AMSI VACATIONRESEARCH SCHOLARSHIPS 2022–23

SET YOUR SIGHTS ON RESEARCH THIS SUMMER

Exploring the Euler Characteristics of Dessins d'Enfants

Paawan Jethva

Supervised by Finnur Lárusson & Daniel Stevenson

The University of Adelaide

28 February 2023

Abstract

Dessins d'enfants are graphs that are of great interest in diverse branches of mathematics including complex geometry, arithmetic algebraic geometry, and Galois theory. The relationship between these branches was established due to a theorem by Belyi in 1978. Famous Mathematician Alexander Grothendieck found these connections astonishing—which roused his interest and the popularisation of dessins d'enfants. In this paper, we define and study the Euler characteristics of dessins. In particular, we define the behaviour of the Euler characteristic under morphisms of dessins.

Contents

1 Introduction

Dessins d'enfants (often shortened to dessins) are graphs that are widely studied because they bridge three prominent branches of mathematics: complex geometry, arithmetic algebraic geometry, and Galois theory. Zapponi (2003) provides a brief summary of some of these connections. In complex geometry, dessins naturally arise from finite coverings of the Riemann sphere by a Riemann surface, ramified over three points only. By Belyi's theorem (Belyi, 2002), every algebraic curve defined over the field $\bar{\mathbb{Q}}$ of algebraic numbers contains an embedded dessin. This theorem was the source of famous mathematician Alexander Grothendieck's fascination towards dessins, who popularised their study in the late nineteenth century. At the time, Grothendieck was studying the Galois group of $\bar{\mathbb{Q}}$ over $\mathbb{Q}$, which acts on the coefficients of the equation of algebraic curves over $\bar{\mathbb{Q}}$ and hence, on dessins—spurring his interest towards them.

Seeing their remarkable relationships to diverse areas of mathematics gives us enough motivation to study their properties. In this paper, we will aim to define and investigate the Euler characteristic of a dessin and thus we will be treating them as purely combinatorial objects. Examples of dessins are shown in fig. 1 which will be elucidated in the next section where we formally define them.

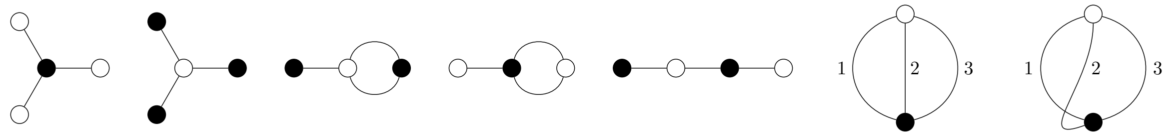


Figure 1: The dessins with three edges. Inspired by Lando and Zvonkin (2004).

Statement of Authorship

This paper reports on the research I conducted during the period of my AMSI Vacation Research Scholarship (VRS). The background was studied collaboratively with fellow VRS students Tom Dee and Lachlan Schilling who also conducted research on dessins d'enfants under the supervision of Finnur Lárusson and Daniel Stevenson. The definition of a dessin was given to us by Finnur Lárusson. However, the following results were independently derived by me with the guidance of my supervisors and inspiration from discussions with my confrères.

2 The Definition of a Dessin d'Enfant

2.1 Informal Characterisation of a Dessin

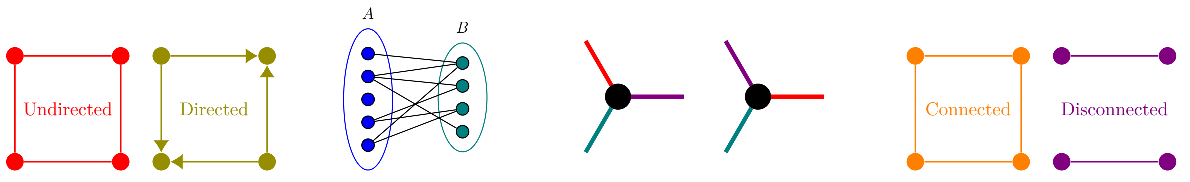
Informally, a dessin d'enfant is a graph that has the following properties: undirectedness, bipartition (represented by colouring vertices black and white), cyclic ordering of edges around the vertices, and connectedness (allowing multiple connections).

An undirected graph is simply a set of vertices (or nodes) and a set of edges that connect two vertices. We call them 'undirected' (or less commonly, 'bidirected') to distinguish them from their 'directed' counterparts which specify a direction for each edge (from a source to a target vertex).

A bipartite graph (often elided to bigraph) is a graph whose vertices can be divided into two sets, say A and B, such that edges only connect vertices from different sets. This is illustrated in fig. 2b. As we can see, drawing them this way makes them unnecessarily tortuous. Therefore, we typically represent vertices from different sets by colours and avoid overlaps between edges where possible for intelligibility. Conventionally, we choose black and white to colour the vertices of a dessin.

In dessins, we also consider the order in which the edges come into the vertex. Since a vertex is circular, the order is cyclic; there is no first or last edge, but each edge has a predecessor and successor. Scilicet, they form a permutation cycle. By convention, we determine the cyclic order going anticlockwise around the vertex. In fig. 2c, we see two vertices that seem to be connected to the same edges. However, the order in which they come into the vertices is different. The first vertex has order (• • •) while the second has (• • •).

A graph is connected if, from any given vertex, there is a path to every other vertex. As a simple counterexample, the graph in fig. 2b has a vertex with no edges hence, it is not connected. Often, however, we encounter graphs that are composed of multiple connected components, as seen in fig. 2d. In the case of a dessin, we allow multiple connections, that is multiple edges between the same two vertices.



(a) Undirected vs directed (b) Bipartite (c) Cyclic ordering (d) Connected vs disconnected

Figure 2: Example graphs displaying the properties of a dessin.

This is an adequate definition by itself; it entirely characterises dessins. However, it is challenging to work with. It is not pleasant to derive properties from such a loose definition. What would a structure-preserving map (morphism) between dessins be? We would need to preserve the four structural elements listed above, and it is rather easy to confirm that undirectedness, bipartition and connectedness are preserved, but it is difficult to discern when cyclic ordering is preserved. For example, we may believe the constant map α and inclusion map β in fig. 3 to be valid morphisms. However, the composition $\beta\alpha : x, y \mapsto x$ fails to preserve cyclic ordering because x comes after y in the source, but in the target, $y \notin \beta(\{x, y\})$ comes after $\beta(y) = x$. The reason we insist on maps to preserve structure is to be able to compose them. Since the composition does not work, we can safely assume that at least one of the component maps is not a morphism. However, we cannot determine which particular map is not a morphism. Rather than exerting more effort to ascertain which map is not a morphism, let us instead construct a more concrete definition of a dessin in terms of objects we may be more familiar with.

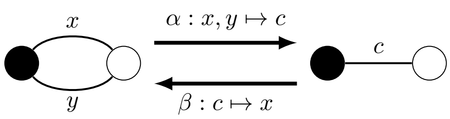


Figure 3: Example of a maps (not necessarily morphisms) between two dessins.

2.2 Formally Defining a Dessin

As we noted, the structural elements other than cyclic ordering are straightforward to understand. So, let us first focus on alternatively characterising cyclic ordering.

We saw earlier that the cyclic ordering of edges around a vertex induces a permutation cycle. We also note that since edges connect a black vertex to a white vertex, two black vertices cannot share an edge. Hence, the black vertices partition the set of edges. This means that the cycles induced by the black vertices, say $c_1, c_2, \ldots, c_n$, are disjoint. So, we can freely multiply these cycles in any order to get a unique permutation in cycle notation. This 'black' permutation is enough to characterise the black vertices, and we can follow the same procedure to characterise the white vertices by a 'white' permutation. Now, to ease the requirement of these permutations being in cycle notation, we can characterise the vertices to be the orbits of the permutations.

To summarise, given two permutations, call them 'black' and 'white', we can correspond their orbits to a vertex of the respective colour, the elements in those orbits to the edges connected to the vertex, and the cycle induced by the orbit to the cyclic ordering of the edges around that vertex. That is, we can correspond the pair of permutations to a graph. The graph is undirected since we do not define a direction for the edges, it is bipartite as we distinguish between the orbits (vertices) of the two permutations, and it defines a cyclic ordering of the edges around each vertex. So, we have managed to characterise two more properties of a dessin in the process of characterising cyclic ordering. We have also partially characterised connectedness because we have eliminated the possibility of isolated vertices (vertices with no connections) by characterising a vertex as an orbit of edges (orbits are nonempty since $x \in \mathrm{Orb}(x)$). We can characterise the remainder of the connectedness property purely in terms of the two pairs of permutations as well. Let B be the set of all orbits of the black permutation and W be the the set of all orbits of the white permutation, then the induced dessin is connected if and only if for all proper subsets $B' \subset B$ and $W' \subset W$, $B' \neq W'$. Again, this is a fine characterisation, but we will experience similar challenges as before with this definition; it is difficult to derive properties using this characterisation. (We omit the proof as we discard the definition.)

So, simply characterising a dessin as a pair of permutations does not lend us with a 'nice' definition. Then, let us try to package these permutations in a 'nicer' object. For example, we can extend the pair of permutations to the subgroup $\langle p_b, p_w \rangle \leq \mathrm{Sym}(E)$, where p_b and p_w are the black and white permutations respectively, E is the edge set of the dessin and $\mathrm{Sym}(E)$ is the symmetric group over E (or equivalently, the group of bijections of E). Then, we can characterise the remainder of the connectedness property by enforcing the subgroup $\langle p_b, p_w \rangle$ to have only one orbit. To clarify, this means from any edge, we can reach any other edge by applying a permutation from $\langle p_b, p_w \rangle$. A permutation $p \in \langle p_b, p_w \rangle$ can be written as a word composed of only p_b, p_w, and their inverses, so applying a permutation p to an edge corresponds to traversing a dessin along edges (switching to the next edge by rotating anticlockwise around a vertex). Since, under our definition, each vertex is connected to at least one edge (isolated vertices do not exist), this implies that from every vertex, we can reach any other edge by traversing edges—which is exactly the definition of connectedness.

To consider orbits of the subgroup $\langle p_b, p_w \rangle$, we implicitly extended the subgroup to a $\langle p_b, p_w \rangle$-set, where the subgroup is acting on the edge set E. Recall that a G-set is a set X with a group action by G which can be characterised as a group homomorphism $G \to \mathrm{Sym}(X)$. Here, we take the group action $\langle p_b, p_w \rangle \to \mathrm{Sym}(E)$ to be the inclusion. To say that $\langle p_b, p_w \rangle$-set only has a single orbit is exactly to say that it is transitive. So, we can define a dessin to be a transitive $\langle p_b, p_w \rangle$-set for suitable permutations p_b and p_w. However, the number of edges $|E|$ and cyclic ordering can vary between dessins. So, we would need a different subgroup for each of these cases. However, all of these subgroups are generated by two elements. Instead of a subgroup $\langle p_b, p_w \rangle$, we can pick the biggest possible group over two generators. That is, the free group over two generators $F_2 = \langle b, w \rangle$ for arbitrary elements b and w. Hence, we obtain our 'nice' definition of a dessin.

Definition 2.1 (Dessin)**.** A *dessin* is a transitive F_2-set where F_2 is the free group over $\{b, w\}$.

Under our initial characterisations of a dessin (as a pair of permutations), the 'empty' dessin (with no edges) could have existed, but under our later characterisation of dessins as groups, they do not since groups are nonempty. In the following remark, we discuss the existence of such an empty dessin under our final definition of a dessin as transitive F_2-sets.

Remark 2.2 (Forbiddance of the empty dessin)**.** For a fixed group G, we typically allow the existence of the trivial G-action on the empty set since the axioms are written in terms of the universal quantifier ('for all') and hence, hold vacuously. However, we typically disallow this trivial action to be transitive. There are three commonly used definitions of transitivity. The action of a group G on a set X is transitive if:

1. for all $x, y \in X$, there is $g \in G$ such that $y = gx$,
2. for all $x \in X$, $\mathrm{Orb}(x) = X$,
3. for one $x \in X$, $\mathrm{Orb}(x) = X$.

It is easy to see that the first two definitions are equivalent, but their equivalence to the final definition fails for the empty set. Again, the first two vacuously hold for the empty set while the final one does not.

We can summarise the first two definitions as "the action has 'at most' one orbit", while the last definition can be stated as "the action has 'exactly' one orbit." The only action that has zero orbits is the trivial action on the empty set. Thus, the discrepancy only arises for the empty set. At this stage, we have two options: to allow the trivial action to be transitive and reject the final definition, or to disallow the the trivial action to be transitive and reject the first two definitions. Most mathematicians opt for the latter because theorems concerning transitive G-sets transpire to be more fruitful when we preclude the trivial action from being transitive. For example, the decomposition of a G-set into a disjoint union of transitive G-sets is unique only if the trivial action is not transitive. nLab authors (2023) details a discussion of this concept for trivial objects in general.

When using one of the first two definitions of transitivity, a prudent author explicitly includes the nonemptiness condition. When using the final definition, the other two are often derived as properties.

We heed the wisdom of more experienced mathematicians and do not allow the trivial action to be transitive. Hence, under definition 2.1, the 'empty' dessin (with no edges) does not exist.

We have a convenient definition of a dessin; let us now summarise how to construct the picture of a dessin from this definition.

Construction 2.3 (Pictorial representation of a dessin)**.** For a dessin:

- the *edge set* E_F is the set that F_2 acts on;
- the *black vertices* are the orbits of the b-action of F_2 and the *white vertices* are the orbits of the w-action;
- the *cyclic ordering* around a vertex is determined by the cycle induced by the corresponding orbit.

This also clarifies the choice of b and w for the generators of F_2; b for 'black' and w for 'white'. We see the construction process in action in the following example.

3 Morphisms Between Dessins

3.1 Defining a Dessin Morphism

We have defined our objects of interest—dessins. Now, we want to define the structure-preserving maps (morphisms) between these objects to further study their properties and their relationship with each other. Luckily, we have defined a dessin to be a (transitive) F_2-set (definition 2.1), and a morphism between two F_2-sets is simply an F_2-equivariant map. Hence, we obtain the following definition.

Definition 3.1 (Dessin morphism)**.** A *dessin morphism* is an F_2-equivariant map. That is, for a dessin morphism $\alpha : F \to G$, $\alpha(pe) = p\alpha(e)$ for all $e \in E_F$ and $p \in F_2$.

Roughly speaking, applying the F_2-action before the morphism is the same as applying it after—which aligns with our usual understanding of 'preserving structure.'

3.2 Properties of Morphisms

Before proceeding to study the properties of dessins, let us take a moment to study the properties of dessin morphism which will aid us in our forthcoming investigation. Our first theorem showcases the rarity dessin morphisms.

Theorem 3.2. *Dessin morphisms are surjective.*

Proof. Suppose $\alpha : F \to G$ is a dessin morphism. Let $y_1 = \alpha(x_1) \in \alpha(E_G) \subseteq E_G$ and $y_2 \in E_G$ for some $x_1 \in E_F$. We want to show that $y_2 = \alpha(x_2)$ for some $x_2 \in E_F$. By transitivity of the F_2-action, there exists $p \in F_2$ such that $y_2 = py_1$. Then, by F_2-equivariance of α, $y_2 = py_1 = p\alpha(x_1) = \alpha(px_1)$. Letting $x_2 = px_1 \in E_F$, we get $y_2 = \alpha(x_2)$. □

Since morphisms need to be surjective this eliminates the possibility of many prospective morphisms. For example, a morphism from a dessin with n edges to a dessin with m edges cannot exist if $n > m$ because it needs to be surjective. In particular, inclusion maps, such as α in fig. 3, do not exist.

In the proof, we only use transitivity of the F_2-action and F_2-equivariance of morphisms; we never use the fact that the group is F_2. So, in fact, this theorem generalises to all transitive G-sets for a fixed group G. Note that the theorem fails for the inclusion morphism of the empty dessin. However, we have no particular need for such a dessin in this paper. So accordingly, we dissallowed it—as stated in remark 2.2.

From this result, we obtain a corollary which characterises isomorphisms of dessins.

Corollary 3.3. *A dessin morphism is injective if and only if it is an isomorphism.*

Proof. Suppose $\alpha : F \to G$ is an injective dessin morphism. Then, by theorem 3.2, it is bijective. Thus, there exists an inverse map $\alpha^{-1} : G \to F$. We want to show that α^{-1} is a dessin morphism. Let $x \in E_F$, $y \in E_G$ and $p \in F_2$. Suppose $\alpha(x) = y$. So, $\alpha(px) = p\alpha(x) = py$. Then, applying the inverse, we get $px = \alpha^{-1}(p\alpha(x)) = \alpha^{-1}(py)$. However, $x = \alpha^{-1}(y)$. Hence, $p\alpha^{-1}(y) = \alpha^{-1}(py)$. □

We now define some features of morphisms which we will use in the next section to study the effect of morphisms on certain properties of dessins. We begin by defining a *fibre* of a dessin morphism.

Definition 3.4 (Fibre of a morphism)**.** A *fibre* of a dessin morphism $\alpha : F \to G$ is the preimage $\alpha^{-1}(e)$ of a single edge $e \in E_G$.

From the above definition and definition 2.1, we obtain the following result about fibres.

Theorem 3.5. *The cardinalities of all fibres of a given dessin morphism are equal. That is, given a morphism $\alpha : F \to G$, for all $y_1, y_2 \in E_G$, $|\alpha^{-1}(y_1)| = |\alpha^{-1}(y_2)|$.*

Proof. Suppose $\alpha : F \to G$ is a dessin morphism. Let $y_1, y_2 \in E_G$ and $x_1 \in E_F$ such that $\alpha(x_1) = y_1$. By transitivity of the F_2-action, there exists $d \in F_2$ such that $y_2 = dy_1$. Let $x_2 = dx_1 \in E_F$. Then, by F_2-equivariance of α, $\alpha(x_2) = \alpha(dx_1) = d\alpha(x_1) = dy_1 = y_2$. So, we have shown that for each $x_1 \in \alpha^{-1}(y_1)$, we can produce $x_2 \in \alpha^{-1}(y_2)$. That is, we can define a surjective function $\alpha^{-1}(y_2) \to \alpha^{-1}(y_1)$ by $x_2 \mapsto x_1$. (Indeed, this is a well-defined function because the F_2 has inverses so, if $x_2 = dx_1 = dx_3$, then $d^{-1}x_2 = x_1 = x_3$.) Therefore, we obtain the inequality $|\alpha^{-1}(y_1)| \le |\alpha^{-1}(y_2)|$ for the cardinality of fibres. However, $y_1 = d^{-1}y_2$ and $x_1 = d^{-1}x_2$. So, by the same argument, $|\alpha^{-1}(y_1)| \ge |\alpha^{-1}(y_2)|$. Hence, $|\alpha^{-1}(y_1)| = |\alpha^{-1}(y_2)|$. □

Again, we only use transitivity and equivariance so, this theorem holds for transitive G-sets in general. From this theorem and theorem 3.2, follows a necessary condition for a dessin morphism to exist.

Corollary 3.6. *If $\alpha : F \to G$ is a dessin morphism, then $|E_G| \,\Big|\, |E_F|$.*

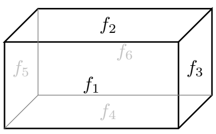
In the following definition, we give a name to the common cardinality of theorem 3.5 for ease of reference.

Definition 3.7 (Degree of a morphism)**.** The *degree* of a dessin morphism is the common cardinality of its fibres.

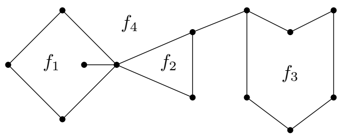
4 Euler Characteristic of a Dessin

4.1 Defining a *Face* of a Dessin

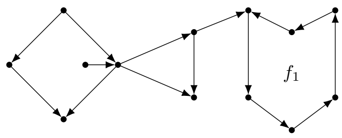
The first major obstacle we encounter in defining the Euler characteristic of a dessin is defining a *face* of a dessin. To motivate a definition of a dessin face, let us revisit faces of objects that we are more familiar with. Our most primitive notion of faces comes from polyhedra such as the cuboid seen in fig. 4a. We know that a face of a polyhedron is simply a two-dimensional region bounded by edges (such that the regions do not overlap). Then, we can consider an undirected graph as shown in fig. 4b. For faces of undirected graphs, we also have to bound two-dimensional regions with edges. However, to bound them, we need to construct a *simple circuit*: a nonempty sequence of distinct edges joining vertices such that only the first and last vertex are the same. As a result, we also obtain an outer face f_4 in fig. 4b. For directed graphs, the faces are again simple circuits but this time, they also have to respect direction—meaning edges must be traversed in their assigned direction. Correspondingly, we lose many faces from fig. 4b to fig. 4c due to our assignation of direction. (Although, we often distinguish between directed faces and faces because both happen to be useful concepts in the theory of directed graphs.)



(a) Polyhedron



(b) Undirected graph



(c) Directed graph

Figure 4: Faces of familiar objects.

So, we have seen that a face is simply a bounded two-dimensional regions however, to bound them, we need to respect the structure of the object under consideration. The primary structure of dessins is cyclic ordering. Thus, we need to construct a simple circuit on a dessin such that each succeeding edge is determined by the cyclic ordering of the corresponding vertex (which is defined anticlockwise). In fig. 5, we see two similar dessin but one has three faces while the other only has one. The face f_1 of the left dessin is traced in red as an example; the rest are left as an exercise. Note that we could start anywhere on the face, but as long as we abide by the cyclic ordering, we will be tracing the same face.

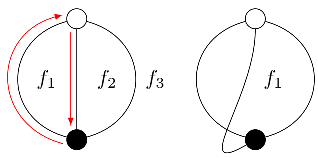


Figure 5: Faces of two dessins with three edges. The face f_1 of the left dessin is traced in red.

One may notice that during the counting of all faces of a dessin, each edge is traversed exactly twice—once in either direction. It would instead be more convenient to have each edge traversed exactly once. Thus, we define a *directed edge* of a dessin which will simplify later definitions and discussions.

Definition 4.1 (Directed edges)**.** A *directed edge* of a dessin is a pair (e, d) for $e \in E$, the edge set of a dessin, and $d \in \{b, w\}$, the generating set of F_2. We call d the *direction* of the directed edge.

We may interpret traversing a directed edge (e, d) as traversing the edge e from the $(c \neq d)$-coloured vertex to the d-coloured vertex as shown in fig. 6. So, d represents the colour of the target vertex which will become obvious from the following definitions. (Note that we only state the colour of one vertex as the other is implied by the bipartition property and we choose to state the target rather than the source colour for convenience in the following definitions.) We may refer to a *directed edge* simply as *edge*, if it is obvious from context that it is directed.

Figure 6: Traversing a directed edge (e, w) (from the b-coloured vertex to the w-coloured vertex).

As aforementioned, a *face* of a dessin is a simple circuit of directed edges in the dessin. In the following definition, we express exactly this notion but in terms of definition 2.1.

Definition 4.2 (Face)**.** A *face* of a dessin is a cyclic permutation of directed edges $\big((e_1, d_1)\,(e_2, d_2)\,\cdots\,(e_n, d_n)\big)$ such that $e_{i+1} = d_i e_i$ and $d_{i+1} \neq d_i$ for all $i \in \mathbb{Z}_n$.

That is, a face of a dessin is a cycle of edges with alternating directions (between black and white). If need be, we may sometimes make a choice of $d_1 = b$ without loss of generality, and write $\big((e_1, b)\,(e_2, w)\,\cdots\,(e_n, w)\big)$ for a general face. Note that $\mathbb{Z}_n$ is the ring of residue classes modulo n.

Remark 4.3. Since we are using 1-based indexing instead of 0-based indexing, we define $n \mod n = n$. That is, we take n instead of 0 as the representative of the residue class $[n] = [0]$ in $\mathbb{Z}_n$.

The benefit of our definition of faces is that we can use induction on them. We have defined a face as a recursively-defined cycle and a cycle is simply a type of periodic sequence. Although, we often distinguish between the two because, for a periodic sequence, we define the initial value, but for cycles we do not. However, we can make any choice of element in the cycle as the initial value to induce a periodic sequence. Formally, we can characterise a cycle of length p as an equivalence class of periodic sequences with period p and p unique elements. Then, we can choose a representative of the class (which corresponds to choosing an initial value), and then perform induction on that, but such formalism is a little unnecessary for our purposes.

Since a face is a recursively-defined cycle (i.e. a periodic sequence), we reduce the information we need to explicitly state and develop shorter notation.

Notation 4.4 (Face)**.** There are three primary notations for faces of dessins that we will adopt. For a face $\big((e_1, d_1)\,(e_2, d_2)\,\cdots\,(e_n, d_n)\big)$, we will write

1. $[e_1, d_1]$ in *initial directed edge notation* which we will abbreviate to *initial edge notation* where, as the name suggests, we simply choose a representative (initial) directed edge from the cycle,

2. $d_1(e_1\, e_2\, \cdots\, e_n)$ in *directed cycle notation* where we write the cycle of edges and apply the direction of the first element in the cycle to the left, or

3. $\big((e_1, d_1)\,(e_2, d_2)\,\cdots\,(e_n, d_n)\big)$ in *full notation* where we write the full cycle of directed edges.

Note that we use round brackets $(\cdots)$ for directed edges and square brackets $[\cdots]$ for an initial edge. As a convention, we will prefer smaller over larger edges and black over white directions, in that order, as representatives.

4.2 Properties of a Dessin Face

We have defined a new feature of a dessin (a face), so why not take a moment to study some of its properties? We begin by defining the *size* of a face in the obvious way as we will want to refer to it in upcoming theorems.

Definition 4.5 (Size of a face). For a dessin face $f = d_1(e_1\, e_2\, \cdots\, e_n)$, the *size* of the face is $|f| = |[e_1, d_1]| = n$.

We obtain an obvious proposition about the size of a face from the definition, which is often useful in confirming that a face has been counted correctly.

Proposition 4.6. *The size of a dessin face is even.*

Proof. Let $f = \big((e_1, b)\,(e_2, w)\,\cdots\,(e_n, w)\big)$ be a face. For each edge $(e_i, b) \in f$, there is an edge $(e_{i+1}, w) \in f$ for $i \in \mathbb{Z}_n$ and vice versa. So, if $n_b, n_w \in \mathbb{N}$ are the number of edges in f in the b and w direction respectively, then we know that $n_b + n_w = n$ and we have shown that $n_b = n_w$. Hence, $n = 2n_b$. □

The following is an important theorem that tells us that faces have no repeat directed edges (meaning faces are well-defined) and they do not overlap with each other.

Theorem 4.7. *For directed edges $(e_1, d_1), (e_2, d_2)$, the relation $(e_1, d_1) \sim (e_2, d_2)$ if $[e_1, d_1] = [e_2, d_2]$ is an equivalence. Importantly, faces partition the set of directed edges of a dessin.*

Proof. Let $(e_1, d_1), (e_2, d_2), (e_3, d_3)$ be directed edges.

Reflexivity: $[e_1, d_1] = [e_1, d_1]$.

Symmetry: if $[e_1, d_1] = [e_2, d_2]$, then $[e_2, d_2] = [e_1, d_1]$.

Transitivity: if $[e_1, d_1] = [e_2, d_2]$ and $[e_2, d_2] = [e_3, d_3]$, then $[e_1, d_1] = [e_3, d_3]$. □

That is, two directed edges are equivalent if they are in the same face. The proof seems trivial due to our notation for a face. However, our notation was lended to us by the recurrence relation. So, in fact, the recurrence relation is the reason why faces partition the set of directed edges. The important point here is that each directed edge contributes to exactly one face. This is a minute but important detail we will need to complete later proofs.

We obtain a useful corollary from this theorem about the sizes of faces. However, to make the corollary more useful, we strengthen it with the result of the following lemma.

Lemma 4.8. *Let E be the edge set and D be the directed edge set of a dessin, then $|D| = 2|E|$.*

Proof. For each edge $e \in E$, there exist only two directed edges $(e, b), (e, w) \in D$. Alternatively, $D = E \times \{b, w\}$ hence, $|D| = |E| \cdot |\{b, w\}| = 2|E|$. □

Now, we prove the corollary of theorem 4.7.

Corollary 4.9. *Let $\{f_1, f_2, \ldots, f_n\}$ be the set of faces of a dessin, E be its edge set and D be its directed edge set, then $\sum_{i=1}^{n} |f_i| = |D| = 2|E|$.*

Proof. By theorem 4.7, faces partition the set of directed edges, so $\sum_{i=1}^{n} |f_i| = |D|$. By lemma 4.8, $|D| = 2|E|$. □

This corollary is often useful in determining if all faces have been counted. To simplify the next theorem, we first define how a morphism acts on a directed edge.

Definition 4.10 (Morphism on directed edges)**.** Let $\alpha : F \to G$ be a dessin morphism. For a directed edge $(x, d) \in D_F$, we define $\alpha(x, d) = (\alpha(x), d) \in D_G$.

The morphism on directed edges is well-defined because the morphism is just acting on the constituent edge of the directed edge. This is purely notation which will make the following theorem more comprehensible. The following theorem informs us about the behaviour of faces under morphisms.

Theorem 4.11. *Let $\alpha : F \to G$ be a dessin morphism and $\Big((x_1, d_1)\,(x_2, d_2)\,\cdots\,(x_n, d_n)\Big)$ a face of F. Then, $\{\alpha(x_1, d_1), \alpha(x_2, d_2), \ldots, \alpha(x_n, d_n)\}$ is the orbit of a face of G.*

Proof. Let $[x_1, d_1] = \Big((x_1, d_1)\,(x_2, d_2)\,\cdots\,(x_n, d_n)\Big)$ be a face in F and $\alpha(x_i) = y_i \in E_G$ for all $i \in \mathbb{Z}_n$. By theorem 4.7, (y_1, d_1) contributes to exactly one face $[y_1, d_1]$. Suppose $(y_i, d_i) \in [y_1, d_1]$. Then, by F_2-equivariance of α, $y_{i+1} = \alpha(x_{i+1}) = \alpha(d_i x_i) = d_i \alpha(x_i) = d_i y_i$. So, $(y_{i+1}, d_{i+1}) = (d_i y_i, d_{i+1})$ where $d_{i+1} \neq d_i$ as they are consecutive in f and directions alternate in a face. Thus, $(y_{i+1}, d_{i+1}) \in [y_1, d_1]$. Hence, by the principle of mathematical induction, $(y_i, d_i) \in [y_1, d_1]$ for all $i \in \mathbb{Z}_n$.

Let $O(f)$ denote the orbit of a face f. We have shown that the image $\alpha(O([x_1, d_1])) \subseteq O([y_1, d_1])$. We will now show $O([y_1, d_1]) \subseteq \alpha(O([x_1, d_1]))$. Suppose $(z_i, d_i) \in O([y_1, d_1])$. Then, by surjectivity of α (theorem 3.2), $(z_i, d_i) = \alpha(x_i, d_i)$ for some directed edge $(x_i, d_i) \in O([x_1, d_1])$. Hence, $\alpha(O([x_1, d_1])) = O([y_1, d_1])$. □

Loosely, this theorem tells us that faces map to faces. Thus, we define that concept formally.

Definition 4.12 (Morphism on faces)**.** Let $\alpha : F \to G$ be a dessin morphism. For a face $[x, d]$ of F, we define $\alpha([x, d]) = [\alpha(x), d]$.

This is well-defined due to theorem 4.11. If theorem 4.11 did not hold, we could have had $x_1 \neq x_2 \in E_F$ with $[x_1, d] = [x_2, d]$ while $[\alpha(x_1), d] \neq [\alpha(x_2), d]$.

So now, we have updated our definition of a dessin morphism from an F_2-equivariant map, say α_E, defined on edges to a family of maps including α_E, and another two maps α_D on directed edges and α_F on the set of faces. There is one structure of a dessin that we have not yet defined the morphism on: vertices. Although this section is on the properties of faces, we need to address the natural question, "Does an analogous result hold for vertices?" so that we could define a map on vertices to add to our morphism. Indeed it does. Vertices are just orbits of edges and in particular they are subsets of edges. Thus, the image of α_E is defined on it. Recall that the image under a map $X \to Y$ induces a map $P(X) \to P(Y)$, where $P(S)$ is the power set of a set S. We just have to make sure that the image of a vertex is indeed a vertex. In the subsequent theorem, we prove this more generally for any orbit.

Theorem 4.13. *Let $\alpha : F \to G$ be a dessin morphism and θ an orbit of the p-action on the edge set E_F for some $p \in F_2$. Then, the image $\alpha(\theta)$ is an orbit of the p-action on the edge set E_G.*

Proof. It is a well-known fact that orbits are preserved under equivariant maps, but suppose we are not familiar with this result. Rather than proving the entire result, we will instead characterise an orbit in a similar way to dessin faces to argue that the proof is analogous to theorem 4.11.

Recall that every orbit $\theta = \{e_1, e_2, \ldots, e_n\}$ induces a cycle which we assume to be $(e_1\, e_2\, \cdots\, e_n)$ without loss of generality. (We can always order θ such that this is true.) Note that for $i \in \mathbb{Z}_n$, $e_{i+1} = pe_i$ in the cycle since θ is an orbit of the p-action. Thus, we can equivalently characterise θ as a cycle of pairs $\big((e_1, p)\,(e_2, p)\,\cdots\,(e_n, p)\big)$ where $e_{i+1} = pe_i$ for $i \in \mathbb{Z}_n$. From here, the proof is analogous to theorem 4.11. $\square$

Now, as a simple corollary, we derive our desired result of the preservation of vertices under morphisms.

Corollary 4.14. *Let $\alpha : F \to G$ be a dessin morphism and v a d-coloured vertex of F. Then, the image $\alpha(v)$ is a d-coloured vertex of G.*

Thus, we define a dessin morphism on vertices which is well-defined by the above corollary (for the same reason that definition 4.12 is well-defined due to theorem 4.11).

Definition 4.15 (Morphism on vertices)**.** Let $\alpha : F \to G$ be a dessin morphism. For a d-coloured vertex $v = \{e_1, e_2, \ldots, e_n\}$, we define $\alpha(v) = \{\alpha(e_1), \alpha(e_2), \ldots, \alpha(e_n)\}$.

We could have opted for a straightforward proof of theorem 4.13. However, our chosen method emphasises the similarity between the definitions of faces and vertices. Thus, as a natural progression, we may whether a face can be characterised by an orbit of some action of F_2. This is precisely the content of the upcoming theorem. To state the theorem, we need to define another feature of a face.

Definition 4.16 (Shadow of a face)**.** Let $f = b(e_1\, e_2\, \cdots\, e_n)$ be a dessin face. The *shadow* of f is the set $\bar{f} = \{e_1, e_3, \ldots, e_{n-1}\}$ of all edges in the b direction.

We choose the name shadow as it is the black projection of a black and white face. We could have equivalently defined an *aureola* of a face to be the set of all edges in the w-direction in the face, but we have no need for both concepts so we make a choice of defining the shadow over an aureola. Before stating the promised theorem, we state an obvious but important proposition to justify our interest in shadows.

Proposition 4.17. *The shadow of a faces uniquely characterises the face. Importantly, if f is the number faces of a dessin and s is the number of its shadows, then $f = s$.*

Proof. Let $\bar{f} = \{e_1, e_3, \ldots, e_{n-1}\}$ be a shadow of a face. The shadow is the set of b-directed edges of a face. Hence, the face corresponding to $\bar{f}$ is $f = [e_1, b]$. It is unique due to theorem 4.7. □

Now, we arrive at the portended theorem which provides us with a convenient way to count faces by courtesy of the above proposition.

Theorem 4.18 (The Eminence in Shadow)**.** *The shadow of a dessin face is an orbit of the wb-action of F_2.*[1]

Proof. Suppose $f = \Big((e_1, b)\,(e_2, w)\,\cdots\,(e_n, w)\Big)$ is a face with shadow $\bar{f} = \{e_1, e_3, \ldots, e_{n-1}\}$. Recall that $e_{i+1} = d_i e_i$ for all $i \in \mathbb{Z}_n$ where d_i is the direction of the corresponding directed edge in the face. So, $e_{i+2} = d_{i-1} d_i e_i$ for all $i \in \mathbb{Z}_n$. Now, recall that $d_{i+1} \neq d_i \in \{b, w\}$ for all $i \in \mathbb{Z}_n$. So, if $i = 2j - 1$ for $j \in \mathbb{Z}_{n/2}$ (namely, i is odd), then $e_{2j+1} = d_{2j-2} d_{2j-1} e_{2j-1} = wbe_{2j-1}$. The final equality holds because we have defined f to have b-directed edges at odd indices and w-directed edges at even indices. (Note that we implicitly used proposition 4.6 to ensure $n/2 \in \mathbb{N}$ while defining $\mathbb{Z}_{n/2}$.) Overall, we found that $e_{2j+1} = wbe_{2j-1} \in \bar{f}$ for all $j \in Z_{n/2}$. Hence, $\bar{f}$ is an orbit of the wb-action of F_2. □

At this point, one may think, "Why not take the 'shadow' as the definition of a face rather than a property?" Orbits are certainly more convenient to work with than definition 4.2. We are hesitant to do so due to two related reasons. First, definition 4.2 is motivated in an intuitive manner. Conversely, defining a face as an orbit of the wb-action may seem arbitrary. Second, the edges in a shadow may not necessarily enclose a two-dimensional region of the pictorial representation of a dessin as exemplified in fig. 7.

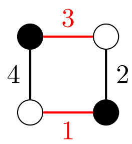


Figure 7: A dessin with a 'disconnected' shadow (shown in red) of the outer face $[1, b]$.

4.3 The Euler Characteristic Under Morphisms

We are finally prepared to study the Euler characteristic of dessins. In this section, we will build towards a theorem explaining the behaviour of the Euler characteristic under morphisms. There are few steps towards the theorem as most of the constituent theorems we will leverage have been proven in the previous section. The first step towards the theorem would be to define the Euler characteristic of a dessin as we are yet to do that. We do this in the usual way.

Definition 4.19 (Euler characteristic of a dessin)**.** Let v be the number of vertices, e the number of edges and f the number of faces of a dessin. Then, its Euler characteristic is $\chi = v - e + f$.

[1]The naming choices are not to be construed as an exemplification of my chūnibyō tendencies. It just so happened that I was watching *The Eminence in Shadow* Aizawa and Nakanishi (2022) at the time I was contemplating the theorem.

Now, we prove a lemma about the effect of morphisms on the number of vertices by leveraging corollary 4.14.

Lemma 4.20. *Let $\alpha : G \to G'$ be a dessin morphism with $\deg(\alpha) = n \in \mathbb{N}$ and v and v' be the number of faces of G and G' respectively. Then, $v' \leq v \leq nv'$.*

Proof. By corollary 4.14 and theorem 3.2, for every vertex v_i' in G', there is at least one vertex v_i in G. Thus, $v' \leq v$. Now, we want to find the maximum number of vertices in G that can be mapped to a single vertex in G'. Let v_i' in G' be a vertex and suppose it has cardinality $|v_i'| = m \in \mathbb{N}$. Then, the cardinality of the preimage is $|\alpha^{-1}(v_i')| = nm$ due to definition 3.7. By corollary 4.14 and theorem 3.2, if v_i is a vertex of G with $\alpha(v_i) = v_i'$, then $|v_i| \geq |v_i'| = m$. Thus, we have converted the original problem into a partition problem. We want to know the maximum number of (nonzero) parts $\geq m$ that partition nm. This is precisely n where we take all parts to be m. Hence, $v \leq nv'$. □

We state a similar lemma for faces but we omit the proof as it is analogous by considering shadows.

Lemma 4.21. *Let $\alpha : G \to G'$ be a dessin morphism with $\deg(\alpha) = n \in \mathbb{N}$ and f and f' be the number of faces of G and G' respectively. Then, $f' \leq f \leq nf'$.*

We now prove the foreshadowed theorem about the effect of morphisms on the Euler characteristic.

Theorem 4.22. *Let $\alpha : G \to G'$ be a dessin morphism with $\deg(\alpha) = n \in \mathbb{N}$ and χ and χ' be the Euler characteristics of G and G' respectively. Then, $\chi \leq n\chi'$.*

Proof. Let v, e, f and v', e', f' be the number of vertices, edges and faces of G and G' respectively. By definition 3.7 and theorem 3.2, $e = ne'$. By lemma 4.20 and lemma 4.21, $v \leq nv'$ and $f \leq nf'$. Hence, $\chi = v - e + f \leq nv' - ne' + nf' = n\chi'$. □

Note that we could also derive a lower bound from lemma 4.20 and lemma 4.21 to get the full inequality $\chi' - (n-1)e' \leq \chi \leq n\chi'$. However, the lower bound is not purely defined in terms of Euler characteristics as we also need to refer to the number of edges e' of G'. The lower bound will not be useful for our purposes in this paper hence, we have excluded it from the theorem.

So we have this theorem which supposedly gives us information about the behaviour of the Euler characteristic under morphisms, but what exactly is this behaviour? It becomes easier to understand this behaviour if we knew a couple more facts which are that the Euler characteristic $\chi \leq 2$ and even for all dessins. We do have empirical evidence of these facts in table 1, but I am deeply embarrassed to admit that I am yet to find a proof under our definitions. Although, proofs do exist under alternative definitions, and it is a simple exercise to establish the equivalence of definitions. Confer Lando and Zvonkin (2004) Definition 1.8 (Topological Map) for a characterisation of graphs (including dessins) as embeddings on surfaces, where they define a *surface* as "a compact oriented two-dimensional topological manifold". Then they define the Euler characteristic of the graph as simply the Euler characteristic of the surface the graph is embedded onto and it is a well-known fact that such surfaces have a nonnegative genus $g \in \mathbb{Z}^{\geq 0}$ and an Euler characteristic $\chi = 2 - 2g$.

Using this fact and theorem 4.22, one can easily deduce that $\chi \leq \chi'$ because χ' is nonpositive most of the time with the only exception of $\chi' = 2$ which can be considered as a special case and confirmed to also hold. Thus, theorem 4.22 is telling us that the Euler characteristic may only decrease under morphisms. However, it actually says a bit more than that. It also tells us the minimum amount the Euler characteristic must be reduced by under the morphism. Loosely speaking, we can think of the Euler characteristic as a measure of complexity (the number of overlaps and their intricacy) of a dessin; the lower the Euler characteristic, the more 'complex' the dessin. Then, our theorem is telling us that morphisms 'simplify' dessins and it also tells us the minimum amount they need to be simplified by. For example, we could have a dessin G with Euler characteristic $\chi = -8$ and say we want to map it to a dessin G' with half the number of edges (so $\deg(\alpha) = \frac{|E_G|}{|E_{G'}|} = 2$), then the new dessin G' has to have an Euler characteristic of $\chi' \geq -4$.

5 Some Statistics of Dessins

While working with dessins; defining morphisms, counting faces, calculating Euler characteristics; we realise that calculations quickly become tedious. Luckily, in the twenty-first century, we have ease of access to quite powerful computers which we can outsource our calculations to. Such was the plan for my code repository (Jethva and Schilling, 2023). However, while we are at the computer, why not try generating all possible dessins (with n edges)? We present some of the statistics for all dessins with up to $n = 7$ edges in this section.

Prior to delving into the statistics, we have a definition. At the end of the previous section, we mentioned that the the Euler characteristic of dessin is $\chi = 2 - 2g$ where we consider the dessin to be embedded in an orientable surface with genus g. We do not have a proof for this fact under our definitions which is why we have not stated it as a theorem. However, we may still define the *genus* of a dessin in terms of the Euler characteristic. This definition will make the following statistics easier to interpret.

Definition 5.1 (Genus of a dessin)**.** The *genus* of a dessin with Euler characteristic χ is $g = 1 - \chi/2$.

No. of Edges	No. of Dessins	$g = 0$	$g = 1$	$g = 2$	$g = 3$
1	1	1			
2	3	3			
3	7	6	1		
4	26	20	6		
5	97	60	33	4	
6	624	291	285	48	
7	4163	1310	2115	708	30

Table 1: Number of dessins with n edges (up to $n = 7$) and their partition by genus g.

In table 1, we record the total number of dessins with n edges for each $n \in \mathbb{N}$ up to $n = 7$, and also the partition of that number by genus. One may question how the dessins of higher genus appear pictorially. Hence, they are exemplified in table 2. Recall that theorem 4.22 told us that morphisms go in the left direction of table (or stay within the same column) and it also told us the minimum amount they need to go in that direction.

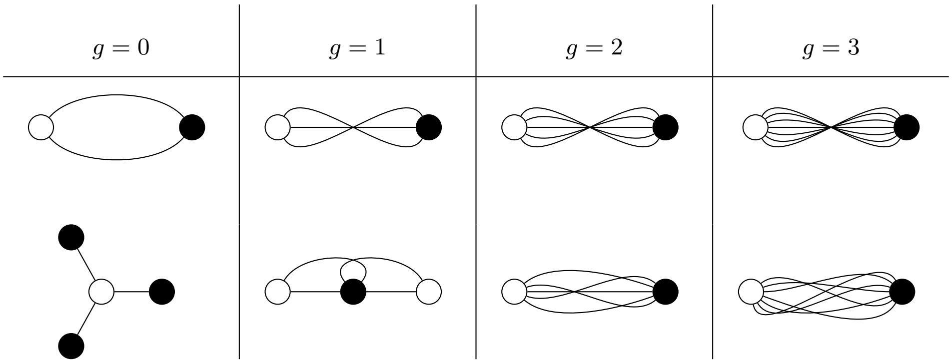


Table 2: Examples of dessins for each genus up to $g = 3$.

We can see some interesting patterns in table 1. Firstly, we see that the genus is always a nonnegative integer $g \in \mathbb{Z}^{\geq 0}$ which suggests the Euler characteristic is ≤ 2 and even for all dessins as we predicted in the previous section. The sequence of the total number of dessins can be another topic of interest. We may enter the sequence on the On-Line Encyclopedia of Integer Sequences (OEIS) to find that it is listed as sequence A057005 with a comment stating it is the "[n]umber of (unlabeled) dessins d'enfants with n edges" (OEIS Foundation Inc., 2000), which corroborates some of our results. Another interesting pattern we may notice in the table is that the maximum genus seems to increase at each odd number of edges. Thus, we may conjecture the following.

Conjecture 5.2. A dessin with genus g has a minimum possible of $2g + 1$ edges.

Again, it pains me to affirm that I am yet to find a proof for this conjecture. I do have a partial proof however. We can simply calculate the genus of the sequence of dessins in the top row of table 2 (dessins with identical b- and w-actions $(1\,2\,\cdots\,n)$) and find that it has the maximum possible genus for its number of edges (that is, if the dessin has $2g + 1$ edges, then it has genus g). However, it is difficult to show that a dessin of that genus cannot occur for a lesser number of edges.

6 Conclusion

In this paper, we successfully defined the Euler characteristic of a dessin in purely combinatorial terms. We then derived some properties of dessins from their Euler characteristics. In particular, we described the behaviour of the Euler characteristic of dessins under morphisms. We ended with an open conjecture about the minimum possible Euler characteristic for a dessin.

Acknowledgements

I would like to express my gratitude towards my supervisors Finnur Lárusson and Daniel Stevenson for their unrelenting support and guidance throughout the research period. I would also like to acknowledge my peers Tom Dee and Lachlan Schilling for their cordial collaboration and compulsive discussions.

References

Aizawa, D. and Nakanishi, K. (2022). *Kage no Jitsuryokusha ni Naritakute!* [trans. *The Eminence in Shadow*]. Nexus.

Belyi, G. (Apr. 2002). "Another proof of the Three Points theorem". *Sbornik: Mathematics* 193. DOI: 10.1070/SM2002v193n03ABEH000633.

Jethva, P. and Schilling, L. (2023). *AMSI VRS Dessins d'Enfants*. https://github.com/TheSurgeonOfDeath/AMSI-VRS-Dessins-d-Enfants.

Lando, S. and Zvonkin, A. (Jan. 2004). *Graphs on Surfaces and Their Applications*. Vol. 141. ISBN: 978-3-642-05523-2. DOI: 10.1007/978-3-540-38361-1.

nLab authors (2023). *too simple to be simple*. https://ncatlab.org/nlab/show/too+simple+to+be+simple. [Revision 18].

OEIS Foundation Inc. (2000). *Sequence A057005*. https://oeis.org/A057005.

Zapponi, L. (2003). "What is a Dessin d'Enfant?" *Notices of the American Mathematical Society* 50 (7), pp. 788–789. URL: https://www.ams.org/notices/200307/what-is.pdf.