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Onset of Turbulence in a Channel Flow

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Abstract

Pipe flow is a common fluid dynamics application in transportation engineering. Inspired by Lord Reynolds [3], we investigate a steady fluid flow moving through a two-dimensional channel. The critical Reynolds number can help us understand the behaviour of the fluid flow in the pipe, whether it is a stable or an unstable flow. The approach to obtain the critical conditions for unstable behaviour is based on a linear stability analysis using the Navier–Stokes equation [1] and numerical methods based on Chebyshev approximations [7].

Contents

1 Introduction

The Navier–Stokes equations describe the motion of fluid flow with respect to time and position, and were first introduced by Claude-Louis Navier in 1822 and developed by George Gabriel Stokes during 1842-1850. These equations enable us to compute the velocity and pressure characteristics of fluid flow and allow us to explore the fluid movement inside a pipe.

When a fluid flows through a pipe, it can display different behaviour, and may be described as a smooth laminar flow or a chaotic turbulent flow (see figure(1)).

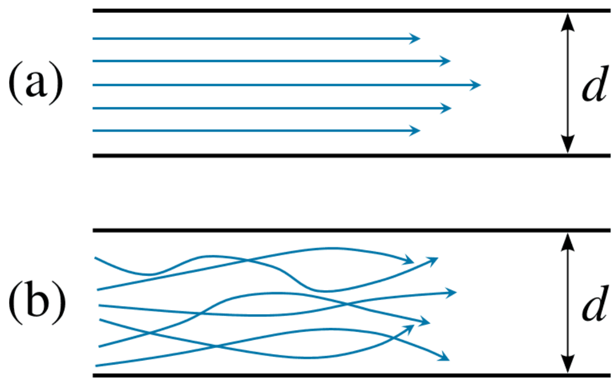

Figure 1: (a) Laminar flow, (b) Turbulent flow. (d = 2h, where h is the half-width) The turbulent flow is chaotic with random movement. [5]

From Lord Reynolds [3], the state of the flow can be determined by the dimensionless Reynolds number

$$Re = \frac{Ud}{\nu},$$

where U is the velocity of the fluid, d is the diameter of the pipe and ν is the kinematic viscosity of the fluid. Laminar flow is smooth and stable, the velocity of each component of the fluid follows the direction of the flow. The flow will remain in a laminar state when it has a sufficiently low Reynolds number. When the Reynolds number gets larger, and exceeds a critical Reynolds number, the flow is going to mix and become unstable. This is called laminar-turbulent transition. The flow will gradually transition to a turbulent state with higher velocity and random movement.

In this paper, we model the fluid flow in a two-dimensional channel, with a few assumptions as a simplification of the flow in a pipe. We obtain the laminar flow by revisiting the Naiver–Stokes equations and apply the no-slip condition. The laminar flow is known as Poiseuille flow. For the linear stability analysis, we perturb the laminar flow with a small normal-mode perturbation to derive the Orr–Sommerfeld equation [2] [6]. We undertake a temporal linear stability analysis of the flow to determine the conditions for the flow to become linearly unstable, which is known as the critical Reynolds number. We obtain the neutral curve by solving the Orr–Sommerfeld equation using Chebyshev derivative matrix approximations.

In the next section, we present the formulation, including the model, base flow, linear stability analysis and the Orr-Sommerfeld equation. In section 3, the temporal stability analysis and the corresponding numerical results are illustrated. And in the last section we draw conclusions for the report.

2 Formulation

2.1 Model

The two-dimensional channel flow model is clearly described by the diagram shown in figure 2.

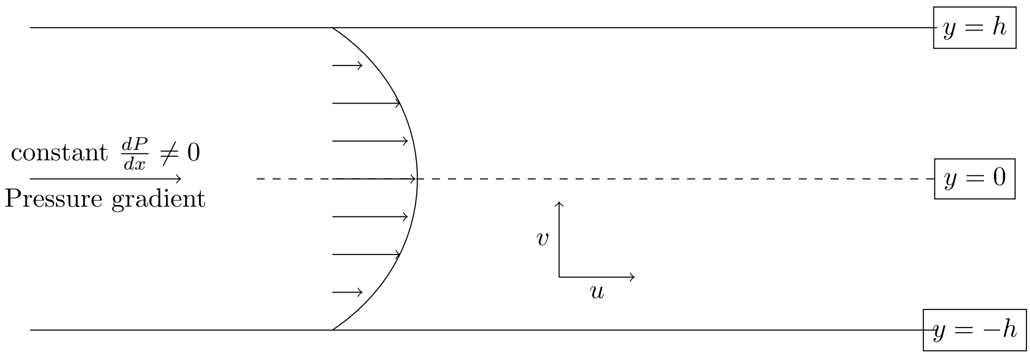

Figure 2: Two dimensional channel flow

We proceed to derive the model for channel flow with the following set up:

- A two dimensional channel with height $2h$. h is the channel half-width.
- The velocity field acting in the x-direction is defined by u and the velocity in the y-direction is defined by v.
- A constant pressure gradient $\frac{dP}{dx}$ is applied at the front of the channel and drives the flow. The constant pressure gradient is applied along the x- direction. This pressure-driven flow is known as Poiseuille flow.

2.2 Base flow

The Navier–Stokes equations in two-dimensions are given as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{2}$$

and the continuity equation [1] is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3}$$

where ν is the kinematic viscosity and ρ is the fluid density.

We can simplify equations (1)-(3) to obtain Poiseuille flow U_B by making the following assumptions [1]:

1. Laminar flow $\boldsymbol{U_B}$ is steady, which implies $\frac{\partial \boldsymbol{u}}{\partial t} = 0$.

2. The flow moves along the x-direction only,

$$V_B = 0 \quad \text{and} \quad \boldsymbol{U_B} = (U_B, 0).$$

3. The no-slip condition is applied on the channel walls, i.e. $U_B = 0$ on $y \pm h$.

4. At the channel centre $y = 0$, the flow achieves the maximum velocity, $U_B = U_0$.

From assumption 2, we know that $V_B = 0$, and equation (3) becomes

$$\frac{\partial U_B}{\partial x} = 0 \implies U_B = U_B(y).$$

The base flow U_B is a function of y only.

Similarly, equation (2) becomes

$$\frac{1}{\rho}\frac{\partial P}{\partial y} = 0 \implies P = P(x).$$

The pressure P is independent of y. Also from our model, $\frac{dP}{dx}$ is a constant.

Now we can look at equation (1), based on the steady flow assumption, we have

$$-\frac{1}{\rho}\frac{dP}{dx} + \nu \frac{d^2 U_B}{dy^2} = 0,$$
$$\frac{d^2 U_B}{dy^2} = \frac{1}{\rho\nu}\frac{dP}{dx}.$$

Recall that $\frac{dP}{dx}$ is a constant, and so this is an ODE for U_B. We can solve this ODE with the following boundary conditions.

From the no-slip assumptions

$$U_B(h) = 0 \quad \text{and} \quad U_B(-h) = 0.$$

The following expression for the base flow U_B is obtained

$$U_B(y) = \frac{P_x}{2\rho\nu}\left(y^2 - h^2\right).$$

We can also obtain U_0 by substituting in $y = 0$,

$$U_0 = -\frac{h^2 P_x}{2\rho\nu}.$$

Thus, the base flow can be simplified as

$$U_B(y) = U_0\left(1 - \frac{y^2}{h^2}\right).$$

Hence, the velocity vector is given by

$$\boldsymbol{U_B} = \left(U_0\left(1 - \frac{y^2}{h^2}\right), 0\right). \tag{4}$$

2.3 Non-Dimensionalisation

Introducing the dimensionless parameters,

$$\hat{u} = \frac{u}{U_0}, \qquad \hat{x} = \frac{x}{h}, \qquad \hat{t} = \frac{U_0 t}{h}, \qquad \hat{v} = \frac{v}{U_0}, \qquad \hat{y} = \frac{y}{h}, \qquad \hat{P} = \frac{P}{\rho U_0^2},$$

and substituting into equations (1)-(3), gives the dimensionless Navier–Stokes equations,

$$\frac{\partial \hat{u}}{\partial \hat{t}} + \hat{u}\frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{u}}{\partial \hat{y}} = -\frac{\partial \hat{P}}{\partial \hat{x}} + \frac{1}{Re}\left(\frac{\partial^2 \hat{u}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{u}}{\partial \hat{y}^2}\right), \tag{5}$$

$$\frac{\partial \hat{v}}{\partial \hat{t}} + \hat{u}\frac{\partial \hat{v}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{v}}{\partial \hat{y}} = -\frac{\partial \hat{P}}{\partial \hat{y}} + \frac{1}{Re}\left(\frac{\partial^2 \hat{v}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{v}}{\partial \hat{y}^2}\right), \tag{6}$$

$$\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} = 0, \tag{7}$$

where the Reynolds number for this channel flow is given as

$$Re = \frac{U_0 h}{\nu}.$$

The non-dimensional Navier–Stokes equations are governed by one parameter, the Reynolds number. We can control the flow behaviour by varying the value of the Reynolds number.

2.4 Linear Stability

We then perform linear stability analysis by adding a linear perturbation to the Poiseuille flow(4),

$$u = U_B + \epsilon u', \quad v = \epsilon v', \quad P = P_B + \epsilon P', \tag{8}$$

where $\epsilon \ll 1$.

The behaviour of the total flow depends on the perturbation. Recall (8), for $\boldsymbol{u} = (u, v)$, if $\boldsymbol{u'} \to \infty$, the flow is linearly unstable, and turbulence ensures. If $\boldsymbol{u'} \to \boldsymbol{0}$, the total flow will return to the base flow U_B, and the flow is linearly stable. To investigate the behaviour of the perturbations, we derive the Orr–Sommerfeld equation.

On substituting equation(8) into the scaled Navier–Stokes equation (5) gives

$$\frac{\partial}{\partial t}(U_B + \epsilon u') + (U_B + \epsilon u')\frac{\partial}{\partial x}(U_B + \epsilon u') + \epsilon v'\frac{\partial}{\partial y}(U_B + \epsilon u') =$$
$$-\frac{\partial}{\partial x}(P_B + \epsilon P') + \frac{1}{Re}\left(\frac{\partial^2}{\partial x^2}(U_B + \epsilon u') + \frac{\partial^2}{\partial y^2}(U_B + \epsilon u')\right).$$

The non-linear ϵ^2 term will vanish due to the small value of ϵ, giving

$$\epsilon\frac{\partial u'}{\partial t} + \epsilon U_B\frac{\partial u'}{\partial x} + \epsilon v'\frac{\partial U_B}{\partial y} = -\epsilon\frac{\partial P'}{\partial x} + \epsilon\frac{1}{Re}\left(\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2}\right),$$

and on cancelling ϵ from both sides, we obtain

$$\frac{\partial u'}{\partial t} + U_B\frac{\partial u'}{\partial x} + v'\frac{\partial U_B}{\partial y} = -\frac{\partial P'}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2}\right). \tag{9}$$

Similarly, we can obtain the following equations from (6) and (7):

$$\frac{\partial v'}{\partial t} + U_B \frac{\partial v'}{\partial x} = -\frac{\partial P'}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2}\right), \tag{10}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0. \tag{11}$$

(Note that all quantities have been scaled into their dimensionless form, i.e. $U_B = \left(1 - \frac{y^2}{h^2}\right)$ and $-1 \leq y \leq 1$.)

2.5 Orr–Sommerfeld equation

We will now introduce our normal mode assumption for the linear perturbations:

$$u' = \hat{u}(y) \exp\left(\mathrm{i}\left(\alpha x - \omega t\right)\right), \tag{12}$$

$$v' = \hat{v}(y) \exp\left(\mathrm{i}\left(\alpha x - \omega t\right)\right), \tag{13}$$

$$P' = \hat{P}(y) \exp\left(\mathrm{i}\left(\alpha x - \omega t\right)\right), \tag{14}$$

where α is the wavelength and ω is the frequency of the wavelike perturbation.

In addition, we introduce the stream function ϕ' that satisfies equation (11), where

$$u' = \frac{\partial \phi'}{\partial y} \quad \text{and} \quad v' = -\frac{\partial \phi'}{\partial x},$$

and ϕ' is defined as

$$\phi' = \hat{\phi}(y) \exp\left(\mathrm{i}\left(\alpha x - \omega t\right)\right).$$

On taking the y-derivative of equation (9) and x-derivative of equation (10), we can eliminate the pressure P' term:

$$\frac{\partial u'}{\partial t \partial y} + \frac{\partial U_B}{\partial y}\frac{\partial u'}{\partial x} + U_B \frac{\partial^2 u'}{\partial x \partial y} + \frac{\partial U_B}{\partial y}\frac{\partial v'}{\partial y} + v'\frac{\partial^2 U_B}{\partial y^2} - \frac{\partial v'}{\partial t \partial x} - U_B \frac{\partial^2 v'}{\partial x^2} = \frac{1}{Re}\left(\frac{\partial^3 u'}{\partial x^2 \partial y} + \frac{\partial^3 u'}{\partial y^3} - \frac{\partial^3 v'}{\partial x^3} - \frac{\partial^3 v'}{\partial y^2 \partial x}\right).$$

Then, on substituting our stream function ϕ', we obtain

$$\frac{1}{Re} D^4 \hat{\phi} + \left(\frac{2\alpha^2}{Re} + \mathrm{i}\omega - U_B \mathrm{i}\alpha\right) D^2 \hat{\phi} + \left(U_B \mathrm{i}\alpha^3 + \frac{\partial^2 U_B}{\partial y^2} - \mathrm{i}\omega\alpha^2 + \frac{\alpha^4}{Re}\right)\hat{\phi} = 0,$$

and on multiplying both sides by i we obtain the Orr–Sommerfeld equation

$$\left(\frac{\mathrm{i}}{Re} D^4 + \left(\alpha U_B - \frac{2\alpha^2 \mathrm{i}}{Re}\right) D^2 + \left(\frac{\mathrm{i}\alpha^4}{Re} - U_B \alpha^3 - \frac{\partial^2 U_B}{\partial y}\alpha\right)\right)\hat{\phi} = \omega\left(D^2 - \alpha^2\right)\hat{\phi}, \tag{15}$$

where D^n represents the n^{th} order derivative with respect to y.

2.6 Boundary Conditions

In section 2.5, we developed the Orr–Sommerfeld equation (15), which is a $4^{\mathbf{th}}$ order ordinary differential equation for the stream function $\hat{\phi}$. Hence, we require four conditions on $\hat{\phi}$ to solve this equation.

Recall the no-slip condition, on the boundary wall, requires the velocity be zero in both the x-direction and the y-direction. That is, when $y = \pm 1$, we obtain

$$u' = 0 \implies \frac{\partial \phi'}{\partial y} = 0 \implies D\hat{\phi} = 0,$$

and

$$v' = 0 \implies -\frac{\partial \phi'}{\partial x} = 0 \implies \hat{\phi} = 0.$$

Therefore, $\hat{\phi} = 0$ and $D\hat{\phi} = 0$ for $y \pm 1$ are the four boundary conditions.

3 Results

Using the Orr–Sommerfeld equation and the corresponding boundary conditions, we use computational methods to obtain the numerical solution.

3.1 Temporal Stability Analysis

Recall our assumption on the wavelike perturbation (equation (12) to (14)) and how the perturbation determines the behaviour of the total flow. We apply a temporal linear stability analysis, where we let the wavelength $\alpha \in \mathbb{R}$ and frequency $\omega \in \mathbb{C}$ for $\omega = \omega_r + \mathrm{i}\omega_{\mathrm{i}}$. Thus, we have the exponential part of the perturbation (equation (12) to (14)) becomes

$$\exp\left(\mathrm{i}\left(\alpha x - \omega t\right)\right) = \exp\left(\mathrm{i}\left(\alpha x - \omega_r t\right)\right)\exp\left(\omega_{\mathrm{i}} t\right).$$

The first exponential represents the wavelike part with imaginary exponents, while the second exponential determines the magnitude of the perturbation.

As $t \to \infty$, if $\omega_{\mathrm{i}} > 0$, then exponential growth occurs and the perturbation will approach infinity leading to linearly unstable behaviour. In another way, if $\omega_{\mathrm{i}} < 0$, then exponential decay happens and the perturbation approaches zero, leading to linearly stable behaviour.

Therefore, the case when $\omega_{\mathrm{i}} = 0$ is what we are interested in, where the flow transitions from being linearly stable to linearly unstable. We aim to investigate the corresponding conditions (i.e. Re, α and ω_r) for the critical point.

3.2 Compute ω

Recall the Orr–Sommerfeld equation (15), which we can write as

$$A\hat{\phi} = \omega B\hat{\phi}, \tag{16}$$

where A and B are matrices made up of the derivatives of $\hat{\phi}$.

The physical derivatives with respect to y are replaced by Chebyshev matrix approximations developed by Trefethen [7]. We can then solve equation (16) by treating it as an eigenvalue problem where ω is considered as the eigenvalues of the matrices A and B, and $\hat{\phi}$ are the corresponding eigenvectors.

For the Chebyshev matrix approximation, the size of the matrix is equal to the number of points used to fit in the Lagrange interpolation denoted by N, which could control the accuracy of the approximation on the physical derivative. The higher rank of the matrix, the more accuracy the approximation is, however the more expensive on computational cost. The numerical result with different N-value approximation is given in the table 1. $N = 100$ is chosen with consideration of trade-off between precision and costs.

N	ω	Re
200	$0.2612 + 0.00000935i$	5815
100	**0.2612+0.00000935i**	**5815**
50	$0.2612 + 0.00000943i$	5815
20	$0.2613 + 0.00000201i$	5530

Table 1: Numerical result for different matrix size.

3.3 Numerical Result

Initially, we consider the case where the wavenumber $\alpha = 1.01$ and the Reynolds number $Re = 5784$. The eigenvalues of equation (16) are computed using MATLAB (see appendix). The resulting frequency ω are plotted in the (ω_r, ω_i)-plane in figure 3.

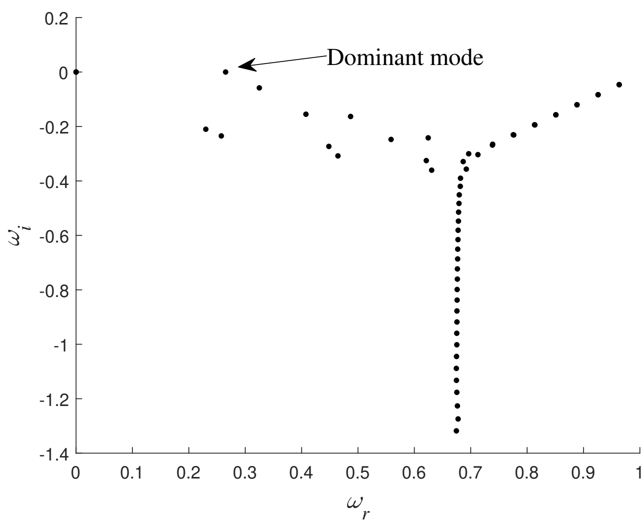

Figure 3: Eigenvalues for $\alpha = 1.01$ and $Re = 5784$

We notice that there is a continuous spectrum on the right-hand side and the discrete spectrum on the left-hand side. According to our temporal stability analysis, the eigenvalue with the least negative imaginary part will dominate the exponential growth of the linear perturbation. In this case the dominate mode is $\omega = 0.265245708327375 + 0.000010632451683i$, which is unstable with positive imaginary part.

Once we have the eigenvalue of the dominate mode, we can obtain the corresponding eigenvectors, which are the stream functions $\hat{\phi}$ plotted in figure 4. The stream functions are plotted using absolute value $|\hat{\phi}|$ and they are normalised to have a maximum of unity. Our boundary conditions can be verified, as at the boundary wall $y = \pm 1$, $\hat{\phi}' = 0$ and the gradient $D\hat{\phi}' = 0$.

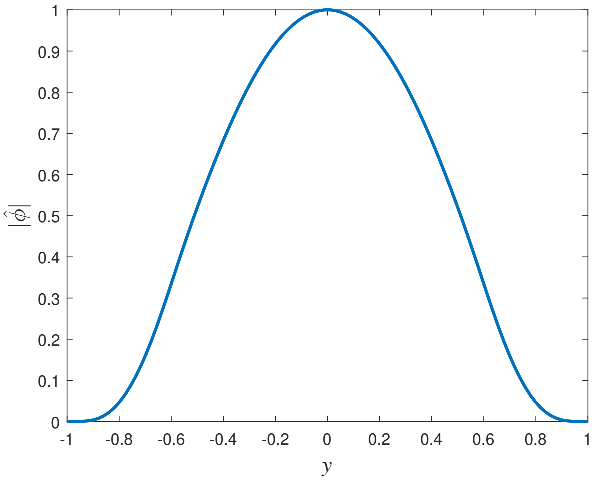

Figure 4: Absolute value of the stream function $\hat{\phi}$ for $\alpha = 1.01$ and $Re = 5784$.

The absolute value of the perturbations $u' = D\hat{\phi}$ and $v' = \mathrm{i}\alpha\phi$ are shown in figure 5 and figure 6 respectively.

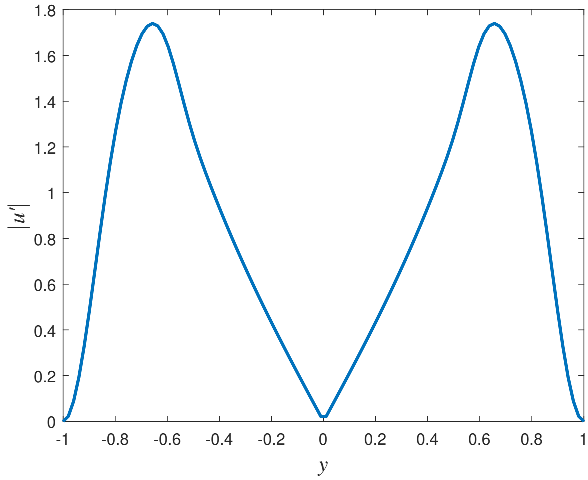

Figure 5: Absolute value of perturbation u' for $\alpha = 1.01$ and $Re = 5784$.

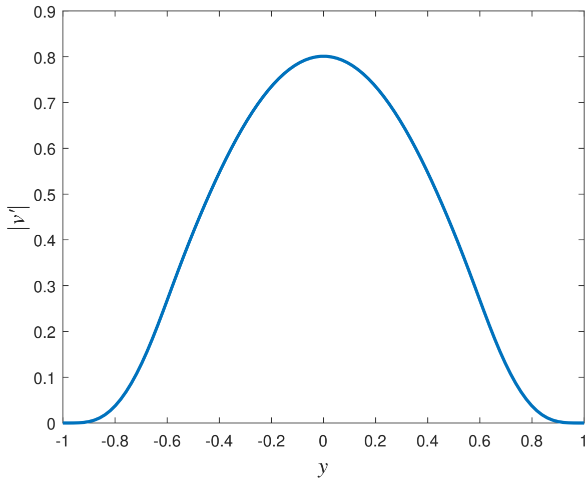

Figure 6: Absolute value of perturbation v' for $\alpha = 1.01$ and $Re = 5784$.

Notice that at the boundary walls, both $u' = 0$ and $v' = 0$, which verifies the no-slip conditions. In figure 5, at $y = 0$, we have the minimum perturbation on x-direction, and it reaches maximum at $y = \pm 0.67$. And in figure 6, we can find a scaled stream function $\hat{\phi}$ with same shape.

We repeat the above analysis to determine neutral conditions (Re, α, ω_r) for linear stability. For a fixed wavenumber α, the Reynolds number Re is increased until the eigenvalue with least negative imaginary part exceeds zero. We then use linear interpolation to compute the Reynolds number matched to $\omega_{\rm i} = 0$. We apply this method for wavenumbers $\alpha \in [0.6, 1.1]$, with stepsize $\Delta\alpha = 0.001$, to determine the corresponding Reynolds number and contruct a neutral curve where $\omega_{\rm i} = 0$, as shown in figure 7 and figure 8.

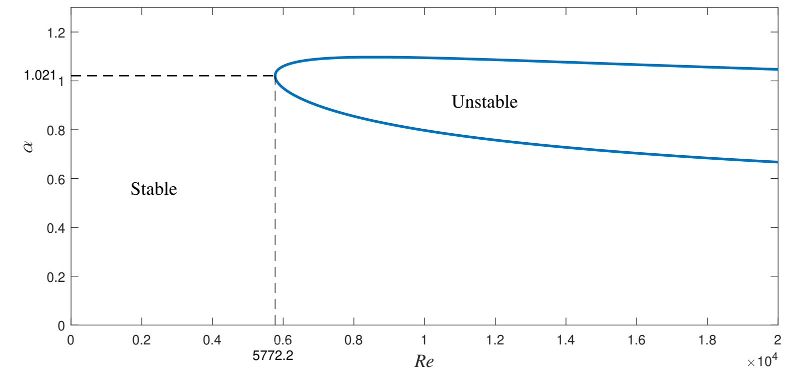

Figure 7: Neutral curve in the (Re, α) plane.

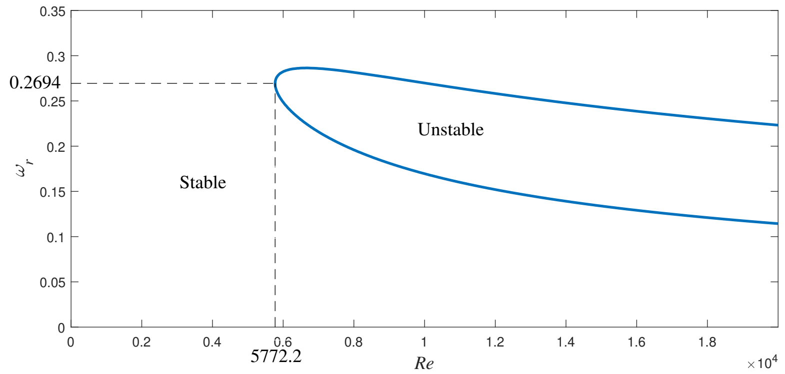

Figure 8: Neutral curve in the (Re, ω_r) plane.

For wavenumber α, frequency ω_r and Reynolds number Re, located inside the curve, the flow is recognized as linearly unstable, while outside the curve the flow is stable. The critical condition is obtained at $\alpha_c = 1.021$, $\omega_{r,c} = 0.2694$ and $Re_c = 5772.2$. We obtained similar results as Schmid and Henningson [4] as, where the Reynolds numbers are identical, the wavenumbers and the real parts of frequency are slightly larger, shown in table 2.

Source	α_c	Re_c	$\omega_{r,c}$
Our result	1.021	5772.2	0.2694
Schimid and Henningson	1.020	5772.2	0.2639

Table 2: Compared result for critical Reynolds number.

4 Conclusion

The linear stability of the two-dimensional channel flow was modelled using the Navier–Stokes equations. With no-slip conditions imposed, the base flow was computed. Using linear stability analysis, the Orr–Sommerfeld equation was obtained and solved via Chebyshev matrix approximations for the derivatives. Critical conditions were found and presented in a neutral curve. The critical Reynolds number was found as $Re_c = 5772.2$ and the corresponding wavenumber $\alpha_c = 1.021$ and frequency $\omega_{r,c} = 0.2694$. These critical conditions are close to Schmid and Henningson's result [4]. In conclusion, the channel flow is laminar and stable when the Reynolds number is sufficiently small ($Re < 5772.2$) but as the Reynolds number grows, the channel flow becomes turbulent and unstable ($Re > 5772.2$) for some specific α.

For further research, a change of geometry might be considered, such as the flow in a cylindrical pipe. Interesting thing is this flow is linear stable for all Reynolds numbers.

5 Appendix

```
% CHEB  compute D = differentiation matrix, x = Chebyshev grid

  function [x,D] = cheb(N)
  if N==0, D=0; x=1; return, end
  x = cos(pi*(0:N)/(N))';
  c = [2; ones(N-1,1); 2].*(-1).^(0:N)';
  X = repmat(x,1,N+1);
  dX = X-X';
  D  = (c*(1./c)')./(dX+(eye(N+1)));        % off-diagonal entries
  D  = D - diag(sum(D'));                    % diagonal entries
```

```
% This code solves the Orr-Sommerfeld equation for the Poiseuille flow in a
% channel, and determines neutral stability conditions
clear;
format long;
% Specify initial conditions
N = 100;          % Chebyshevs
as = 1.01;        % Initial alpha
ae = 1.01;        % Final alpha
da = 0.001;       % alpha stepsize
Res = 5000;       % Initial Reynolds number
delR = 10;           % Reynolds number stepsize

[Y,D] = cheb(N-1);  % Call cheb.m to setup derivative matrices
D2 = D^2;           % d2/dy2
D4 = D^4;           % d4/dy4

% Base flow
U = 1 - Y.^2;
D2U = -2;

% Imaginary value
ci = complex(0,1);

% We want to store neutral conditions
ntcrv = [];

% Loop through range of alpha values
for alpha = as:da:ae

alpha
Re = Res

% Initial guess for dominant eigenvalue
eval = -ci;

% Store all eigenvalues for each alpha - allows us to determine neutral
```

```
% conditions
feval = [];

% Solve Orr-Sommerfeld equation if imag(omega)<)
while imag(eval)<0

% Setup matrices for Orr-Sommerfeld equation
L = zeros(N);
K = zeros(N);

for i = 1:N
    for j = 1:N
        L(i,j) = [(alpha*U(i) - 2*ci*alpha^2/Re)*D2(i,j) + ci*D4(i,j)/Re];
        K(i,j) = D2(i,j);
        if i==j
            L(i,j) = L(i,j) + [ci*alpha^4/Re - U(i)*alpha^3 - D2U*alpha];
            K(i,j) = K(i,j) - alpha^2;
        end
    end
end

% Setup boundary conditions
for j = 1:N
    L(1,j) = 0;
    L(2,j) = 0;
    L(N-1,j) = 0;
    L(N,j) = 0;
    K(1,j) = 0;
    K(2,j) = D(1,j);
    K(N-1,j) = D(N,j);
    K(N,j) = 0;
end
K(1,1) = 1;
K(N,N) = 1;

% Determine eigenvalues 'e' and eigenvectors 'f' of the Orr-Sommerfeld
```

```
% equaition
[d,e] = eig(L,K);

% Determine the dominant mode
e = diag(e);
ind = 0;
for i = 1:N
    if imag(e(i)) > imag(eval)&&(real(e(i))>0.01)
        eval = e(i);
        ind = i;
    end
end

% Store eigenvalue
feval = [feval;Re eval];

% Update Reynolds number
Re = Re+delR

end

eval = -ci;

% While loop has ended, so determine critical Reynolds number via linear
% interpolation
Ren = Re-delR*(2*imag(feval(end))-imag(feval(end-1)))/(imag(feval(end))-imag
    (feval(end-1)));

% Determine critical real part of omega (again via linear interpolation)
fevaln = real(feval(end))-(real(feval(end))-real(feval(end-1)))*(Ren-Re)/
    delR;

% Store neutral conditions
ntcrv = [ntcrv; alpha Ren real(fevaln)]

% Update Reynolds number for next alpha value
```

```
Res = Re-100;

end
```

6 Reference

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