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An Introduction to Supergeometry

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Abstract

An extension to the standard model of particle physics is the theory of supersymmetry. In the current standard model of particle physics, two families of particles, bosons and fermions, are treated separately with commuting and anti-commuting coordinates respectively. Supersymmetry is a spacetime symmetry between the two. For this, it is useful to introduce the idea of a *supermanifold*, a space that has both commuting and anti-commuting coordinates. This report details the different constructions of a supermanifold and describes connections between them.

1 Introduction

A smooth manifold is the generalisation of a surface to higher dimensions; it is a space that locally looks like Euclidean space. One of the aims of supergeometry is to generalise this idea of a manifold to encompass extra coordinates in an attempt to provide supersymmetry and other physical theories with a convenient mathematical groundwork. As much of supergeometry makes use of established differential geometry, we direct the reader to [KN96],[dC92], and [Lee18] for any unfamiliar terminology.

Supersymmetry, first introduced in the 1970s, is a theory of particle physics developed to unify the treatment of two types of elementary particles, fermions and bosons. Fermions are matter-carrying particles while bosons are the force carriers in our current standard model of particle physics [Var04]. Classically, bosons are dealt with in standard coordinates while fermions anti-commuting coordinates. A prediction made by the theory of supersymmetry is the existence of 'partner' particles: for each boson (resp. fermion), there exists a partner fermionic (resp. bosonic) particle. For example, the electron and its partner, the *selectron* [Rog07].

The theory of supersymmetry is built upon supergeometric objects, the most foundational of which is a supermanifold. Supermanifolds were defined independently by Batchelor, Berezin, DeWitt, Kostant, Leites, and Rogers in [Bat80], [Ber87], [Dew84], [Kos77],[Lei80], and [Rog80] respectively. There exist two main constructions of supermanifolds; the concrete and the algebro-geometric approach. Batchelor, DeWitt and Rogers follow the concrete approach, describing supermanifolds as spaces that locally resemble *flat superspace*, the model space in supergeometry. In contrast, Berezin, Kostant and Leites describe the algebro-geometric approach, defining a supermanifold as a topological space equipped with a sheaf of $\mathbb{Z}/2\mathbb{Z}$-graded algebras. This is analogous to how we can view a classical smooth manifold as a ringed space $(M, \mathcal{O}_M)$ that is locally isomorphic to Euclidean space with its sheaf of smooth functions, $(\mathbb{R}^n, \mathcal{C}^{\infty}_{\mathbb{R}^n})$. For more details on this, see Appendix A.

This report will introduce the theory of supermanifolds using both the classical and algebro-geometric approaches, discussing when the two are equivalent.

1.1 Statement of Authorship

The bulk of the presented material is adapted from [LC07], [Rog80], [Rog07], and [Var04] and is cited appropriately. All results and their proofs can be found in the listed references.

2 Batchelor-DeWitt-Rogers Approach to Supermanifolds

There are three main constructions of a concrete supermanifold, the differences between them are the various topologies and differentiable functions defined upon them.

We will pass by the preliminaries of superlinear algebra as most definitions evolve naturally from classical linear algebra when introducing a $\mathbb{Z}/2\mathbb{Z}$-grading. To get a feel for how this works, we will introduce perhaps the most important construction, a *super vector space*.

Definition 2.1. A *super vector space*, V, is a vector space which can be decomposed as follows:

$$V = V_0 \oplus V_1,$$

where $v \in V_0$ are said to be 'even' while $v \in V_1$ are 'odd' elements. We introduce a parity function on the homogeneous elements,

$$p(v) = |v| = \begin{cases} 0, & v \in V_0 \\ 1, & v \in V_1. \end{cases}$$

The main difference between linear algebra and superlinear algebra is the commutation factor that keeps the parity of objects consistent: whenever commuting two objects, a and b, we find $ab = (-1)^{|a||b|}ba$. To see the rigorous construction of superlinear algebra, consult [Rog07] or [LC07].

2.1 Supernumbers and Superspace

The concrete approach aims to realise a supermanifold as locally resembling flat superspace, an analogue to $\mathbb{R}^n$. To understand flat superspace, we must first define the Grassman algebra.

Definition 2.2. For each positive integer k, we define the *Grassman algebra* (sometimes referred to as the exterior algebra) with k generators, $\beta_1, \cdots, \beta_L$, to be the unital algebra generated by all linear combinations of the generators, their products and an identity element, 1. We demand that the generators are anti-commuting. That is, $\beta_i\beta_j = -\beta_j\beta_i$ for all $i, j = 1, \cdots, L$. We denote this algebra by $\Lambda(\beta_1, \cdots, \beta_L)$.

A generic element of $\Lambda(\beta_1, \cdots, \beta_L)$ is of the form,

$$X = X_0 + \sum_{i=1}^{L} X_i\beta_i + \sum_{i<j} X_{ij}\beta_i\beta_j + \cdots + \sum_{1\leq i_1<\cdots<i_{L-1}<L} X_{i_1\cdots i_{L-1}}\beta_{i_1}\cdots\beta_{i_{L-1}} + X_{1\cdots L}\beta_1\cdots\beta_L \tag{2.1}$$

where the $X_0, \cdots, X_{\mu_1\cdots\mu_k}$ are elements of $\mathbb{R}$. We can write this more compactly using multi-index notation:

$$X = \sum_{\mu\in M_L} X_\mu\beta_\mu \tag{2.2}$$

where $M_n = \{\mu = (\mu_1, \cdots, \mu_k) \mid 1 \leq \mu_1 < \cdots < \mu_k \leq L\}$ is the set of multi-indices of up to length L, $\beta_\mu = \beta_{\mu_1}\cdots\beta_{\mu_k}$, and where $X_\mu \in \mathbb{R}$. Note that the empty multi-index will be denoted $\emptyset$ and $\beta_\emptyset = 1$.

As we will come to see, it can be useful to not restrict ourselves to a finite number of generators. This report will deal with three superalgebras; B_L, B_∞ and W_∞. B_L denotes the Grassman algebra with L generators while

W_∞ denotes an algebra generated by a countably infinite number of anti-commuting generators. We now define B_∞, which is a subalgebra of W_∞.

Definition 2.3. Let B_∞ be the vector space ℓ_1 of infinite sequences of real numbers $(x_1, x_2, \cdots)$ such that $\sum_{i=1}^{\infty} |x_i| < \infty$. With the usual ℓ_1 norm, B_∞ becomes a Banach space. We now define a multiplication, turning B_∞ into a Banach algebra. Let $M_\infty = \bigcup_{L=1}^{\infty} M_L$. Define β_μ to be the sequence in B_∞ with $x_\mu = 1$ and all other entries 0. It is convinient to write a correspondence between $\mathbb{Z}_+$ and M_∞. Take $r \in \mathbb{Z}_+$ and $\mu \in M_\infty$, then $r \leftrightarrow \mu$ if $r = \frac{1}{2}(2^{\mu_1} + \cdots + 2^{\mu_k})$. That is, an element $(x_1, x_2, x_3, \cdots) \in B_\infty$ can be written as $(x_{(1)}, x_{(2)}, x_{(1,2)}, \cdots)$. We then define β_μ to be the sequence with $x_\mu = 1$ and all other entries 0.

Multiplication is then defined by $\beta_\emptyset \beta_\mu = \beta_\mu \beta_\emptyset = \beta_\mu$, $\beta_{(i)}\beta_{(j)} = -\beta_{(j)}\beta_{(i)}$, and $\beta_\mu = \beta_{(\mu_1)} \cdots \beta_{(\mu_k)}$. An arbitrary element can thus be written as

$$(x_{(1)}, x_{(2)}, x_{(12)}, \cdots) = \sum_{\mu \in M_\infty} x_\mu \beta_\mu$$

One can extend the definition of multiplication by linearity and continuity to all of B_∞, making it a Banach algebra.

Let $L < \infty$. We can give B_L the structure of a supercommutative algebra by noticing that $B = B_{L,0} \oplus B_{L,1}$ where

$$B_{L,0} := \left\{ x \in B_L \,\middle|\, x = \sum_{\mu \in M_{L_0}} x_\mu \beta_\mu \right\},$$

$$B_{L,1} := \left\{ \xi \in B_L \,\middle|\, \xi = \sum_{\mu \in M_{L_1}} \xi_\mu \beta_\mu \right\},$$

where M_{L_0} and M_{L_1} are the sets of multi-indices of even and odd length respectively. By convention, we say that the multi-index $\emptyset$ is of even length. Similarly, B_∞ and W_∞ form supercommutative algebras.

Remark. From now on, unless otherwise mentioned, L can be taken to be finite or infinite.

Definition 2.4. We now define *flat superspace* to be $B_L^{m,n} := \underbrace{B_{L,0} \times \cdots \times B_{L,0}}_{\text{m times}} \times \underbrace{B_{L,1} \times \cdots \times B_{L,1}}_{\text{n times}}$. A typical element will be denoted $(x; \xi) = (x_1, \cdots, x_m, \xi_1, \cdots, \xi_n)$.

There is a natural algebra homomorphism which we often refer to as the *body map*, $\varepsilon : B_L \to \mathbb{R}$ by mapping

$$X = \sum_{\mu \in M_L} X_\mu \beta_\mu \mapsto X_\emptyset.$$

We can extend the body map to $\varepsilon_{m,n} : B_L^{m,n} \to \mathbb{R}^m$ by $(x_1, \cdots, x_m, \xi_1, \cdots, \xi_n) \mapsto (\varepsilon(x_1), \cdots, \varepsilon(x_m))$. We can define the inverse of elements in B_L with non-zero body elements. Given $X \in B_L$ with $\varepsilon(X) \neq 0$, we define

$$X^{-1} = \frac{1}{\varepsilon(X)} \sum_{i=0}^{\infty} (-1)^i \left(\frac{s(X)}{\varepsilon(X)} \right)^i.$$

Here $s(X) := X - \varepsilon(X)1$, is often referred to as the *soul* map.

Remark. When L is finite the soul is nilpotent; however, this may not be the case when L is infinite.

For any L (finite or infinite), we equip B_L with a complete norm, $\|\cdot\|_{B_L}$ defined for $X = \sum_{\mu \in M_L} X_\mu \beta_1^{\mu_1} \cdots \beta_L^{\mu_L}$ by $\|X\|_{B_L} = \sum_{\mu \in M_L} |X_\mu|$. The paper [Rog07] shows that $\|1\|_{B_L} = 1$ and $\|XY\|_{B_L} \leq \|X\|_{B_L} \|Y\|_{B_L}$, giving B_L the structure of a Banach algebra.

We can extend the definition of our norm to flat superspace,

$$\|(x_1, \cdots, x_{m+n})\|_{B_L^{m,n}} := \|x_1\|_{B_L} + \cdots + \|x_{m+n}\|_{B_L}.$$

There are several different topologies that can be placed upon $B_L^{m,n}$. We will focus on only two: the product topology and the DeWitt topology.

Definition 2.5. A subset $U \subset B_L^{m,n}$ is said to be open in the *DeWitt topology* if there exists a subset $V \subset \mathbb{R}^m$ such that $\varepsilon_{m,n}^{-1}(V) = U$.

Definition 2.6. The product topology on $B_L^{m,n}$ is the coarsest topology that ensures the projection maps onto $B_{L,0}$ and $B_{L,1}$ are continuous with respect to the finite vector space topology.

2.2 Functions on Flat Superspace

In this section, we develop the notion of a superfunction and discuss the superdifferentiability of such functions.

The following proposition is proven in [Rog80] and gives a first example of the difference between L being finite and infinite.

Proposition 2.1. *Suppose $\alpha_r \in B_L$ for $r = 1, \cdots, m+n$, is such that $\sum_{i=1}^m h_i \alpha_i + \sum_{j=1}^n \eta_j \alpha_{j+m} = 0$ for all $(h,k) \in B_L^{m,n}$. Then,*

(a) if $1 \leq i \leq m$, $\alpha_i = 0$,

(b) if $1 \leq j \leq n$ and L is finite, then $\alpha_{j+m} = \lambda \beta_{12\cdots L}$ for $\lambda \in \mathbb{R}$, and

(c) if $1 \leq j \leq n$ and L is infinite, then $\alpha_{j+m} = 0$.

We follow [Rog07] and define the analogue of smooth functions, G^∞ functions, in a very similar manner to how classical calculus is developed. This then leads us to a more elegant definition of supersmooth functions.

Definition 2.7. Let $U \subset B_L^{m,n}$ be open in the product topology and consider $f : U \to B_L$.

(a) f is said to be G^0 on U if f is continuous.

(b) f is said to be G^1 on U if there exist $m+n$ continuous functions $\partial_k f : U \to B_L$, $k = 1, \cdots, m+n$ and a function $\gamma : B_L^{m,n} \to B_L$ such that for $(x,\xi), (x+y, \xi+\eta) \in U$,

$$f(x+y, \xi+\eta) = f(x,\xi) + \sum_{i=1}^m y_i (\partial_i f)(x,\xi) + \sum_{j=1}^n \eta_j (\partial_{j+m} f)(x,\xi) + \|(x,\xi)\|_{B_L^{m,n}} \gamma(y,\eta)$$

where $\|\gamma(y,\eta)\|_{B_L} \to 0$ as $\|(y,\eta)\|_{B_L^{m,n}} \to 0$.

(c) $f \in G^p(U)$ if f is G^1 on U and the functions $\partial_l f \in G^{p-1}(U)$.

(d) $f \in G^\infty(U)$ if $f \in G^p(U)$ for all $p \in \mathbb{Z}_+$.

(e) Let $g : U \to B_L^s$ for some $s \in \mathbb{Z}_+$. Let p_k denote the projection of a point $(x_1, \cdots, x_{m+n}) \in U$ to x_k. We say g is G^r on U if $p_k \circ g$ is G^r on U for all $k \in \{1, \cdots, s\}$.

Remark. In the above definition, when L is finite (b) doesn't define unique odd partial derivatives. To see this, Proposition 2.1 (b) implies that $\partial_{j+m} f(x;\xi) = \lambda \beta_{1\cdots L}$ for all $j = 1, \cdots, n$ and some $\lambda \in \mathbb{R}$.

Example 2.1. Let $f : B_L^{2,2} \to B_L$ be defined by $(x_1, x_2, \xi_1, \xi_2) \mapsto c x_1 x_2^2 \xi_1 \xi_2$, where $L > 1$. Then, f is G^∞ with

$$\partial_1 f = c x_2^2 \xi_1 \xi_2, \quad \partial_2 f = 2 c x_1 \xi_1 \xi_2, \quad \partial_3 f = c x_1 x_2^2 \xi_2, \quad \partial_4 f = -c x_1 x_2^2 \xi_1.$$

Fix $L < \infty$. We now extend B_L-valued smooth functions on $\varepsilon_{m,n}(U)$, which we denote the class of by $C^\infty(\varepsilon_{m,n}(U)) \otimes B_L$, to G^∞ functions. This process resembles analytic continuation from complex analysis, and indeed we call it *Grassman analytic continuation.*

Definition 2.8. Suppose $U \subset B_L^{m,n}$ is open in the product topology. Given $f \in C^\infty(\varepsilon_{m,n}(U))$, we define the *Grassman analytic continuation* of f to be

$$\widehat{f}(x;\xi) := \sum_{i_1=0}^{\infty} \cdots \sum_{i_m=0}^{\infty} \frac{1}{i_1! \cdots i_m!} \partial_1^{i_1} \cdots \partial_m^{i_m} f(\varepsilon_{m,n}(x;\xi)) \times s(x_1)^{i_1} \times \cdots \times s(x_m)^{i_m}. \tag{2.3}$$

These sums terminate due to the nilpotency of the soul and so one obtains an injective algebra homomorphism $\widehat{\ }: C^\infty(\varepsilon_{m,n}(U)) \to G^\infty(U)$. It turns out that every function in $G^\infty(U)$ takes the form

$$f(x;\xi) = \sum_{\mu \in M_n} \widehat{f_\mu}(x) \xi_\mu, \tag{2.4}$$

where $f_\mu \in C^\infty(\varepsilon_{m,n}(U)) \otimes B_L$. We now witness the second shortcoming of the finitely generated Grassman algebra. As the odd terms $\xi_1, \cdots, \xi_n$ are nilpotent, if the length of μ is greater than n then ξ_μ vanishes and the f_μ are completely undetermined. We can conclude that the map

$$C^\infty(\varepsilon_{m,n}(U)) \otimes B_L \otimes \Lambda(\xi_1, \cdots, \xi_n) \to G^\infty(U), \quad f \otimes \xi_\mu \mapsto \widehat{f_{\mu_1 \cdots \mu_n}} \xi_\mu \tag{2.5}$$

is an isomorphism of superalgebras if and only if $n \le L$, and is an epimorphism otherwise.

We now see that we could have defined G^∞ functions abstractly:

Definition 2.9. Let $U \subset B_L^{m,n}$ be open in the product topology. A function $f : U \to B_L$ is called G^∞ if there exist functions $f_\mu \in C^\infty(\varepsilon_{m,n}(U)) \otimes B_L$, such that

$$f(x;\xi) = \sum_{\mu \in M_n} \widehat{f_\mu}(x) \xi_\mu$$

for all $(x;\xi) \in U$.

This definition lends itself nicely to defining derivatives on B_∞. Grassman analytic continuation is defined such that having infinite generators still makes sense, the only thing we must clarify is the meaning of a smooth function on $V \subset \mathbb{R}^m$ that map into B_∞.

Definition 2.10. Let $V \subset \mathbb{R}^m$ be open and let $f : V \to B_\infty$. Then $f \in C^\infty(V) \otimes B_\infty$ if for every $L \in \mathbb{Z}_{\geq 0}$, $\mathcal{P}_L \circ f \in C^\infty(V) \otimes B_L$, where $\mathcal{P}_L$ is the projection of B_∞ onto B_L by setting all generators β_j with $j > L$ to 0.

We now define what we mean by differentiating a supersmooth function on B_∞.

Definition 2.11. Suppose that $f \in G^\infty(U)$. Then, for $i = 1, \cdots, m$, the even derivative, $\partial_i^E f$, is defined to be the $G^\infty(U)$ function

$$\partial_i^E f(x;\xi) := \sum_{\mu \in M_n} \widehat{\frac{\partial}{\partial \varepsilon(x_i)} f_\mu}(x)\xi_\mu.$$

For $j = 1, \cdots, n$, the odd derivative, $\partial_j^O f$ is defined by

$$\partial_j^O f(x;\xi) := \sum_{\mu \in M_n} (-1)^{|f_\mu|} p_{j,\mu} \widehat{f_\mu}(x)\xi_{\mu/j},$$

where we define

$$p_{j,\mu} := \begin{cases} (-1)^{l+1}, & \text{if } j = \mu_l, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \xi_{\mu/j} := \begin{cases} \xi_{\mu_1} \cdots \xi_{\mu_{l-1}} \xi_{\mu_{l+1}} \cdots \xi_{\mu_k}, & \text{if } j = \mu_l, \\ 0, & \text{otherwise.} \end{cases}$$

As in the classical case, we can prove the linearity of derivatives as well as the Leibniz rule and chain rule.

Example 2.2. Let $f : B_L^{1,2} \to B_L$ be defined by $(x, \xi_1, \xi_2) \mapsto x^2 + \xi_1$. Then f is G^∞. To see this, define the following maps;

$$\begin{aligned} f_\emptyset &: \mathbb{R} \to \mathbb{R}, \quad f_\emptyset(y) = y^2, \\ f_1 &: \mathbb{R} \to \mathbb{R}, \quad f_1(y) = 1, \end{aligned}$$

and $f_2, f_{12} : \mathbb{R} \to \mathbb{R}$ by $\varepsilon(x) \mapsto 0$. We extend the domain of these functions via Grassman analytic continuation:

$$\begin{aligned} \widehat{f_\emptyset}(x;\xi) &= f_\emptyset(\varepsilon_{1,2}(x;\xi)) + \frac{\partial}{\partial \varepsilon(x)} f_\emptyset(\varepsilon_{1,2}(x;\xi)) \times s(x) + \frac{1}{2}\frac{\partial^2}{\partial \varepsilon(x)^2} f_\emptyset(x;\xi) \times s(x)^2, \\ &= \varepsilon(x)^2 + 2\varepsilon(x)s(x) + s(x)^2, \\ &= x^2, \end{aligned}$$

and $\widehat{f_1}(x;\xi) = 1, \widehat{f_2}(x;\xi), \widehat{f_{12}}(x;\xi) = 0$. Then,

$$\widehat{f_\emptyset}(x;\xi) + \widehat{f_1}(x;\xi)\xi_1 + \widehat{f_2}(x;\xi)\xi_2 + \widehat{f_{12}}(x;\xi)\xi_1\xi_2 = x^2 + \xi_1 = f(x;\xi).$$

We can thus take the even and odd partial derivatives as defined above,

$$\begin{aligned} \partial_1^E f(x;\xi) &= \widehat{\frac{\partial}{\partial \varepsilon(x)} f_\emptyset}(x;\xi) + \widehat{\frac{\partial}{\partial \varepsilon(x)} f_1}(x;\xi)\xi_1, \\ &= 2\varepsilon(x) + 2s(x) = 2x, \\ \partial_1^O f(x;\xi) &= (-1)^{|f_1|}(-1)^2 \widehat{f_1}(x;\xi)\xi_{1/1}, \\ &= -\widehat{f_1}(x;\xi) = -1. \end{aligned}$$

Finally, we define a more restrictive class of functions, H^∞, which have real-valued coefficient functions in the expansion (2.4).

Definition 2.12. Fix $L < \infty$. Let $U \subset B_L^{m,n}$ be open. $H^\infty(U)$ is the subalgebra of $G^\infty(U)$ where for $f \in H^\infty(U)$, there exists functions $f_\mu \in C^\infty(\varepsilon_{m,n}(U))$ such that

$$f(x;\xi) = \sum_{\mu \in M_n} \widehat{f_\mu}(x)\xi_\mu. \tag{2.6}$$

Now (2.5) restricts to an algebra epimorphism

$$C^\infty(\varepsilon_{m,n}(U)) \otimes \Lambda(\xi_1, \cdots, \xi_n) \to H^\infty(U). \tag{2.7}$$

When $n \leq L$, the above is an isomorphism.

2.3 Concrete Supermanifolds

The following sections define a supermanifold following the classical theory, adding a smooth structure to a topological space. We can produce a variety of different supermanifolds by requesting the transition functions live in different function classes, using different topologies or having L be finite and infinite. As we will see, this dramatically changes the structure of the supermanifold.

2.3.1 G^∞ Supermanifolds

This approach to supermanifolds is due to Rogers [Rog80]. We will place no restrictions on the value of L, and endow $B_L^{m,n}$ with the product topology.

Definition 2.13. Let M be a Hausdorff topological space.

(a) An (m, n) open chart on $\mathcal{M}$ over B_L is a pair (U, φ) with $U \subset \mathcal{M}$ and $\varphi : U \to V \subset B_L^{m,n}$ is a homeomorphism.

(b) Given $r \in \mathbb{N} \cup \{\infty\}$, an (m, n)-G^r structure on $\mathcal{M}$ over B_L is a collection $\{(U_\alpha, \varphi_\alpha) \mid \alpha \in I\}$ such that

  (i) $M = \bigcup_{\alpha \in I} U_\alpha$,

  (ii) for all $\alpha, \beta \in I$ with $U_\alpha \cap U_\beta \neq \emptyset$, then $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is G^r, and

  (iii) the structure is maximal. That is there is no larger collection of open charts.

(c) An (m, n)-G^r supermanifold over $B_L^{m,n}$ is a Hausdorff topological space $\mathcal{M}$ equipped with an (m, n)-G^r structure.

Remark. If $r = \infty$, then by replacing G^r with H^∞, we obtain an H^∞ supermanifold.

2.3.2 DeWitt Supermanifolds

We make note that in the following definition, we haven't given our proposed supermanifold any topology. The DeWitt topology is inherited through the transition functions. DeWitt's original definition in [Dew84] defines differentiable functions algebraically that turn out to be equivalent to the G^∞ functions defined above.

The infinite-dimensional superalgebra present in DeWitt's supermanifold is not B_∞; instead, he makes use of the space W_∞, the algebra generated by countably infinite anti-commuting objects. This has ramifications when defining differentiability as W_∞ cannot be endowed with the structure of a Banach algebra as B_L can.

Definition 2.14. Fix $L < \infty$. Let $\mathcal{M}$ be a set, and let $m, n \in \mathbb{N}$. $\mathcal{M}$ is an (m,n)-dimensional G^∞ *DeWitt supermanifold* if there exists a maximal G^∞ atlas $\{(V_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ where $V_\alpha \subset \mathcal{M}$, and $\varphi_\alpha : \mathcal{M} \to B_L^{m,n}$ (or W_∞) are bijections such that

(a) $\bigcup_{\alpha \in A} V_\alpha = \mathcal{M}$,

(b) for each $\alpha, \beta \in A$ such that $V_\alpha \cap V_\beta \neq \emptyset$, the map

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(V_\alpha \cap V_\beta) \to \varphi_\beta(V_\alpha \cap V_\beta)$$

is G^∞.

The difference between DeWitt's supermanifold and Rogers' G^∞ supermanifold is the topology. DeWitt's coarse topology restricts most of the information from the odd components while Roger's doesn't.

Example 2.3. As usual, the boring example of the model space is a supermanifold. That is, $B_L^{m,n}$ or $W_\infty^{m,n}$ are our first supermanifolds, where $\{(\mathcal{M}, \mathrm{id})\}$ forms the G^∞ atlas.

Example 2.4. Any open set $V \subset B_L^{m,n}, W_\infty^{m,n}$ in the DeWitt topology is a supermanifold since we have the atlas $\{(V, \iota)\}$ where ι is the inclusion into $B_L^{m,n}$ or $W_\infty^{m,n}$.

Example 2.5. The first non-trivial example is that of real super projective space, $\mathbb{SR}P^{m,n}$. Let

$$U := (\varepsilon_{m+1}, n)(\mathbb{R}^{m+1} \setminus \{0\}) \subset B_L^{m+1,n}$$

and define an equivalence relation $\sim$ on U by $(x;\theta) \sim (\tilde{x}, \tilde{\theta})$ if and only if there exists an invertible, even element $\ell \in B_L$ such that

$$x^i = \ell \tilde{x}^i, \quad i = 1, \cdots, m,$$
$$\theta^j = \ell \tilde{\theta}^j, \quad j = 1, \cdots, n.$$

We define $\mathbb{SRP}^{m,n} = U/\sim$. For brevity, we will continue through this example with $B_L^{m,n}$. It works analogously with $W_\infty^{m,n}$. This space is given the structure of a G^∞ DeWitt supermanifold by defining the following atlas. For $i = 1, \cdots, m+1$, let

$$V_i = \{[x;\theta] \mid (x;\theta) \in U, \varepsilon(x^i) \neq 0\},$$

with coordinate maps $\varphi_i : V_i \to B_L^{m,n}$ defined by

$$[(x;\theta)] \mapsto \left(\frac{x_1}{x^i}, \cdots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \cdots, \frac{x_m}{x^i}, \frac{\theta^1}{x^i}, \cdots \frac{\theta^n}{x^i} \right).$$

We can give the above definition of a supermanifold the DeWitt topology by requiring each coordinate map is a homeomorphism as follows:

Theorem 2.1. *Let $\mathcal{M}$ be a G^∞ DeWitt supermanifold with a complete atlas $\{(V_\alpha, \varphi_\alpha)\}$. Let Γ_{DeWitt} be the collection of subsets $U \subset \mathcal{M}$ such that, for all $\alpha \in A$, $\varphi_\alpha(V_\alpha \cap U)$ is open in $B_L^{m,n}$ or $W_\infty^{m,n}$ in the DeWitt topology. Then, Γ_{DeWitt} is a topology on $\mathcal{M}$.*

We now introduce the idea of the underlying 'body' manifold. This is particularly useful in building intuition.

Theorem 2.2. *Let $\mathcal{M}$ be a G^∞ DeWitt supermanifold with atlas $\{(V_\alpha, \varphi_\alpha) \mid \alpha \in A\}$. Then,*

(a) The relation $\sim$ defined on $\mathcal{M}$ by $p \sim q$ if and only if there exists some $\alpha \in A$ such that $p, q \in V_\alpha$ and $\varepsilon_{m,n}(\varphi_\alpha(p)) = \varepsilon_{m,n}(\varphi_\alpha(q))$, is an equivalence relation.

(b) The body manifold defined to be $\mathcal{M}_\emptyset = \mathcal{M}/\sim$ has the structure of a real smooth manifold with an atlas $\{(V_{[\emptyset]\alpha}, \varphi_{[\emptyset]\alpha}) \mid \alpha \in A\}$, where $V_{[\emptyset]\alpha} = \{[p] : p \in V_\alpha\}$, and $\varphi_{[\emptyset]\alpha} : V_{[\emptyset]\alpha} \to \mathbb{R}^m$ is defined by $[p] \mapsto \varepsilon_{m,n} \circ \varphi_\alpha(p)$.

Example 2.6. One would hope that the body of $\mathbb{SR}P^{m,n}$ would be $\mathbb{R}P^m$ given the similarity in the definitions and indeed it is, yet we must be careful as most classical constructions do not carry over to a super counterpart.

That being said, on super projective space, we place the equivalence relation τ, defined by $[(x;\theta)]\tau[(y;\gamma)]$ if and only if there exists some $i \in \{1, \cdots, m+1\}$ such that $\varepsilon_{m,n}(\varphi_i([x;\theta])) = \varepsilon_{m,n}(\varphi_i([y;\gamma]))$. This is equivalent to having

$$\left(\varepsilon\left(\frac{x_1}{x^i}\right), \cdots, \varepsilon\left(\frac{x^{i-1}}{x^i}\right), \varepsilon\left(\frac{x^{i+1}}{x^i}\right), \cdots, \varepsilon\left(\frac{x^m}{x^i}\right)\right) = \left(\varepsilon\left(\frac{y^1}{y^i}\right), \cdots, \varepsilon\left(\frac{y^{i-1}}{y^i}\right), \varepsilon\left(\frac{y^{i+1}}{y^i}\right), \cdots, \varepsilon\left(\frac{y^m}{y^i}\right)\right).$$

In other words, this equivalence relation is grouping together all elements of our supermanifold that have the same underlying real components. One can readily see that the ensuing chart maps will be exactly those of $\mathbb{R}P^m$.

Remark. All DeWitt supermanifolds are G^∞ supermanifolds but the converse is not true. A counter-example to this is the supertorus which cannot be given a DeWitt supermanifold structure [Rog80].

2.3.3 Batchelor Supermanifolds

Batchelor in [Bat80] takes yet another approach to defining a supermanifold that is similar to both DeWitt's and Roger's construction. Fix L to be finite. The key differences between Batchelor's approach to DeWitt's and Roger's approaches are that

(i) $\varphi_\alpha : U_\alpha \to \varphi(U_\alpha)$ is a homeomorphism where $\varphi(U_\alpha) \subset B_L^{m,n}$ is open in the DeWitt topology (as in DeWitt's definition), yet Batchelor restricts the DeWitt topology to $\varphi_\alpha(U_\alpha)$ where DeWitt places the product topology on $\varphi_\alpha(U_\alpha)$ itself.

(ii) The transition functions are H^∞, not G^∞.

2.4 Functions of Supermanifolds

In classical differential geometry, we define a smooth function on a smooth manifold $f : M \to \mathbb{R}$, to be a map such that when composed with the chart maps, is smooth on the model space, $\mathbb{R}^n$. We adopt a similar approach in the super case.

Definition 2.15. Let $U \subset M$ be an open subset of a supermanifold (of any type) M in the respective topology. Then a function $f : U \to B$ is said to be G^∞ (resp. H^∞) if for each $\alpha \in A$ such that $U \cap V_\alpha \neq \emptyset$, the function $f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U \cap V_\alpha) \to \mathbb{R}_S$ is G^∞ (resp. H^∞).

3 Berezin-Kostant-Leites Approach to Supermanifolds

We will now discuss the theory of differentiable supermanifolds using language from algebraic geometry. In Appendix A, we present a brief introduction to the theory of sheaves and ringed spaces. These definitions are fundamental to understanding the more modern approach to supergeometry. There are two slightly different definitions of a supermanifold, one from Berezin and Leites in [Ber87] and [Lei80], and the other due to Kostant in [Kos77]. The two definitions are easily seen to be equivalent. The remainder of this section follows [LC07] and [Var04].

As we can define a smooth manifold by its associative algebra of smooth functions, we will proceed similarly when defining supermanifolds. We first require the notion of superrings and superalgebras.

3.1 Preliminary Definitions

Definition 3.1. A *super ring* $R = R_0 \oplus R_1$ is local if it admits a unique homogeneous maximal ideal I. That is, $I = (I \cap R_0) \oplus (I \cap R_1)$.

Definition 3.2. A *super ringed space* is a pair $(M, \mathcal{O}_M)$ consisting of a topological space M, and a sheaf of supercommutative superrings $\mathcal{O}_M$. If in addition, the stalk $\mathcal{O}_{M,x}$ is a local super ring for all $x \in M$, we say that $(M, \mathcal{O}_M)$ is a *superspace* or *local super ringed space.*

As we define the morphisms of ringed spaces and locally ringed spaces, we define analogously their super counterparts. We note that in the super setting, morphisms must respect the parity of elements.

Example 3.1. Let M be a smooth manifold with the sheaf of smooth functions on M, $\mathcal{C}^\infty_M$. We can define the sheaf of supercommutative $\mathbb{R}$-algebras by the assignment

$$V \mapsto \mathcal{O}_M(V) := \mathcal{C}^\infty_M(V) \otimes \Lambda(\xi_1, \cdots, \xi_q)$$

where $V \subset M$ is open. In a sense, we can view elements of these rings as *superfunctions*:

$$f(x, \xi) = \sum_{\mu \in M_q} f_\mu(x) \xi_\mu \tag{3.1}$$

where $f_\mu \in C^\infty(V)$ and $\mu \in M_q$ is a multi-index as usual.

We find $(M, \mathcal{O}_M)$ is a superringed space and in fact forms a superspace, where the maximal ideal of $\mathcal{O}_{M,x}$ is generated by the maximal ideal of $\mathcal{C}^\infty_{M,x}$ and the odd indeterminates $\xi_1, \cdots, \xi_q$. Importantly, in the case where $M = \mathbb{R}^n$, we obtain *flat superspace*

$$\mathbb{R}^{p,q} := (\mathbb{R}^p, \mathcal{C}^\infty_{\mathbb{R}^p} \otimes \Lambda(\xi_1, \cdots, \xi_q)).$$

This will form the model space for our definition of a supermanifold

Definition 3.3. Given an open subset $|U| \subset |S|$ where $(|S|, \mathcal{O}_S)$ is a superspace, we always have a superspace $(|U|, \mathcal{O}_S|_{|U|})$ called the *open subspace* associated with $|U|$.

The following example defines the general linear supergroup, an extremely important space in the development of supergeometry. This turns out to be a Lie supergroup.

Example 3.2. Let $M^{p,q} = \mathbb{R}^{p^2+q^2, 2pq}$ denote the superspace corresponding to the vector space of $((p,q)\times(p,q))$-matrices. As a super vector space,

$$(M_{p,q})_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\}, \quad (M_{p,q})_1 = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \right\},$$

where A, B, C, D are $(p \times p), (p \times q), (q \times p), (q \times q)$-matrices respectively. We have $p^2 + q^2$ even coordinates, t^{ij} where we can take $i, j \in \{1, \cdots, p\}$ or $i, j \in \{p+1, \cdots, p+q\}$, corresponding to the matrices A and D. We also have $2pq$ odd coordinates, ξ^{kl} where we can take $k \in \{1, \cdots, p\}$ and $l \in \{p+1, \cdots, p+q\}$ or $k \in \{p+1, \cdots, p+q\}$ and $l \in \{1, \cdots, p\}$.

We then define the structure sheaf of $M_{p,q}$ by the assignment

$$V \mapsto \mathcal{O}_{M_{p,q}}(V) := \mathcal{C}^\infty_{M_p \times M_q}(V) \otimes \Lambda(\xi_{kl})$$

for all open $V \subset M_p \times M_q$. The superspace $M_{p,q}$ is called the superspace of *supermatrices*. Consider the open set $U \subset M_p \times M_q$ which has $\det(t^{ij}) \neq 0$ for $i, j \in \{1, \cdots, p\}$ or $i, j \in \{p+1, \cdots, p+q\}$. We can define the open subspace of $M_{p,q}$ associated with the open set U to be $GL_{p,q} := \{U, \mathcal{O}_{M_{p,q}|_U}\}$, the *general linear supergroup*. It turns out this space has a Lie supergroup structure.

Remark. The sections of a sheaf of superalgebras are often not legitimate functions since superrings contain many nilpotent elements which vanish on the topological space. Thus, we must view these geometric objects using algebraic geometry to not lose information. This is different to the classical manifold case in which the sections are functions and so the topological map defines the sheaf morphism which is not the case for superspace.

3.2 Supermanifolds and Their Local Structure

Definition 3.4. Let $\mathcal{C}^\infty_U$ be the sheaf of smooth functions on the domain $U \subset \mathbb{R}^p$. Define the *superdomain* $U^{p,q}$ to be the superspace $(U, \mathcal{C}^\infty_{\mathbb{R}^m}|_U \otimes \Lambda(\xi_1, \cdots, \xi_q))$.

Definition 3.5. A superspace $\mathcal{M} = (|M|, \mathcal{O}_M)$ is called a (Berezin-Leites) *supermanifold* if

(i) $|M|$ is a locally compact, second countable, Hausdorff topological space, and

(ii) for each $x \in |M|$, there exists an open neighbourhood $U \ni x$ such that there is an isomorphism

$$(U, \mathcal{O}_{M|_U}) \to U^{m,n} \subset \mathbb{R}^{m,n}$$

for fixed m, n, where $U^{m,n}$ is a superdomain of $\mathbb{R}^{m,n}$.

Unlike in the concrete approach, it isn't clear what it means to evaluate a superfunction at a point, or even what points are in our model space. The evaluation of f at a point $(x_1, \cdots, x_p) \in U$ leads us to a value $f(x;\xi) \in \mathbb{R} \otimes \Lambda(\xi_1, \cdots, \xi_q)$. It is shown in [LC07] that an element $s \in R \otimes \Lambda(\xi_1, \cdots, \xi_q)$ with

$$s = s_0 + \sum_a s_a \xi_a + \sum_{a,b} s_{a,b} \xi_a \xi_b + \cdots$$

where $s_0, s_a, \cdots \in R$, is invertible if and only if s_0 is invertible in the unital commutative ring R.

Taking this result with $R = \mathcal{C}^\infty(U)$, we find that a superfunction of the form (3.1) is invertible if and only if f_0 is invertible.

We define the value of a superfunction $f \in \mathcal{C}_U^\infty \otimes \Lambda(\xi_1, \cdots, \xi_q)$ at a point $x \in U$ to be the unique value $k \in \mathbb{R}$ such that $f - k$ is not invertible in any neighbourhood of $x \in U$. This construction of the evaluation of a superfunction leads to an alternate definition for the body of a supermanifold.

Kostant's definition of a supermanifold differs in two aspects to the above construction; he assumes that M is a topological manifold, requiring the extra structure of an atlas, and he assumes that for a ringed space to be a supermanifold, there must exist a defined body manifold.

Definition 3.6. A smooth (Kostant) supermanifold of dimension (m, n) is a pair $\mathcal{M} = (M, \mathcal{O})$, where M is a topological manifold and $\mathcal{O}$ is a sheaf of supercommutative algebras over M such that

(a) there exists an open cover $\{U_\alpha \mid \alpha \in I\}$ where for each $\alpha \in I$

$$\mathcal{O}(U_\alpha) \simeq C^\infty(U_\alpha) \otimes \Lambda(\mathbb{R}^n),$$

(b) if $\mathfrak{R}$ is the sheaf of nilpotents in $\mathcal{O}$, then $(M, \mathcal{O}/\mathfrak{R})$ is isomorphic to $(M, \mathcal{C}^\infty)$.

The two definitions are equivalent since the local isomorphism of ringed spaces defines a family of homeomorphisms from subsets of M to $\mathbb{R}^n$. This is exactly what is required for a topological manifold. Furthermore, [LC07] shows that every supermanifold has a well-defined body manifold.

Definition 3.7. An open neighbourhood $U \subset M$ such that $\mathcal{O}(U) \simeq C^\infty(U) \otimes \Lambda(\mathbb{R}^n)$ is called a *splitting neighbourhood*.

Condition (b) in definition 3.6 implies that there is a unique homomorphism $b : \mathcal{O}(U) \to C^\infty(U)$, defined by $f \mapsto f_\emptyset$. This mapping commutes with the restriction maps of our sheaf. Given a splitting neighbourhood, U, there exists subalgebras $C(U), D(U)$ of $\mathcal{O}(U)$ with $C(U) \simeq C^\infty(U)$ and $D(U) \simeq \Lambda(\mathbb{R}^n)$ such that

$$\mathcal{O}(U) = C(U) \otimes D(U) \simeq C^\infty(U) \otimes \Lambda(\mathbb{R}^n).$$

Kostant in [Kos77] shows that $b|_{C(U_\alpha)}$ is an isomorphism of superalgebras.

Definition 3.8. The following defines a system of local coordinates on a supermanifold $\mathcal{M} = (M, \mathcal{O})$.

(a) $U \subset M$ is called a coordinate neighbourhood of $\mathcal{M}$ if it is a splitting neighbourhood and is a coordinate neighbourhood of M.

(b) Let U be a coordinate neighbourhood of $\mathcal{M}$ and $\mathcal{O}(U) = C(U) \otimes D(U) \simeq C^\infty(U) \otimes \Lambda(\mathbb{R}^n)$. Then there exists a system of odd coordinates $(\xi_1, \cdots, \xi_n)$ on U. We say $(x_1, \cdots, x_m)$ is a system of even coordinates (U) if $(b(x_1), \cdots, b(x_m))$ is a system of coordinates on M.

(c) A system of coordinates on a coordinate neighbourhood of $\mathcal{M}$ is given by a system of even and odd coordinates, $\{x_i; \xi_j\}$ for $1 \le i \le m,\ 1 \le j \le n$.

Proposition 3.1. *Suppose that U and V are two coordinate neighbourhoods on a supermanifold $\mathcal{M} = (M, \mathcal{O})$, with $U \cap V \neq 0$. Let $\{x_i; \xi_j\}$, $\{y_i; \eta_j\}$ be the respective systems of coordinates on U and V. Then, the restriction of these coordinates to $U \cap V$ forms a coordinate system. Furthermore, we may express the restrictions of $\{y_i; \eta_j\}$ in terms of $\{x_i; \xi_j\}$ as follows:*

$$y_j = \sum_{\mu \in M_{N,0}} P_{i\mu} \xi_\mu, \quad 1 \le i \le m$$
$$\eta_j = \sum_{\mu \in M_{N,1}} Q_{j\mu} \xi_\mu, \quad 1 \le j \le m$$

where each $P_{i\mu}$ and $Q_{j\mu}$ is a uniquely determined element of $C(U \cap V)$.

Remark. The above expressions for restricted coordinates are vital to the construction of an H^∞ supermanifold from a Berezin-Kostant-Leites supermanifold.

4 Equivalence of the Definitions

The following result is sketched in [Rog80].

Theorem 4.1. *Given a Berezin-Kostant-Leites supermanifold $(X, \mathcal{O})$, we can construct a Batchelor supermanifold Y. Furthermore, given a Batchelor supermanifold, M, we may construct a sheaf of supercommutative algebras, $\mathcal{H}^\infty$, such that $(M_\emptyset, \mathcal{H}^\infty)$ is a Berezin-Kostant-Leites supermanifold.*

Remark. The above result shows that the definition of a Batchelor supermanifold is equivalent to the definitions of Berezin, Kostant and Leites.

Proof. We begin with an (m, n)-dimensional Batchelor supermanifold, M, showing it can be endowed with a sheaf of superfunctions such that it has the structure of a Berezin-Kostant-Leites supermanifold. Let $\{(U_\alpha, \varphi_\alpha) \mid \alpha \in I\}$ be a complete H^∞ altas on M. For each $\alpha \in I$, let $V_\alpha \subset M_\emptyset$ be such that $\varepsilon_{m,n}(U_\alpha) = V_\alpha$. Let $\mathcal{H}^\infty(V_\alpha) := H^\infty(U_\alpha)$ define a sheaf on $M_\emptyset$, the body manifold of M. The proof that this in fact forms a sheaf is analogous to other sheaves of functions.

We find from (2.7) that $H^\infty(U_\alpha)$ is isomorphic to $C^\infty(V_\alpha)\otimes\Lambda(\mathbb{R}^n)$ since $n \leq L$. It can be shown to respect restrictions and so we have an isomorphism of sheaves. Hence, $(M_\emptyset, \mathcal{H}^\infty)$ has the structure of a Berezin-Kostant-Leites supermanifold.

Now, let $(X, \mathcal{O})$ be a Berezin-Kostant-Leites supermanifold. We aim to construct a Batchelor supermanifold. By definition, X has an open cover $\{U_\alpha\}_{\alpha\in I}$ such that $\mathcal{O}(U_\alpha) = C(U_\alpha)\otimes D(U_\alpha) \cong C^\infty(U_\alpha)\otimes\Lambda(\mathbb{R}^n)$ for subalgebras $C(U_\alpha)$ and $D(U_\alpha)$ of $\mathcal{O}(U_\alpha)$. We define local coordinates on these subalgebras; $x_1^\alpha, \cdots, x_m^\alpha$ on $C(U_\alpha)$ and $\xi_1^\alpha, \cdots, \xi_n^\alpha$ on $D(U_\alpha)$.

For each $\alpha, \beta \in I$ with $U_\alpha \cap U_\beta \neq \emptyset$, define $x_i^{\alpha\beta} := \rho_{U_\alpha, U_\alpha\cap U_\beta} x_i^\alpha$ for $1 \leq i \leq m$. Define similarly $\xi_j^{\alpha\beta}$ for $1 \leq j \leq n$. Here ρ denotes the sheaf restriction maps. By Proposition 3.1, we have unique elements $P_{i\mu}^{\alpha\beta}, Q_{j\mu}^{\alpha\beta} \in C(U_\alpha \cap U_\beta)$ such that

$$x_i^{\beta\alpha} = \sum_{\mu\in M_{N,0}} P_{i\mu}^{\alpha\beta}\xi_\mu^{\alpha\beta}, \quad 1 \leq i \leq m,$$
$$\xi_j^{\beta\alpha} = \sum_{\mu\in M_{N,1}} Q_{j\mu}^{\alpha\beta}\xi_\mu^{\alpha\beta}, \quad 1 \leq j \leq n.$$

For each $\alpha \in I$, define $\phi_\alpha : U_\alpha \to \mathbb{R}^m$ to be the coordinate map on X, corresponding to our local coordinates $\{x_i^\alpha\}_{i=1}^m$. That is, $\phi_\alpha = (b(x)_1^\alpha, \cdots, b(x)_m^\alpha)$ where $b : \mathcal{O}(U_\alpha) \to C^\infty(U_\alpha)$ is an isomorphism. Now, given $\alpha, \beta \in I$, let $S_{\alpha\beta} := \varepsilon_{m,n}^{-1}(\phi_\alpha(U_\alpha \cap U_\beta)) \subset B_L^{m,n}$. The following maps will be shown to be transition maps.

Given $\alpha, \beta \in I$, define the mapping $\tau_{\beta\alpha} : S_{\alpha\beta} \to S_{\beta\alpha}$ by

$$p_i \circ \tau_{\beta\alpha}(y;\eta) := \sum_{\mu\in M_{n,0}} \left(\widehat{b(P_{i\mu}^{\alpha\beta}) \circ \phi_\alpha^{-1}}\right)(y)\eta_\mu, \quad 1 \leq i \leq m,$$
$$p_{j+m} \circ \tau_{\beta\alpha}(y;\eta) := \sum_{\mu\in M_{n,1}} \left(\widehat{b(P_{j\mu}^{\alpha\beta}) \circ \phi_\alpha^{-1}}\right)(y)\eta_\mu, \quad 1 \leq j \leq n.$$

Here p_i is the projection to the i^{th} coordinate. The following lemma was proven in [Rog80].

Lemma 4.1. *The $\tau_{\alpha\beta}$ defined above are transition functions. That is,*

(a) suppose $\alpha, \beta, \gamma \in I$. Then $\tau_{\gamma\beta} \circ \tau_{\beta\alpha} = \tau_{\gamma\alpha}$,

(b) for all $\alpha \in I$, $\tau_{\alpha\alpha} = \mathrm{id}$,

(c) for all $\alpha, \beta \in I$, $\tau_{\alpha\beta} \circ \tau_{\beta\alpha} = \mathrm{id}$, and

(d) $\tau_{\beta\alpha} : S_{\alpha\beta} \to S_{\beta\alpha}$ is a homeomorphism.

We remark that by construction, the transition function $\tau_{\beta\alpha}$ is H^∞ on $S_{\alpha\beta}$.

To construct our atlas, for each $\alpha \in I$, define $Z_\alpha = \{\alpha\} \times S_{\alpha\alpha}$. For brevity, $S_{\alpha\alpha}$ will be denoted by S_α. Define a map $\Omega_\alpha : Z_\alpha \to S_\alpha$ by $(\alpha, p) \mapsto p$ for all $p \in S_\alpha$. Let $Z = \bigcup_{\alpha\in I} Z_\alpha$ and define the following relation: for $z_1, z_2 \in Z$, we say $z_1 \; R \; z_2$ if and only if $\tau_{\alpha\beta} \circ \Omega_\alpha(z_1) = \Omega_\beta(z_2)$. It can be shown that R is an equivalence relation on Z. Define $Y := Z/R$ and let $Y_\alpha := \{[z] \mid z \in Z_\alpha\}$. This quotient essentially patches together the S_α sets. We define the mapping $\varphi_\alpha : Y_\alpha \to S_\alpha$ by $[z] \mapsto \Omega_\alpha(z)$. This is a bijective mapping, where z is the unique representative of $[z]$ in Z_α.

Also, given $[z] \in Y_\alpha \cap Y_\beta$, we notice $\varphi_\alpha([z]) = \Omega_\alpha(z)$ and $\varphi_\beta([z]) = \Omega_\alpha(z')$, with $z \; R \; z'$. Hence, $\varphi_\beta([z]) = \tau_{\beta\alpha} \circ \Omega_\alpha(z) = \tau_{\beta\alpha} \circ \varphi_\alpha([z])$ and so $\varphi_\beta \circ \varphi_\alpha^{-1} = \tau_{\beta\alpha}$. This gives $\varphi_\beta \circ \varphi_\alpha^{-1} \in H^\infty(\varphi_\alpha(Y_\alpha \cap Y_\beta))$, as required.

Thus, $\{(Y_\alpha, \varphi_\alpha) \mid \alpha \in I\}$ gives Y the structure of a Batchelor supermanifold. □

Remark. [Rog80] shows that Y with the sheaf of functions we constructed for an arbitrary Batchelor supermanifold is in fact isomorphic to $(M, \mathcal{O})$.

Type of Supermanifold	**Structure**	**Topology on** $B_L^{m,n}$	**Values for L**	**Transition Functions**	**Equivalent To**
Kostant-Berezin-Leites Supermanifold	Manifold with a sheaf	-	-	-	Batchelor
DeWitt Supermanifold	Set with an atlas	Coarse (DeWitt)	Finite or infinite	G^∞	-
Rogers Supermanifold	Set with an atlas	Fine (Product)	Finite or infinite	G^∞ or H^∞	-
Batchelor Supermanifold	Set with an atlas	Coarse (DeWitt)	Finite	H^∞	Kostant-Berezin-Leites

Table 1: Summary of the definitions of supermanifolds [Rog80].

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A Sheaf Theory

The theory of sheaves provides an elegant way to view differentiable manifolds and more general geometric objects. It is paramount in defining a supermanifold and so we introduce some of the basic theory here. This section follows closely the presentation in [EH00]

Definition A.1. Let $|M|$ be a topological space. A *presheaf* of commutative algebras $\mathcal{F}$ on $|M|$ is an assignment $U \mapsto \mathcal{F}(U)$, where U is open in $|M|$ and $\mathcal{F}(U)$ is a commutative algebra, such that the following holds:

1. If $U \subset V$ are two open sets in $|M|$, there exists a restriction morphism $r_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$, such that

    (a) $r_{U,U} = \text{id}$

(b) $r_{W,U} = r_{V,U} \circ r_{W,V}$ for all $U \subset V \subset W$.

If in addition the following holds, we call $\mathcal{F}$ a sheaf.

2. Given an open covering $\{U_i\}_{i \in I}$ of U and a family $\{f_i\}_{i \in I}$, $f_i \in \mathcal{F}(U_i)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there exists a unique $f \in \mathcal{F}(U)$ with $f|_{U_i} = f_i$.

The elements in $\mathcal{F}(U)$ are called *sections* over U; when $U = |M|$, we call such elements *global sections.*

Definition A.2. Let $\mathcal{F}$ be a presheaf on the topological space $|M|$ and let $x \in |M|$. Define the *stalk* $\mathcal{F}_x$ of $\mathcal{F}$ at the point x as the disjoint union of all pairs (U, s), where U is an open neighbourhood of x and s a section over U, modulo the equivalence relation: $(U, s) \cong (V, t)$ if and only if there exists a neighbourhood $W \subset U \cap V$ containing x, such that $s|_W = t|_W$.

The elements in $\mathcal{F}_x$ are called *germs of sections.*

Definition A.3. Let $\mathcal{F}, \mathcal{G}$ be presheaves on $|M|$. A *morphism of presheaves* $\phi : \mathcal{F} \to \mathcal{G}$ is a collection of morphisms $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for all open sets in $|M|$ such that the follow diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\phi_V} & \mathcal{G}(V) \\ {\scriptstyle r_{V,U}}\downarrow & & \downarrow{\scriptstyle r_{V,U}} \\ \mathcal{F}(V) & \xrightarrow{\phi_U} & \mathcal{G}(U) \end{array}$$

A morphism of sheaves is a morphism of the underlying presheaves. Any morphism of presheaves induces a morphism on the stalks: $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$.

Remark. A morphism of sheaves is *injective* if the induced stalk morphisms are all injective. Similarly, we can define surjectivity, however, we must be careful as we may have a surjective sheaf morphism $\phi : \mathcal{F} \to \mathcal{G}$ yet for some U, ϕ_U may not be surjective. Curiously, $U \mapsto \ker \phi(U)$ will always define a sheaf but $U \mapsto \operatorname{im} \phi(U) = \mathcal{F}(U)/\mathcal{G}(U)$ is only a presheaf in general. We now introduce the concept of *sheafification.*

Definition A.4. Let $\mathcal{F}$ be a presheaf on some topological space $|M|$. Define the *étalé space* of $\mathcal{F}$ to be the disjoint union $\sqcup_{x \in |M|} \mathcal{F}_x$. For every open set $U \subset |M|$ and each section $s \in \mathcal{F}(U)$, define the map

$$\hat{s}_U : U \to \sqcup_{x \in |U|} \mathcal{F}_x, \quad \hat{s}_U(x) = s_x.$$

We give to the étalé space the finest topology that makes $\hat{s}_U$ continuous for all open U and all sections s. Define then $\mathcal{F}_{et}$ to be the assignment

$$U \mapsto \mathcal{F}_{et}(U) = \{\hat{s}_U \ : \ U \in |M|, s \in \mathcal{F}(U)\}.$$

This is an explicit construction of the sheafification of $\mathcal{F}$.

Definition A.5. Let $\mathcal{F}$ be a presheaf on $|M|$. The *sheafification* of $\mathcal{F}$ is the unique sheaf $\widetilde{\mathcal{F}}$ and a morphism of presheaves $\varphi : \mathcal{F} \to \widetilde{\mathcal{F}}$ such that for all $x \in |M|$, $\varphi_x : \mathcal{F}_x \to \widetilde{\mathcal{F}}_x$ is an isomorphism.

Definition A.6. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of rings on some topological space $|M|$. If we have an injective morphism of sheaves such that $\mathcal{G}(U) \subset \mathcal{F}(U)$ for all open U, we define the *quotient sheaf* $\mathcal{F}/\mathcal{G}$ to be the sheafification of the image presheaf $U \mapsto \mathcal{F}(U)/\mathcal{G}(U)$.

Definition A.7. A *ringed space* is a pair $M = (|M|, \mathcal{F})$ consisting of a topological space $|M|$ and a sheaf of commutative rings $\mathcal{F}$. If each stalk, $\mathcal{F}_x$, is a local ring (it has a unique maximal ideal) then we say M is a *locally ringed space.*

Definition A.8. A morphism of ringed spaces $\phi : (|M|, \mathcal{F}) \to (|N|, \mathcal{G})$ is a pair $(|\phi|, \phi^*)$ where $|\phi| : |M| \to |N|$ is a topological space morphism and a $\phi^* : \mathcal{G} \to \phi_* \mathcal{F}$ is a sheaf morphism. Here $\phi_* \mathcal{F}$ is a sheaf defined on $|N|$ by $(\phi_* \mathcal{F})(U) = \mathcal{F}(|\phi|^{-1}(U))$ for every open $U \subset |N|$.

Every morphism of ringed spaces induces a morphism on the stalks, $\phi_x : \mathcal{G}_{|\phi|(x)} \to \mathcal{F}_x$ for all $x \in |M|$. If we have two locally ringed spaces, the morphism ϕ is called a locally ringed space morphism if $\phi_x^{-1}(\mathfrak{m}_{M,x}) = \mathfrak{m}_{N,|\phi|(x)}$, where $\mathfrak{m}_{M,x}, \mathfrak{m}_{N,|\phi|(x)}$ are the maximal ideals of the stalks $\mathcal{F}_x$ and $\mathcal{G}_{|\phi|(x)}$ respectively.

We now present an alternate definition of a smooth manifold using the theory of sheaves and locally ringed spaces.

Definition A.9. Let M be a Hausdorff, second countable topological space. Let $\mathcal{O}_M$ be a sheaf of commutative algebras on M such that $(M, \mathcal{O}_M)$ is a locally ringed space. We say $(M, \mathcal{O}_M)$ is a smooth manifold of dimension n if it is locally isomorphic as a locally ringed space to $(\mathbb{R}^n, \mathcal{C}_{\mathbb{R}^n})$, where $\mathcal{C}_{\mathbb{R}^n}$ is the sheaf of smooth functions on $\mathbb{R}^n$.

We want to see that the two definitions of a smooth manifold are equivalent. This fact requires three key ideas which we will now present as lemmas.

Lemma A.1. *Let M be a Haussforff, second countable topological space, equipped with a maximal smooth atlas $\mathcal{A} := \{(U_i, \phi_i)\}$. Then, we may construct a locally ringed space $(M, \mathcal{C}_M^\infty)$ where $\mathcal{C}_M^\infty$ is the sheaf of smooth functions on M.*

Proof. First consider the assignment $U \mapsto C^\infty(U)$ for each $U \subset M$ open, where $C^\infty(U)$ is the $\mathbb{R}$-algebra of smooth functions on U. Taking the regular restriction of functions, we may define our restriction morphism: $\mathrm{res}_{V,U}(f) = f|_U$ for open sets $U \subset V \subset M$. It follows that $\mathcal{C}_M^\infty$ is a presheaf. To see that it is in fact a sheaf, take an open cover of U, $\{U_i\}_{i \in I}$ and a family of smooth functions $\{f_i\}_{i \in I}$ with agreement between functions on the intersection of our sets. The topological pasting lemma gives a unique continuous function f such that $f|_{U_i} = f_i$ exactly as required. This f is smooth precisely as it restricts to smooth functions on U.

$(M, \mathcal{C}_M)$ is a ringed space by definition. For each $x \in M$, we define a mapping from the stalk $\mathcal{C}_{M,x} \to \mathbb{R}$ by $[(U, f)] \mapsto f(x)$. This is a surjective map as the constant functions are smooth. We know that the kernel of this map, K, is an ideal and so we can take the quotient of rings: $\mathcal{C}_{M,x}/K \cong \mathbb{R}$, the isomorphism following from the first isomorphism theorem for rings. As the above quotient ring is isomorphic to a field, the ideal K must be maximal. It is readily seen that any element not in K must be invertible and so it is the unique maximal ideal. This shows that $(M, \mathcal{C}_M)$ is a locally ringed space. □

Lemma A.2. $(M, \mathcal{C}_M^\infty)$ *is locally isomorphic as a locally ringed space to* $(\mathbb{R}, \mathcal{C}_{\mathbb{R}}^\infty)$.

Proof. We want to show that for all $x \in M$, there exists an open neighbourhood $U_i \ni x$ in M such that $\Phi : (U, \mathcal{C}_U^\infty) \to (V, \mathcal{C}_V^\infty)$ is an isomorphism of ringed spaces. Here $\mathcal{C}_U^\infty$ is the restriction sheaf to the subset

$U \subset M$, and $V \subset \mathbb{R}^n$ is open. That is, we want to find for every $x \in M$, a homeomorphism $\varphi : U \to V$ and an isomorphism of sheaves,

$$\Phi^* : \mathcal{C}^\infty_V \to \Phi_* \mathcal{C}^\infty, \quad (\Phi_* \mathcal{C}^\infty_U)(W) := \mathcal{C}^\infty_U(\varphi(W))$$

for some open $W \subset V$.

Indeed, for any $x \in M$, we have some $U_i \subset M$ that is open and contains x. Moreover, we immediately have the homeomorphism $\phi_i : U_i \to \phi_i(U_i) =: V_i$. We will define the sheaf isomorphism Φ^* by its constituent ring isomorphisms, $\Phi^*_{V_i} : \mathcal{C}^\infty_{V_i}(W) \to \mathcal{C}^\infty_{U_i}(\phi_i^{-1}(W))$ where $W \subset V_i$. For smooth f defined on $V_i \subset \mathbb{R}^n$, define $\Phi^*_{V_i}$ by $f \mapsto f \circ \phi_i$, the pullback. This clearly defines a ring isomorphism. Thus, we have constructed a sheaf isomorphism Φ^*.

As the ringed spaces we are dealing with are both locally ringed spaces, each stalk is a local ring and so our ring isomorphisms $\Phi^*_{V_i}$ will map units to units, preserving the local ring property. □

Theorem A.1. *Given a Haussdorff, second countable topological space M and a sheaf of commutative algebras $\mathcal{O}_M$ such that $(M, \mathcal{O}_M)$ is a locally ringed space isomorphic to $(\mathbb{R}^n, \mathcal{C}^\infty_{\mathbb{R}^n})$, we can always obtain a maximal smooth atlas on M so that $(M, \mathcal{C}^\infty_M) \cong (M, \mathcal{O}_M)$ where $\mathcal{C}^\infty_M$ is the sheaf of smooth functions on M.*

Proof. We are given a local isomorphism between $(M, \mathcal{O}_M)$ and $(\mathbb{R}^n, \mathcal{C}^\infty_{\mathbb{R}^n})$. That is, we have for each $x \in M$, some $U_i \ni x$ open in M and an isomorphism $\Phi : (U_i, \mathcal{O}_{U_i}) \to (V_i, \mathcal{C}^\infty_{V_i})$ consisting of a homeomorphism $\phi_i : U_i \to V_i$ and a sheaf isomorphism $\Phi^* : \mathcal{C}^\infty_{V_i} \to \Phi_* \mathcal{O}_{U_i}$. We can build our atlas from the topological maps on the open subsets of M: $\{(U_i, \phi_i)\}$. We need only check the transition maps are smooth.

Consider two open sets $U, V \subset M$ and their respective locally ringed space isomorphisms, Φ_1 and Φ_2. We will denote the respective topological maps by φ_1, φ_2 and the respective sheaf isomorphisms by Φ_1^*, Φ_2^*. Consider the restricted isomorphisms,

$$\Phi_1 : (U \cap V, \mathcal{O}_{U \cap V}) \to \left(\varphi_1(U \cap V), \mathcal{C}^\infty_{\varphi_1(U \cap V)}\right)$$
$$\Phi_2 : (U \cap V, \mathcal{O}_{U \cap V}) \to \left(\varphi_2(U \cap V), \mathcal{C}^\infty_{\varphi_2(U \cap V)}\right).$$

We can see that $\Phi_1 \circ \Phi_2^{-1}$ will be an isomorphism too. Define the open sets $W \subset \varphi_1(U \cap V)$ and $\widetilde{W} = \left(\varphi_1 \circ \varphi_2^{-1}\right)^{-1}(W)$. Then, the ring morphism $(\Phi_1 \circ \Phi_2^{-1})^*_W : C^\infty(W) \to C^\infty(\widetilde{W})$ is given by $f \mapsto f \circ \varphi_1 \circ \varphi_2^{-1}$. To see this, assume the contrary. Without loss of generality, we can translate f such that $(\Phi_1 \circ \Phi_2^{-1})^*_W(f)|_x = 0$ and $f \circ \varphi_1 \circ \varphi_2^{-1}(x) \neq 0$. In other words, $[f] \in \mathcal{C}^\infty_{\varphi_1 \circ \varphi_2^{-1}(x)}$ is a non-zero, invertible germ and $[(\Phi_1 \circ \Phi_2^{-1})^*_W(f)] \in \mathcal{C}^\infty_x$ is the zero germ. This is a contradiction as ring isomorphisms map units to units.

Finally, we have that $f \circ \varphi_1 \circ \varphi_2^{-1} : \widetilde{W} \to W \to \mathbb{R}$ is smooth for all $f : W \to \mathbb{R}$ and so $\varphi_1 \circ \varphi_2^{-1}$ is smooth.

This shows that transition maps in our atlas are smooth and so $(M, \mathcal{O}_M)$ has the structure of a smooth manifold. □

Remark. In the sheaf definition, we demand only a sheaf of commutative algebras whereas when coming from the classical definition of a smooth manifold, we constructed a sheaf of smooth functions. These end up being isomorphic as sheaves.

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