3) - x_2^2x_3(\lambda_3 - \lambda_2)]\lambda_3z_3 + [x_1^2x_2(\lambda_2 - \lambda_1) - x_3^2x_2(\lambda_3 - \lambda_2)]\lambda_2z_2\\ &\quad + [x_2^2x_1(\lambda_2 - \lambda_1) - x_3^2x_1(\lambda_1 - \lambda_3)]\lambda_1z_1.\end{aligned}$$

Moreover

$$\begin{aligned}[\tilde{N}, Z] &= [z_2x_2x_3(\lambda_3 - \lambda_2) - z_1x_1x_3(\lambda_1 - \lambda_3)]\lambda_3e_3 - [z_3x_2x_3(\lambda_3 - \lambda_2) - z_1x_1x_2(\lambda_2 - \lambda_1)]\lambda_2e_2\\ &\quad + [z_3x_1x_3(\lambda_1 - \lambda_3) - z_2x_1x_2(\lambda_2 - \lambda_1)]\lambda_1e_1.\end{aligned}$$

So

$$\begin{aligned}\langle [\tilde{N}, Z], X\rangle &= [z_2x_2x_3^2(\lambda_3 - \lambda_2) - z_1x_1x_3^2(\lambda_1 - \lambda_3)]\lambda_3 - [z_3x_2^2x_3(\lambda_3 - \lambda_2) - z_1x_1x_2^2(\lambda_2 - \lambda_1)]\lambda_2\\ &\quad + [z_3x_1^2x_3(\lambda_1 - \lambda_3) - z_2x_1^2x_2(\lambda_2 - \lambda_1)]\lambda_1.\end{aligned}$$

And

$$
\begin{aligned}
\langle [Z,X],\tilde{N}\rangle &= (z_1x_2 - z_2x_1)\lambda_3x_1x_2(\lambda_2-\lambda_1) - (z_1x_3 - z_3x_1)\lambda_2x_1x_3(\lambda_1-\lambda_3) \\
&\quad + (z_2x_3 - z_3x_2)\lambda_1x_2x_3(\lambda_3-\lambda_2). \\
&= z_2x_2^2x_1(\lambda_2-\lambda_1)\lambda_3 - z_2x_1^2x_2(\lambda_2-\lambda_1)\lambda_3 - z_1x_1x_3^2(\lambda_1-\lambda_3)\lambda_2 + z_3x_1^2x_3(\lambda_1-\lambda_3)\lambda_2 \\
&\quad + z_2x_2x_3^2(\lambda_3-\lambda_2)\lambda_1 - z_3x_2^2x_3(\lambda_3-\lambda_2)\lambda_1.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\langle \nabla_X\tilde{N}, Z\rangle = \frac{1}{2}[&z_1(x_1x_2^2(\lambda_2-\lambda_1)\lambda_1 - x_1x_3^2(\lambda_1-\lambda_3)\lambda_1 + x_1x_3^2(\lambda_1-\lambda_3)\lambda_3 \\
&- x_1x_2^2(\lambda_2-\lambda_1)\lambda_2 + x_2^2x_1(\lambda_2-\lambda_1)\lambda 3 - x_1x_3^2(\lambda_1-\lambda_3)\lambda_2) \\
&+ z_2(-x_1^2x_2(\lambda_2-\lambda_1)\lambda_2 + x_2x_3^2(\lambda_3-\lambda_2)\lambda_2 - x_2x_3^2(\lambda_3-\lambda_2)\lambda_3 \\
&+ x_1^2x_2(\lambda_2-\lambda_1)\lambda_1 - x_1^2x_2(\lambda_2-\lambda_1)\lambda 3 + x_2x_3^2(\lambda_3-\lambda_2)\lambda_1) \\
&+ z_3(x_1^2x_3(\lambda_1-\lambda_3)\lambda_3 - x_2^2x_3(\lambda_3-\lambda_2)\lambda_3 + x_2^2x_3(\lambda_3-\lambda_2)\lambda_2 \\
&- x_1^2x_3(\lambda_1-\lambda_3)\lambda_1 + x_1^2x_3(\lambda_1-\lambda_3)\lambda 2 - x_2^2x_3(\lambda_3-\lambda_2)\lambda_1)].
\end{aligned}
$$

This result implies

$$
\begin{aligned}
\nabla_X\tilde{N} = \frac{1}{2}[&(x_1x_2^2((\lambda_2-\lambda_1)\lambda_1 - (\lambda_2-\lambda_1)\lambda_2 + (\lambda_2-\lambda_1)\lambda_3) \\
&+ (x_1x_3^2(-(\lambda_1-\lambda_3)\lambda_1 + (\lambda_1-\lambda_3)\lambda_3 - (\lambda_1-\lambda_3)\lambda_2)))e_1 \\
&+ (x_1^2x_2(-(\lambda_2-\lambda_1)\lambda_2 + (\lambda_2-\lambda_1)\lambda_1 - (\lambda_2-\lambda_1)\lambda_3) \\
&+ (x_2x_3^2((\lambda_3-\lambda_2)\lambda_2 - (\lambda_3-\lambda_2)\lambda_3 + (\lambda_3-\lambda_2)\lambda_1)))e_2 \\
&+ (x_1^2x_3((\lambda_1-\lambda_3)\lambda_3 - (\lambda_1-\lambda_3)\lambda_1(\lambda_1-\lambda_3)\lambda_2) \\
&+ (x_2^2x_3(-(\lambda_3-\lambda_2)\lambda_3 + (\lambda_3-\lambda_2)\lambda_2 - (\lambda_3-\lambda_2)\lambda_1)))e_3].
\end{aligned}
$$

Therefore, the i-th component of $\nabla_X(\kappa N)$ equals

$$
\frac{1}{2}x_i^2(x_j^2(\lambda_j-\lambda_i)(\lambda_i-\lambda_j+\lambda_k) + x_k^2(\lambda_k-\lambda_i)(\lambda_i-\lambda_k+\lambda_j)),
$$

where $\{i, j, k\} = \{1, 2, 3\}$.

The first component of $(\nabla_X \nabla_X X \times X)$ equals

$$\begin{aligned}
&\frac{x_2x_3}{2}[x_1^2(\lambda_2-\lambda_1)(\lambda_1-\lambda_2-\lambda_3)+x_3^2(\lambda_3-\lambda_2)(\lambda_1+\lambda_2-\lambda_3)] \\
&-\frac{x_2x_3}{2}[x_1^2(\lambda_1-\lambda_3)(-\lambda_1+\lambda_2+\lambda_3)+x_2^2(\lambda_3-\lambda_2)(-\lambda_1+\lambda_2-\lambda_3)] \\
=&\frac{x_2x_3}{2}[x_1^2(\lambda_1-\lambda_2-\lambda_3)(\lambda_2-\lambda_1+\lambda_1-\lambda_3)+x_3^2(\lambda_3-\lambda_2)(\lambda_1+\lambda_2-\lambda_3) \\
&-x_2^2(\lambda_3-\lambda_2)(-\lambda_1+\lambda_2-\lambda_3)] \\
=&\frac{x_2x_3(\lambda_2-\lambda_3)}{2}[x_1^2(\lambda_1-\lambda_2-\lambda_3)+x_3^2(-\lambda_1-\lambda_2+\lambda_3)+x_2^2(-\lambda_1+\lambda_2-\lambda_3)].
\end{aligned}$$

The second component of $(\nabla_X \nabla_X X \times X)$ equals

$$\begin{aligned}
&\frac{x_1x_3}{2}[x_1^2(\lambda_1-\lambda_3)(-\lambda_1+\lambda_2+\lambda_3)+x_2^2(\lambda_3-\lambda_2)(-\lambda_1+\lambda_2-\lambda_3)] \\
&-\frac{x_1x_3}{2}[x_2^2(\lambda_2-\lambda_1)(\lambda_1-\lambda_2+\lambda_3)+x_3^2(\lambda_1-\lambda_3)(-\lambda_1-\lambda_2+\lambda_3)] \\
=&\frac{x_1x_3}{2}[x_2^2(\lambda_1-\lambda_2+\lambda_3)(-\lambda_3+\lambda_2-\lambda_2+\lambda_1)+x_1^2(\lambda_1-\lambda_3)(-\lambda_1+\lambda_2+\lambda_3) \\
&-x_3^2(\lambda_1-\lambda_3)(-\lambda_1-\lambda_2+\lambda_3)] \\
=&\frac{x_2x_3(\lambda_1-\lambda_3)}{2}[x_2^2(\lambda_1-\lambda_2+\lambda_3)+x_1^2(-\lambda_1+\lambda_2+\lambda_3)+x_3^2(\lambda_1+\lambda_2-\lambda_3)].
\end{aligned}$$

The third component of $(\nabla_x \nabla_X X \times X)$ equals

$$\begin{aligned}
&\frac{x_1x_2}{2}[x_2^2(\lambda_2-\lambda_1)(\lambda_1-\lambda_2+\lambda_3)+x_3^2(\lambda_1-\lambda_3)(-\lambda_1-\lambda_2=\lambda_3)] \\
&-\frac{x_1x_2}{2}[x_1^2(\lambda_2-\lambda_1)(\lambda_1-\lambda_2-\lambda_3)+x_3^2(\lambda_3-\lambda_2)(\lambda_1+\lambda_2-\lambda_3)] \\
=&\frac{x_1x_2}{2}[x_3^2(\lambda_1+\lambda_2-\lambda_3)(\lambda_3-\lambda_1-\lambda_3+\lambda_2)+x_2^2(\lambda_2-\lambda_1)(\lambda_1-\lambda_2+\lambda_3) \\
&-x_1^2(\lambda_2-\lambda_1)(\lambda_1-\lambda_2-\lambda_3)] \\
=&\frac{x_1x_2(\lambda_2-\lambda_1)}{2}[x_3^2(\lambda_1+\lambda_2-\lambda_3)+x_2^2(\lambda_1-\lambda_2+\lambda_3)+x_1^2(-\lambda_1+\lambda_2+\lambda_3)].
\end{aligned}$$

It follows that $(\nabla_X \nabla_X X) \times X = \tau \nabla_X X$, where

$$\tau = f(X) = \frac{1}{2}[x_1^2(-\lambda_1+\lambda_2+\lambda_3)+x_2^2(\lambda_1-\lambda_2+\lambda_3)+x_3^2(\lambda_1+\lambda_2-\lambda_3)] \tag{4}$$

To complete the proof, we need to find those $X \in \mathfrak{g}$ for which $f(X) = 0$ and $\kappa N \neq 0$. This is because if $\kappa N = 0$, it leads to $\kappa = 0$ which is a straight line. Here $f(X)$ is given in (4) and κN, in (3). We consider several possible cases.

- Case 1: $\lambda_1 = \lambda_2 = \lambda_3 = \alpha$

  This implies that $\kappa N = 0$

  Thus, there is no solution. We note that in this case, $\mu_1 = \mu_2 = \mu_3 = \frac{1}{2}\alpha$, and so all three principle Ricci curvatures are equal to $\frac{1}{4}\alpha^2$. Hence, the Ricci sign signature is either $(0, 0, 0)$ or $(+, +, +)$.

- Case 2: Suppose of the λ_i are equal, and the third one is different.

  Without loss of generality, let $\lambda_1 = \lambda_2 = \alpha$ and $\lambda_3 = \beta$ where $\alpha \neq \beta$.

  Therefore, by (3)

  $$\kappa N = x_2 x_3(\beta - \alpha)e_1 + x_1 x_3(\alpha - \beta)e_2, \quad f(X) = \beta x_1^2 + \beta x_2^2 + (2\alpha - \beta)x_3^2 = 0.$$

  If $\beta(2\alpha - \beta) > 0$, then the cone $f(X)$ is trivial (a single point $X = 0$), and so $\kappa N = 0$. Suppose $\beta(2\alpha - \beta) = 0$. If $\beta = 0$, then $2\beta - \alpha \neq 0$, and so the cone $f(X) = 0$ is the plane $x_3 = 0$. If $2\alpha - \beta = 0$, then $\beta \neq 0$, and so the cone $f(X) = 0$ is the line $x_1 = x_2 = 0$. In both cases we have $\kappa N = 0$.

  Finally, if $\beta(2\alpha - \beta) < 0$, then the cone $f(X) = 0$ is a (non-trivial) circular cone in $\mathfrak{g}$. As $\kappa N = 0$ if and only if either $x_3 = 0$ or $x_1 = x_2 = 0$, all the points on the cone $f(X) = 0$ except for the origin $X = 0$ correspond to vectors generating non-geodesic circles in the group G.

  Note that in this case $\mu_1 = \mu_2 = \frac{1}{2}\beta$, $\mu_3 = \alpha - \frac{1}{2}\beta$, and so the principal Ricci curvatures are $r_1 = r_2 = \frac{1}{4}\beta(2\alpha - \beta)$, $r_3 = \frac{1}{4}\beta^2$. The Ricci signature cannot be $(0, 0, -)$, and it is $(-, -, +)$ exactly when $\beta(2\alpha - \beta) < 0$.

- Case 3: $\lambda_1, \lambda_2, \lambda_3$ are pairwise different.

  This implies that $\kappa N = 0$ if and only if at least two of the x_i's are zeros.

  Therefore,

  $$f(X) = \mu_1 x_1^2 + \mu_2 x_2^2 + \mu_3 x_3^2 = 0.$$

  First suppose that all three μ_i are not zero. If they have the same sign, then the cone $f(X) = 0$ is trivial. If two of the μ_i are positive, and the third one is negative (or vice versa), then the cone $f(X) = 0$ is non-trivial excluding the vertex $X = 0$ and $\kappa N \neq 0$. This is because $\lambda_1, \lambda_2, \lambda_3$ are pairwise different and $X \neq 0$.

Note that in this case, two of r_i's are negative and the third one is positive, which shows that the Ricci signature is $(-,-,+)$.

Now suppose that at least one of the μ_i is zero. Since $\lambda_i's$ are different, so we have at most 1 of $\mu_i = 0$. The reason is that if there are 2 of $\mu_i = 0$, 2 of μ_i are equal.

Without loss of generality, we can assume that $\mu_3 = 0$, so that $\lambda_3 = \lambda_2 + \lambda_1$. Note that $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$.

Thus,

$$\mu_1 = \frac{1}{2}(-\lambda_1 + \lambda_2 + \lambda_1 + \lambda_2) = \lambda_2 \quad \text{and} \quad \mu_2 = \frac{1}{2}(\lambda_1 - \lambda_2 + \lambda_1 + \lambda_2) = \lambda_1. \tag{5}$$

Therefore, $\mu_1, \mu_2 \neq 0$ and $f(X) = \lambda_2 x_1^2 + \lambda_1 x_2^2$.

We have

$$\begin{aligned}
\lambda_2 x_1^2 + \lambda_1 x_2^2 &= 0 \\
\lambda_2 x_1^2 &= -\lambda_1 x_2^2 \\
x_1^2 &= -\frac{\lambda_1}{\lambda_2} x_2^2 \\
x_1 &= \pm\Big(\sqrt{-\frac{\lambda_1}{\lambda_2}}\Big) x_2.
\end{aligned}$$

If λ_1 and λ_2 have the same sign, then the cone $f(X) = 0$ is the line $x_1 = x_2 = 0$, and so $\kappa N = 0$ by (3).

Suppose $\lambda_1 \lambda_2 < 0$. Then, the equations $x_1 = \pm(\sqrt{-\frac{\lambda_1}{\lambda_2}})x_2$ are the solution, which is the union of 2 planes in $\mathbb{R}^3$ excluding the line $x_1 = x_2 = 0$.

Note that in this case, two of r_i's are zero and the third one is negative, which shows that the Ricci signature is $(0, 0, -)$.

This completes the proof of the Theorem.

5 Conclusion

It has been shown that the Frenet formulas and the covariant derivative ∇_X by the Koszul formula plays a vital role for the proof of the Theorem. Different properties between λ_i's result in different Ricci sign signatures; however, when the signature of Ricci form is either $(+,-,-)$

or $(0, 0, -)$, the curve is left-invariant circle which is not geodesic for a three dimensional, unimodular, metric lie group.

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