The Category of Dessins d'Enfants: Investigating Monomorphisms and Epimorphisms

Thomas Dee

Supervised by Finnur Lárusson & Daniel Stevenson

University of Adelaide

Abstract

When viewed as a combinatorial object, a dessin d'enfant (named by Grothendieck to mean 'child's drawing') can be described as a bipartite graph with cyclic ordering on its vertices. In this project, we study dessins from a category theoretic perspective. To do so, we first establish the category of dessins by characterising dessins as transitive F_2-sets, and the morphisms between them as equivariant maps. Second, we investigate the nature of the epimorphisms and monomorphisms in this category. It is shown that all morphisms are epimorphisms, and all monomorphisms are isomorphisms in the category of dessins.

Contents

1 Introduction

This project is about first, establishing a category of dessins d'enfants and second, describing monomorphisms and epimorphisms in this category. Category theory is an abstract area of pure mathematics, it is concerned with the study of structure and how it can be preserved. Category theory can be used to better understand the more nuanced subject of dessins d'enfants. Named in French by Alexander Grothendieck in the 1980s, a dessin d'enfant (child's drawing) is a type of bipartite graph. Grothendieck's study of dessins was motivated by the study of the absolute Galois group (González-Diez & Jaikin-Zapirain, 2015). Indeed, dessins can be considered 'graphs embedded on surfaces' (Guillot, 2014). The theory of dessins is both far reaching and beyond the scope of this project; we view dessins simply as combinatorial objects.

1.1 Statement of Authorship

This research was completed as a group project, the group was composed of three undergraduate students from the University of Adelaide; Paawan Jethva, Lachlan Schilling, and the author. Our supervisors were Prof Finnur Lárusson and Dr Danial Stevenson. All three students contributed to the development of the ideas that are presented below. And both supervisors provided a lot of guidance, particularly in establishing the category of dessins. The supervisors were able only to briefly proofread this report because the author left them such little time to do so.

2 Some Relevant Category Theory

To understand what category theory is about and why it is studied, consider the following metaphor. The relationship that category theory has with other fields of pure maths, like topology and group theory, is different to the sort of relationship these fields have with each other. Category theory is high above the landscape of maths, taking 'a bird's eye view' (Leinster, 2014). From this less focused perspective, we can study the underling structure in an area of interest and we can see analogous structures in unrelated areas. Thus, despite its lack of nuance, category theory is a powerful tool with which to study mathematics.

Structure preserving maps are a pervasive dogma in pure mathematics. In category theory, these are called morphisms, they are fundamental to the study of categories. The concept is further generalised by functors and natural transformations. Furthermore, dessins d'enfants can be characterised as natural transformations between functors, however in this report, a different perspective is taken. Consequently, we need only to consider the definition of a category.

Definition 1. A **category** $\mathscr{C}$ is made up of the following:

- A collection of objects $\mathrm{Ob}(\mathscr{C})$,
- with for each $A, B \in \mathrm{Ob}(\mathscr{C})$, a collection of morphisms from A to B, called $\mathrm{Hom}_{\mathscr{C}}(A, B)$,
- including, an identity morphism $\mathrm{id}_A \in \mathrm{Hom}_{\mathscr{C}}(A, A)$ for each $A \in \mathrm{Ob}(\mathscr{C})$,
- whereby, for each morphism $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathscr{C}}(B, C)$, there is a morphism $g \circ f : A \to C$ called the **composite** of f and g.

And a category satisfies these two axioms:

C1. Composition of morphisms is **associative**: That is, if $f \in \mathrm{Hom}_{\mathscr{C}}(A,B)$, $g \in \mathrm{Hom}_{\mathscr{C}}(B,C)$, and $h \in \mathrm{Hom}_{\mathscr{C}}(C,D)$ then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

C2. The identities satisfy the following **identity law**. For each $f \in \mathrm{Hom}_{\mathscr{C}}(A,B)$, $f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f$.

Notation. For brevity, we take $A \in \mathscr{C}$ to mean $A \in \mathrm{Ob}(\mathscr{C})$. And we take $g \in \mathscr{C}(A,B)$ or $g : A \to B$ to mean $g \in \mathrm{Hom}_{\mathscr{C}}(A,B)$.

3 Dessins d'Enfants

For the purpose of this project, dessins d'enfants are best understood as graphs with special structure. This is exemplified in Figure 1, where a diagram is used to represent a dessin. Indeed, a dessin satisfies the definition of a graph (Definition 2), but the extra structure shown in the diagram is important, this is formalised in Definition 3.

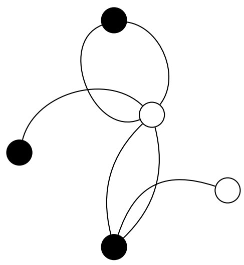

Figure 1: Diagram of a dessin.

Definition 2. A **graph** is a non-empty, finite set of vertices and a finite *family* of edges. Each edge is an unordered pair of vertices.

Definition 3. A **dessin d'enfant** (or simply **dessin**) is a *graph* which satisfies the following properties:

1. Bipartite: The set of vertices is partitioned (black and white). Each edge connects a black vertex to a white vertex.
2. Connected: Between any two vertices, is a path (a sequence of edges between adjacent vertices where no vertex appears more than once).
3. Cyclic ordering: At each vertex, the anticlockwise ordering of the edges incident to that vertex are specified.
4. There is no dessin with a single vertex.

Remark. A dessin can have multiple edges, but due to the bipartite condition, a dessin cannot have loops.

Remark. There are no empty (no edges) dessins; the empty dessin with zero vertices is already disqualified by Definition 2, which specifies the set of vertices is *non-empty*. All empty dessins with one vertex are disqualified by Definition 3. And the empty dessins with more than one vertex are disqualified by connectedness in Definition 3.

Remark. In this project, all dessins have a finite number of vertices and a finite number of edges. This is made sure by Definition 2. However, note that in some other cases a local finiteness is sufficient.

3.1 A Characterisation of Dessins as Transitive F_2-sets

To proceed in establishing the category of dessins, it is reasonable to find a characterisation of dessins that is concise and easy to work with. It will be explained in sections 3.1.1-3.1.3 that dessins are characterised by transitive F_2-sets.

3.1.1 The Set of Edges of a Dessin

A dessin can have multiple edges between the same two vertices. Consequently, if the edges are considered unordered pairs of vertices, then a dessin with multiple edges has a multiset of edges. However, each edge can be considered a distinct object, this gives rise to a set of edges.

Definition 4. The **edge set** E_F of a dessin F, is the set whose elements are the edges of F.

Example 1. Consider Figure 2(a), there is a diagram of a dessin with multiple edges. Never mind a multiset; Figure 2(b) shows how the edges are considered distinct elements of the set $\{e_1,\ e_2,\ e_3,\ e_4,\ e_5\}$.

Figure 2: A diagram of a dessin (a) where the names of the edges are made explicit (b).

3.1.2 Black and White Permutations of a Dessin

Given any dessin F, by definition, we can take the cyclic (anticlockwise) ordering of the edges at each vertex. Therefore, associated with each vertex v, is a cyclic permutation on the subset of E_F which exclusively contains all the edges incident to that vertex v. For example, the left most black vertex in Figure 2(b) is associated with the ‘cycle’ $(e_1\ e_3\ e_2)$, and the white vertex is associated with $(e_1\ e_2\ e_3\ e_5\ e_4)$.

Now consider the following important fact: *every* edge is incident to *one and only one* black vertex and *one and only one* white vertex. Thus, the collection of ‘cycles’ associated with the black vertices are disjoint, and likewise for the white vertices. Additionally, the product of all the ‘cycles’ associated with the black vertices is a well-defined permutation on the edge set E_F, and likewise for the white vertices. (Note that ‘the product’ is reasonable because this multiplication is commutative and associative, and therefore unique). This gives rise to two important characteristics of a dessin.

Definition 5. The **black permutation** of a dessin F, is the unique permutation on E_F that is given by the cyclic ordering of the edges on the black vertices of F. The **white permutation** of F is analogous.

Remark. The construction of the black and white permutations is described in the two preceding paragraphs.

3.1.3 Transitive F_2-sets

In the beginning of this section, we define some general mathematical concepts which will be called upon to establish dessins as transitive F_2-sets. Finally, Theorem 1 formalises our characterisation of dessins.

Definition 6. F_2 is the *free group* on two generators, $\{b, w\}$.

Remark. F_2 can be considered the group generated by the set $\{b, w\}$ which satisfies the minimal requirements for a group - that is, F_2 has no unnecessary baggage. See Appendix A for a formal definition of free groups. And see Appendix B for a construction of F_2. However, it should suffice to know that the elements of F_2 are all the unique *words* of $\{b, w\}$, (such as $b, w, bw, b^{-1}wwbw^{-1}$) after *reduction*, (for example $ww^{-1}b = b$). The group operation is concatenation of these words and the identity is the empty word.

Definition 7. Let G be a group. The **action** of G on the set X is a map $\alpha : G \times X \to X$,

$$\alpha : (g, x) \mapsto g \cdot x, \quad \text{for all} \quad g \in G \quad \text{and} \quad x \in X,$$

which satisfies the following conditions:

1. $\alpha(e, x) = x$ where e in the identity in G.
2. For all $g, h \in G$ and all $x \in X$, $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$.

The pair (X, α) is called a ***G*-set**.

Remark. Defined above is a *left* group action and subsequently, a left G-set. A right group action $X \times G \to X$ is analogous, but changes the order in which products act. For the purpose of this project, we consider only left group actions and take 'group action' to mean 'left group action'.

Definition 8. A group action of G on X is **transitive** if for any $x, y \in X$, there exists $g \in G$ such that $g \cdot x = y$.

Theorem 1. *A dessin G is characterised as a transitive F_2-set (E_G, α), where E_G is a non-empty, finite set (of edges) and $\alpha : F_2 \times E_G \to E_G$ is a group action.*

Proof. **Given a dessin, we can construct a transitive F_2-set on the set of edges:**

Let G be a dessin. Therefore, E_G is a non-empty, finite set of edges. Take the black permutation of G to be $\phi : E_G \to E_G$ and take the white permutation of G to be $\psi : E_G \to E_G$. Now define an F_2-action

$\alpha : F_2 \times E_G \to E_G$ where for all $e \in E_G$,

$$\alpha : (b, e) \mapsto \phi(e),$$
$$\alpha : (w, e) \mapsto \psi(e).$$

Now consider that by condition 2 in Definition 7, for all $e \in E_G$ and $x \in F_2$, $\alpha(x, \alpha(x^{-1}, e)) = \alpha(xx^{-1}, e) = \alpha(\text{empty word}, e) = e$, by condition 1 in Definition 7. But by definition of α, $\alpha(b, \alpha(b^{-1}, e)) = \phi(\alpha(b^{-1}, e))$, therefore $\phi(\alpha(b^{-1}, e)) = e$.

Similarly, for all $e \in E_G$, $\alpha(x^{-1}, \alpha(x, e)) = e$ by conditions 1 & 2 of Definition 7. But $\alpha(b^{-1}, \alpha(b, e)) = \alpha(b^{-1}, \phi(e))$ by the definition of α, therefore $\alpha(b^{-1}, \phi(e)) = e$. Thus, $\alpha(b^{-1}, e) = \phi^{-1}(e)$, analogously $\alpha(w^{-1}, e) = \psi^{-1}(e)$.

This group action is well-defined because all $x \in F_2$ can be written as products of $b, w, b^{-1}w^{-1} \in F_2$. Therefore, using the second condition in Definition 7, it can be established what (x, e) is mapped to for all $x \in F_2$ and $e \in E_G$.

Finally, it can be shown this F_2-set is transitive. Let $e_1, e_2 \in E_G$, e_1 is connected to some vertex v_1 and e_2 is connected to some vertex v_2. G is a connected graph, so between v_1 and v_2, there is a path along the edges. We can construct a product of permutations ϕ, ψ which follows this sequence from edge to edge such that we permute e_1 to e_2. We begin with e_1 and at each edge in the path, the next edge is connected to the same black vertex and/or the same white vertex as the current edge. Thus, we apply the necessary permutation (ϕ or ψ) as many times as the cyclic ordering at that vertex of interest requires to permute the current edge to the next edge on the path. Composing each new permutation on the right. Finally we reach an edge which is connected to the same vertex as e_2, by apply the required repetitions of a permutation (which permutation depends on the colour of the vertex and the number of repetitions depends on the cyclic ordering at this vertex), we arrive at e_2. The final composition of permutations can be simplified.

We end up with a composition of permutations $p_n \cdots p_1(e_1) = e_2, \;\; n \in \mathbb{N}$, where p_i is one of ϕ or ψ. And n is the number of permutations, including repetitions. This composition can be rewritten according to condition 2 in Definition 7:

$$\begin{aligned} p_n \cdots p_1(e_1) &= p_n(\cdots p_1(e_1) \cdots), \\ &= \alpha(f_n, \cdots \alpha(f_2, \alpha(f_1, e_1)) \cdots), \text{ where } \alpha(f_i, e) = p_i(e) \;\; \forall e \in E_G, \text{ and } i \in 1, \ldots, n. \end{aligned}$$

Now consider that for all $i \in 1, \ldots, n$, $f_i = b$ or w since $p_i = \phi$ or ψ. Therefore $f_n \cdots f_1$ is a word of $\{b, w\}$, hence $f_n \cdots f_1 \in F_2$. Additionally, $\alpha(f_n, \cdots \alpha(f_2, \alpha(f_1, e_1)) \cdots) = \alpha(f_n \cdots f_1, e_1)$ by condition 2 of Definition 7. Thus, there exists $z = f_n \cdots f_1 \in F_2$ such that $z \cdot e_1 = e_2$. Hence, this non-empty, finite F_2-set is transitive.

Given a non-empty, finite set E and an F_2-action on E, we can construct a unique dessin G:

Let $\alpha : F_2 \times E \to E$ be a group action on E where E is non-empty and finite. Therefore, E is countable and can be expressed as $E = \{e_1, \ldots, e_n\}, \;\; n \in \mathbb{N}$. We can take E to be the finite set of edges of G.

Recall that F_2 is generated by $\{b, w\}$. Define the black permutation of G as the unique permutation

$$\phi = \begin{pmatrix} e_1 & \cdots & e_n \\ \alpha(b, e_1) & \cdots & \alpha(b, e_n) \end{pmatrix}.$$

Similarly, define the white permutation of G as the unique permutation

$$\psi = \begin{pmatrix} e_1 & \cdots & e_n \\ \alpha(w, e_1) & \cdots & \alpha(w, e_n) \end{pmatrix}.$$

It is a fact from group theory that we can write each permutation as a product of disjoint cycles. Now include in each permutation, all the cycles of length 1 so that each permutation makes explicit what happens to all elements of E. Consider that each cycle is unique up the cyclic ordering of the elements in that cycle. And each product of disjoint cycles is unique up to which disjoint cycles are being multiplied. So the number of cycles in each resulting permutation is unique. Consistent with how the black and white permutations are constructed in section 3.1.2, the number of disjoint cycles in ϕ is the number of black vertices in G. And the number of disjoint cycles in ψ is the number of white vertices in G. It follows that each disjoint cycle is associated with a distinct vertex in G and each vertex in G is associated with a distinct disjoint cycle in either ϕ or ψ; the elements in a disjoint cycle are all the edges incident on the associated vertex.

Thus, a unique bipartite graph G, with cyclic ordering, and a finite, non-empty set of edges, has been constructed and it can now be shown that G is connected.

Let v_1 and v_2 be vertices in G, by the construction, each vertex must be connected to at least one edge. Take $e_1 \in E$ incident to v_1 and $e_2 \in E$ incident to v_2. α is a transitive F_2-action on the set of edges E. Therefore, for $e_1, e_2 \in E$, there exists $z \in F_2$ such that $\alpha : (z, e_1) \mapsto e_2$. This means, that by applying a composition of black and white permutations (and their inverses), we can permute e_1 to e_2. Therefore, there is a sequence of edges from e_1 to e_2. In this sequence, consecutive edges are connected to a common vertex. Hence G is connected. ■

Remark. It is evident that a dessin has two components. The first is the edge set. The second is the F_2-action which describes the underlying structure by giving the black and white permutations. Two dessins may have the same edge set, but a different F_2-action on this set, thus a different underlying structure. Conversely, two dessins may have the same underlying structure but different edge sets. The reader may find it useful to consider a dessin with some structure but nondescript edges (such as Figure 2(a)) as a representative of its isomorphism class.

Remark. Dessins were introduced in Definition 3 as graphs with extra structure. It follows from Theorem 1, that dessins can be considered *sets* with extra structure.

Upon considering Theorem 1, the question my be raised: Why characterise a dessin in terms of an F_2-set if it carries so much redundant information about the dessin? Indeed, to characterise a dessin, we need only to know the edge set and the black and white permutations. For example, we do not need to know the permutation that $(wwb^{-1}b) \in F_2$ would be mapped to by the F_2-action. However, there are a number of benefits. First, it offers a simple way to ensure connectedness through transitivity. Second, by using an F_2-set for all dessins, we can easily establish maps between dessins (see Section 4.1) because each group-set is acted on by the same group.

4 The Category of Dessins d'Enfants

Now we establish the category of dessins, denoted **Des**. The objects of **Des** are dessins and all dessins are objects of **Des**. The morphisms in **Des** are expected to *preserve structure* in some way. After characterising dessins as transitive F_2-sets, it is reasonable to let the morphisms between dessins be equivariant maps between F_2-sets. Indeed, it will be shown that equivariant maps preserve cyclic ordering of the vertices.

4.1 Morphisms Between Dessins

If a morphism between two dessins F and G is an equivariant map between the F_2-sets which characterise F and G, then it induces a map between the edge sets E_F and E_G, which satisfies the condition in Definition 9.

Definition 9. Let G be a group. And let (X, α) and (Y, β) be G-sets where the group action of G on set X is $\alpha : G \times X \to X$ and the group action of G on set Y is $\beta : G \times Y \to Y$. An equivariant map $\theta : X \to Y$ satisfies the condition that for all $g \in G$ and $x \in X$, $\theta(\alpha(g \cdot x)) = \beta(g \cdot \theta(x))$.

Lemma 2. *Let F and G be dessins with a map $\theta : F \to G$ which maps each edge of F to an edge of G. If θ is an equivariant map, then the cyclic ordering of the edges is preserved.*

Proof. By Theorem 1, F is characterised as an F_2-set with group action $\alpha : F_2 \times E_F \to E_F$. Similarly G is characterised as an F_2-set with group action $\beta : F_2 \times E_G \to E_G$.

Let $x \in F_2$, and let $e \in E_F$ where E_F is the edge set of F. By transitivity, $\theta(\alpha(x, e)) = \beta(x, \theta(e))$. That is, whether we permute any given edge $e \in E_F$ according to the F_2-action on F by $x \in F_2$ and use θ to map the resulting edge to an edge in G, or we map e by θ to an edge in G and then permute the resulting edge according to the F_2-action on G by x, the resulting edge is the same. Recall from the proof of Theorem 1, $\alpha(b, e)$ maps e according to the black permutation on the edges of F, $\alpha(w, e)$ maps e according to the white permutation of F, $\beta(b, e)$ maps e according to the black permutation on G, and $\beta(w, e)$ maps e according to the white permutation on G. By taking $x = b$, we see the cyclic ordering at the black vertices is 'preserved' and by taking $x = w$, we see the cyclic ordering at the white vertices is 'preserved'. ∎

4.2 Formalising Des

We have a collection of objects and the collections of maps between them. We can now show that these form a category.

Theorem 3. *The collection of all possible dessins and the equivariant maps between them form a category called* ***Des****.*

Proof. We can show **Des** satisfies Definition 1. By Theorem 1, non-empty, finite, F_2-sets characterise dessins. So every dessin is considered a pair $F = (E_F, \alpha)$ where E_F is a non-empty, finite set of edges and α is a transitive F_2-action on E_F.

Collection of Objects. We have a collection of objects $\mathrm{Ob}(\mathbf{Des})$ which consists only of every non-empty, finite, transitive F_2-set.

Morphisms. Let $F, G \in \mathrm{Ob}(\mathbf{Des})$, the (potentially empty) collection of maps from F to G is the collection of all the equivariant maps from $F = (E_F, \alpha)$ to $G = (E_G, \beta)$.

Composition. Let $F, G, H \in \mathrm{Ob}(\mathbf{Des})$ and let $\sigma : F \to G$ and $\theta : G \to H$ be equivariant maps. Take the composition $\theta \circ \sigma : F \to H$, it is shown below that $\theta \circ \sigma$ is an equivariant map. Consider that $F = (E_F, \alpha)$, $G = (E_G, \beta)$, and $H = (E_H, \gamma)$. Let $e \in E_F$ and $f \in F_2$

$$\begin{aligned}\gamma(f, \theta \circ \phi(e)) &= \gamma(f, \theta(\sigma(e))),\\ &= \theta(\beta(f, \sigma(e))), \text{ by equivariance of } \theta,\\ &= \theta(\sigma(\alpha(f, e))), \text{ by equivariance of } \sigma,\\ &= \theta \circ \sigma(\alpha(f, e)).\end{aligned}$$

Thus, the composition of any two maps in **Des** is a map in **Des**.

Identity. Let $F \in \mathrm{Ob}(\mathbf{Des})$, therefore $F = (E_F, \alpha)$ is an F_2-set where $\alpha : F_2 \times E_F \to E_F$ is the F_2-action on the set E_F. Define a map $\theta : F \to F$ as for all $e \in E_F$, $\theta(e) = e$. This induces the identity map $\mathrm{id} : E_F \to E_F$. We can show this is an equivariant map. Let $e \in E_F$, $f \in F_2$, then $\theta(\alpha(f, e)) = \alpha(f, e)$ and $\alpha(f, \theta(e)) = \alpha(f, e)$ by the definition of θ. Therefore $\theta(\alpha(f, e)) = \alpha(f, \theta(e))$. Hence, θ is an equivariant map. Thus, each object $F \in \mathbf{Des}$ has an identity map $\mathrm{id}_F : F \to F$.

C1. Let $F, G, H, I \in \mathrm{Ob}(\mathbf{Des})$ and let $\sigma : F \to G$, $\theta : G \to H$, $\delta : H \to I$ be equivariant maps. We can take the following compositions $\delta \circ (\theta \circ \sigma) : F \to I$ and $(\delta \circ \theta) \circ \sigma : F \to I$. And we can show they are equal.

Let $e \in E_F$ where E_F is the underlying set in the F_2-set F. Consider that the induced maps of edge sets is are composed associatively.

$$\theta \circ \sigma : e \mapsto \theta(\sigma(e)), \quad \delta : \theta(\sigma(e)) \mapsto \delta(\theta(\sigma(e))), \quad \therefore \quad \delta \circ (\theta \circ \sigma)(e) = \delta(\theta(\sigma(e))).$$

$$\sigma : e \mapsto \sigma(e), \quad \delta \circ \theta : \sigma(e) \mapsto \delta(\theta(\sigma(e))), \quad \therefore \quad (\delta \circ \theta) \circ \sigma(e) = \delta(\theta(\sigma(e))).$$

Thus, $\delta \circ (\theta \circ \sigma) = (\delta \circ \theta) \circ \sigma$.

C2. Let $F, G \in \mathrm{Ob}(\mathbf{Des})$ and let $\theta : F \to G$ be an equivariant map. Take $\mathrm{id}_F : F \to F$, the identity morphism on F and take $\mathrm{id}_G : G \to G$, the identity morphism on G. We can take the compositions $\theta \circ \mathrm{id}_F : F \to G$ and $\mathrm{id}_G \circ \theta : F \to G$. Let $e \in E_F$ where E_F is the underlying set in the F_2-set F.

$$\theta \circ \mathrm{id}_F(e) = \theta(\mathrm{id}_F(e)) = \theta(e).$$

$$\mathrm{id}_G \circ \theta(e) = \mathrm{id}_G(\theta(e)) = \theta(e).$$

Thus, $\theta \circ \mathrm{id}_F = \theta = \mathrm{id}_G \circ \theta$.

It is proved we have a category of dessins. ■

Corollary 4. *Let $F, G \in \mathbf{Des}$ where $\theta : F \to G$ is a morphism in $\mathbf{Des}$. $\theta : F \to G$ induces a map between edges $\theta : E_F \to E_G$. Where $F = (E_F, \alpha)$ and $G = (E_G, \beta)$.*

Proof. This comes from Theorem 3. $\theta : F \to G$ is simply a map between dessins, but it is characterised as an equivariant map between two transitive F_2-sets. Thus, we say θ is a map between dessins which *is* equivariant and specifies a map $\theta : E_F \to E_G$ on the edge sets of the dessins. ■

Remark. In Section 5, $\theta : E_F \to E_G$ is referred to as the equivariant 'map of edges' in relation to the morphism between dessins, $\theta : F \to G$.

5 Morphisms in Des

Now that the category of dessins has been established, we can investigate the nature of the morphisms in this category.

5.1 Epimorphisms in Des

Definition 10. Let $\mathscr{C}$ be a category and $A, B \in \mathscr{C}$, a morphism $\theta : A \to B$ is an **epimorphism** if for all objects $C \in \mathscr{C}$ and maps $\sigma : B \to C$ and $\sigma' : B \to C$,

$$\sigma \circ \theta = \sigma' \circ \theta \implies \sigma = \sigma'.$$

This is known as right cancellation.

Remark. Epimorphisms are analogous to surjective functions between sets.

Lemma 5. *For all morphisms in $\mathbf{Des}$, the map of edges is surjective.*

Proof. Let $F, G \in \mathbf{Des}$ and morphism $\theta : F \to G$. By Theorem 1, we must have $F = (E_F, \alpha)$ and $G = (E_G, \beta)$ where E_F is the edge set of F, E_G is the edge set of G, $\alpha : F_2 \times E_F \to E_F$ is the F_2-action on E_F, and $\beta : F_2 \times E_G \to E_G$ is the F_2-action on E_F.

$\theta : F \to G$ is an equivariant map between these F_2-sets. This induces a map $\theta : E_F \to E_G$. Let $g \in E_G$. There exists $f' \in E_F$ and $g' \in E_G$ such that $\theta : f' \mapsto g'$. Now consider that by transitivity, there exists $x \in F_2$ such that $\beta : (x, g') \mapsto g$, thus $\beta(x, \theta(f')) = g$. But θ is an equivariant map, therefore $\theta(\alpha(x, f')) = g$. Therefore, there exists $f = \alpha(x, f') \in E_F$ such that $\theta(f) = g$. Hence, $\theta : E_F \to E_G$ is as surjective map of edges. ■

Theorem 6. *All morphism in* ***Des*** *are epimorphisms.*

Proof. Let $F, G, H \in$ **Des** with morphisms $\theta : F \to G$, $\sigma : G \to H$, and $\sigma' : G \to H$. Assume that $\sigma \circ \theta = \sigma' \circ \theta$, thus for all $e \in E_F$, $\sigma \circ \theta(e) = \sigma' \circ \theta(e)$. Now let $g \in E_G$, by Lemma 5 the induced map of edges $\theta : E_F \to E_G$ is surjective. Therefore, there exists $f \in E_F$ such that $\theta(f) = g$. Consider that

$$\sigma(g) = \sigma(\theta(f)) = \sigma \circ \theta(f) \quad \text{and} \quad \sigma'(g) = \sigma'(\theta(f)) = \sigma' \circ \theta(f).$$

But $\sigma \circ \theta(f) = \sigma' \circ \theta(f)$. Therefore $\sigma(g) = \sigma'(g)$ for all $g \in E_G$. That is, $\sigma = \sigma'$. Thus, it has been shown that for any $H \in$ **Des** with $\sigma : G \to H$ and $\sigma' : G \to H$, if $\sigma \circ \theta = \sigma' \circ \theta$, then $\sigma = \sigma'$. Hence, θ satisfies Definition 10 and is therefore an epimorphism. Thus, all morphisms in **Des** are epimorphisms. ■

5.2 Monomorphisms and Isomorphisms in Des

Definition 11. Let $\mathscr{C}$ be a category and $A, B \in \mathscr{C}$, a morphism $\theta : A \to B$ is a **monomorphism** if for all objects $C \in \mathscr{C}$ and maps $\sigma : C \to A$ and $\sigma' : C \to A$,

$$\theta \circ \sigma = \theta \circ \sigma' \implies \sigma = \sigma'.$$

This is known as right cancellation.

Remark. Monomorphisms are analogous to injective functions between sets.

Definition 12. Let $\mathscr{C}$ be a category and $A, B \in \mathscr{C}$, a morphism $\theta : A \to B$ is an **isomorphism** if there exists another morphism $\theta' : A \to B$ such that

$$\theta \circ \theta' = \mathrm{id}_B \quad \text{and} \quad \theta' \circ \theta = \mathrm{id}_A.$$

Remark. This means a morphisms is an isomorphism if it has an inverse.

Lemma 7. *A morphism in* ***Des*** *is an isomorphism if and only if the map of edges in bijective.*

Proof. Let $F, G \in$ **Des** and $\theta : F \to G$.

($\Rightarrow$) Assume $\theta : F \to G$ is an isomorphism. Therefore, there exists a morphism $\theta' : G \to F$ such that

$\theta \circ \theta' = \mathrm{id}_G$ and $\theta' \circ \theta = \mathrm{id}_F$. Now let $e_1, e_2 \in E_F$ and assume $\theta(e_1) = \theta(e_2)$, this implies

$$\begin{aligned} \theta'(\theta(e_1)) &= \theta'(\theta(e_2)), \\ \therefore\ \theta' \circ \theta(e_1) &= \theta' \circ \theta(e_2), \\ \therefore\ \mathrm{id}_F(e_1) &= \mathrm{id}_F(e_2). \end{aligned}$$

Thus, $e_1 = e_2$. Thus the map of edges is injective, but since θ is a morphism in **Des**, by Lemma 5, the map of edges is also surjective. Hence, the map of edges is bijective.

($\Leftarrow$) Assume $\theta : F \to G$ is a morphism where the map of edges is surjective. Define $\sigma : E_G \to E_F$ as for all $g \in E_G$, $\sigma : g \mapsto f \in E_F$ such that f is the unique edge in E_F which $\theta : f \mapsto g$. This can always be done because θ is a bijective map on edges. We can show σ is an equivariant map between dessins $F = (E_F, \alpha)$ and $G = (E_G, \beta)$. Let $g \in E_G$ and $x \in F_2$. Consider that there exists a unique $f \in E_F$ such that $\theta(f) = g$, because θ is a bijective map.

$$\sigma(\beta(x, g)) = \sigma(\beta(x, \theta(f))) = \sigma(\theta(\alpha(x, f))) = \alpha(x, f).$$
$$\alpha(x, \sigma(g)) = \alpha(x, \sigma(\theta(f))) = \alpha(x, f).$$

Thus, $\sigma(\beta(x, g)) = \alpha(x, \sigma(g))$, so σ is an equivariant map of F_2-set, $G = (E_G, \beta)$ and $F = (E_F, \alpha)$. And we have: for all $g \in E_G$, $\theta \circ \sigma(g) = g$ and for all $f \in E_F$, $\sigma \circ \theta(f) = f$. Thus, there exists $\sigma : G \to F$ such that $\theta \circ \sigma = \mathrm{id}_G$ and $\sigma \circ \theta = \mathrm{id}_F$. Hence, θ is an isomorphism. ■

Lemma 8. *A morphism in **Des** is a monomorphism if and only if the map of edges is injective.*

Proof. ($\Leftarrow$) Let $F, G \in$ **Des** and $\theta : F \to G$ a morphism such that the map of edges is injective. Therefore, for all $e_1, e_2 \in E_F$, $\theta(e_1) = \theta(e_2) \implies e_1 = e_2$. Let $H \in$ **Des** and $\sigma, \sigma' : H \to F$. Assume $\theta \circ \sigma = \theta \circ \sigma'$, this implies

$$\begin{aligned} \forall\, h \in H,\ \theta \circ \sigma(h) &= \theta \circ \sigma'(h), \\ \therefore\ \theta(\sigma(h)) &= \theta(\sigma'(h)), \\ \implies \sigma(h) &= \sigma'(h), \text{ by injectivity of } \theta. \end{aligned}$$

Thus, $\beta = \beta'$. Therefore, for all $H \in$ **Des** and $\beta, \beta' : H \to F$, it has been shown $\theta \circ \sigma = \theta \circ \sigma' \implies \sigma = \sigma'$. Hence, θ is a monomorphism.

($\Rightarrow$) Let $F, G \in$ **Des** and $\theta : F \to G$ a monomorphism. Let $e_1, e_2 \in E_F$ and assume $\theta(e_1) = \theta(e_2)$. Consider that $F = (E_F, \alpha)$ where (E_F, α) is a transitive F_2-set. Consider defining a new F_2-action $\beta : F_2 \times E_F \to E_F$, defined for all $f \in E_F$ and $x \in F_2$ as

$$\beta(x, f) = \gamma \circ \alpha(x, \gamma(f)),$$

where $\gamma : E_F \to E_F$ is defined as

$$\gamma(f) = \begin{cases} f, & f \neq e_1, e_2, \\ e_2, & f = e_1, \\ e_1, & f = e_2. \end{cases}$$

It is shown in Appendix C that β is indeed an F_2-action. Therefore (E_F, β) is an F_2-set and it is shown below that this is a transitive F_2-set. Let $a, b \in E_F$. There are several cases to consider:

1. Assume $a, b \notin \{e_1, e_2\}$ and choose $x \in F_2$ such that $\alpha(x, a) = b$, this is always possible by the transitivity of α.
$$\therefore \quad \beta(x, a) = \gamma(\alpha(x, \gamma(a))) = \gamma(\alpha(x, a)) = \alpha(x, a) = b.$$
2. Assume $a = e_1$ and $b \notin \{e_1, e_2\}$, and by the transitivity of α, choose $x \in F_2$ such that $\alpha(x, e_2) = b$.
$$\therefore \quad \beta(x, a) = \gamma(\alpha(x, \gamma(a))) = \gamma(\alpha(x, e_2)) = \alpha(x, e_2) = b.$$
3. Assume $a = e_2$ and $b \notin \{e_1, e_2\}$, and by the transitivity of α, choose $x \in F_2$ such that $\alpha(x, e_1) = b$.
$$\therefore \quad \beta(x, a) = \gamma(\alpha(x, \gamma(a))) = \gamma(\alpha(x, e_1)) = \alpha(x, e_1) = b.$$
4. Assume $b = e_1,\ a \in E_F$ and choose $x \in F_2$ such that $\alpha(x, \gamma(a)) = e_2$, this is possible by the transitivity of α and because $\gamma(a) \in E_F$.
$$\beta(x, a) = \gamma(\alpha(x, \gamma(a))) = \gamma(e_2) = e_1.$$
5. Assume $b = e_2,\ a \in E_F$ and choose $x \in F_2$ such that $\alpha(x, \gamma(a)) = e_1$, this is possible by the transitivity of α and because $\gamma(a) \in E_F$.
$$\beta(x, a) = \gamma(\alpha(x, \gamma(a))) = \gamma(e_1) = e_2.$$

All possible cases have been considered, and in each, there exists $x \in F_2$ such that $\beta(x, a) = b$ for all $a, b \in E_F$. Therefore $H = (E_F, \beta)$ is a non-empty, finite, transitive F_2-set, thus by Theorem 3, $H \in \mathbf{Des}$.

Now let us establish a morphism $\gamma : H \to F$ where the induced map of edges $\gamma : E_F \to E_F$ is defined as before. This is shown below to be an equivariant map. But, first consider this property of $\gamma : H \to F$. Let $e \in E_F$,

$$\gamma \circ \gamma(e) = \begin{cases} \gamma(e) = e, & \text{when } e \neq e_1, e_2, \\ \gamma(e_2) = e_1, & \text{when } e = e_1, \\ \gamma(e_1) = e_2, & \text{when } e = e_2, \end{cases} .$$

Thus, $\gamma \circ \gamma(e) = e$ for all $e \in E_F$.

Now to show that $\gamma : H \to F$ is an equivariant map. Let $e \in E_F$ and $x \in F_2$,

$$\begin{aligned}\gamma(\beta(x,e)) &= \gamma(\gamma(\alpha(x,\gamma(e)))),\\ &= \gamma \circ \gamma(\alpha(x,\gamma(e))),\\ &= \alpha(x,\gamma(e)).\end{aligned}$$

Thus, $\gamma : H \to F$ is indeed an equivariant map.

Now consider a third map, $\delta : H \to F$ defined as for all $e \in E_F$, $\delta : e \mapsto e$.

And consider the following two compositions $\theta \circ \gamma : H \to G$ and $\theta \circ \delta : H \to G$. For all $e \in E_F$, $\theta \circ \delta(e) = \theta(e)$. And for all $e \in E_F \setminus \{e_1, e_2\}$, $\theta \circ \gamma(e) = \theta(e)$, additionally $\theta \circ \gamma(e_1) = \theta(e_2)$ and $\theta \circ \gamma(e_2) = \theta(e_1)$. Consider that it was assumed at the very beginning that $\theta(e_1) = \theta(e_2)$. Therefore $\theta \circ \gamma(e_1) = \theta(e_1)$ and $\theta \circ \gamma(e_2) = \theta(e_2)$. Thus, for all $e \in E_F$, $\theta \circ \gamma = \theta \circ \delta$.

But θ is a monomorphism, therefore $\gamma = \delta$, thus the following is true:

$$\begin{aligned}e_1 &= \gamma(e_2), \text{ by definition of } \gamma,\\ &= \delta(e_2), \text{ by } \gamma = \delta,\\ &= e_2 \text{ by definition of } \delta.\end{aligned}$$

Therefore, for all $e_1, e_2 \in E_F$, it has been shown that $\theta(e_1) = \theta(e_2) \implies e_1 = e_2$. Hence, θ induces an injective map of edges. ■

Theorem 9. *A morphism in* ***Des*** *is a monomorphism if and only if it is an isomorphism.*

Proof. (⇒) Let $F, G \in$ **Des** and $\theta : F \to G$ is a monomorphism. Therefore, the map of edges in injective by Lemma 8. But the map of edges is also surjective by Lemma 5. Thus, the map of edges is bijective, therefore by Lemma 7, $\theta : F \to G$ is an isomorphism.

(⇐) Let $F, G \in$ **Des** and $\theta : F \to G$ is an isomorphism. By Lemma 7, this implies the map of edges is bijective. Thus, the map of edges in injective, therefore by by Lemma 8, $\theta : F \to G$ is a monomorphism. ■

6 Conclusion

The category of dessins (**Des**) has been established and its morphisms have been investigated further. Theorem 6 tells us that all morphisms in **Des** are epimorphism. Theorem 9 shows that all monomorphisms in **Des** are in fact isomorphisms. These results indicate that morphisms in **Des** – the cyclic order preserving maps between dessins – are rare. These results are limited by the characterisation of dessins which took dessins to only be finite and excluded empty dessins.

References

González-Diez, G. & Jaikin-Zapirain, A. (2015). Tha absolute galois group acts fathfully on regular dessin and on beauville surfaces. *Proceedings of the London Mathematical Society*, 111(4).

Guillot, P. (2014). An elementary approach to dessins d'enfants and the grothendieck–teichmüller group. *Enseignement Mathématique*, 60(3), 293–375.

Leinster, T. (2014). *Basic Category Theory*, volume 143 of *Cambridge Studies in Advanced Mathematics*. Cambridge, UK: Cambridge University Press.

A Free Groups

Definition 13. Let X be a set. A free group F is called *free on X* if there is a set map $\psi : X \to F$ and for any group G and set map $\phi : X \to G$, there is a unique homomorphism $\alpha : F \to G$ such that $\alpha \circ \psi = \phi$.

B Constructing F_2

We can construct F_2, the free group on the set $\{b, w\}$ in the following way:

1. Add an inverse for each element.
2. With the set $\{b, w, b^{-1}, w^{-1}\}$, make all possible multiplications (e.g. $bwb^{-1}wwb$) and *reduce* these products by cancelling where appropriate (e.g. $wbb^{-1}wb = wwb$).
3. Take this set of words (e.g. $\{b, w, bw, wwb, bwb^{-1}w^{-1}, \dots\}$), including the empty word as the identity. Take the group operation as concatenation of words, this is a binary operation.
4. And the set of words of $\{w, b\}$ is clearly non-empty and contains an identity. Concatenation of words is associative. We can construct an inverse for each word by reversing the order of the symbols and replacing each symbol with its inverse, this must also be a word in F_2. Thus we have a group.

Additionally, it can be proven that F_2 satisfies Definition 13, but this is trivial.

C Subsidiary Proof: β in the proof of Lemma 8 is a valid F_2-action on E_F

Proof. To prove β is a valid group action of F_2 on E_F, we can show it satisfies Definition 7. $\beta : F_2 \times E_F \to E_F$ is defined as, for all $x \in F_2$, and all $f \in E_F$,

$$\beta : (x, f) \mapsto \gamma \circ \alpha(x, \gamma(f)),$$

where $\gamma : E_F \to E_F$ is defined as

$$\gamma(f) = \begin{cases} f, & f \neq e_1, e_2, \\ e_2, & f = e_1, \\ e_1, & f = e_2, \end{cases}$$

for some chosen $e_1, e_2 \in E_F$. And where $\alpha : F_2 \times E_F \to E_F$ is a group action and thus satisfies the two properties in Definition 7.

We can show β satisfies the first condition in Definition 7. Let $0 \in F_2$ denote the empty word, this is the identity in F_2. Let $f \in E_F$. If $f \notin \{e_1, e_2\}$, then

$$\beta(0, f) = \gamma \circ \alpha(0, \gamma(f)) = \gamma \circ \alpha(0, f) = \gamma(f) = f.$$

If $f = e_1$, then

$$\beta(0, f) = \gamma \circ \alpha(0, \gamma(e_1)) = \gamma \circ \alpha(0, e_2) = \gamma(e_2) = e_1.$$

If $f = e_2$, then

$$\beta(0, f) = \gamma \circ \alpha(0, \gamma(e_2)) = \gamma \circ \alpha(0, e_1) = \gamma(e_1) = e_2.$$

Thus, for all $f \in E_F$, $\beta(0, f) = f$.

We can show β satisfies the second condition in Definition 7. First recall that it was shown in the proof of Lemma 8 that $\gamma : E_F \to E_F$, satisfies the property $\gamma \circ \gamma(f) = f$ for all $f \in E_F$. This is a consequence of its definition only. Let $f \in E_F$ and $x, y \in F_2$,

$$\begin{aligned} \beta(x, \beta(y, f)) &= \gamma(\alpha(x, \gamma(\gamma(\alpha(y, \gamma(f)))))), \\ &= \gamma(\alpha(x, \alpha(y, \gamma(f)))), \quad \because\ \gamma(\gamma(e)) = e,\ \forall\, e \in E_F, \\ &= \gamma(\alpha(xy, \gamma(f))), \quad \because\ \alpha \text{ is a group action on } E_F, \\ &= \beta(xy, f). \end{aligned}$$

Thus, for all $x, y \in F_2$ and all $f \in E_F$, $\beta(x, \beta(y, f)) = \beta(xy, f)$.

Therefore, it is shown that β satisfies the definition of an F_2-action on E_F. So (E_F, α) is an F_2-set. ■