The Riemann–Roch Theorem and Dirac Operators

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Abstract

In this report we discuss the Riemann–Roch theorem, which gives a formula for the dimension of a certain space of meromorphic functions on a compact Riemann surface. From a classical perspective, we prove the formula using the ideas of cohomology and exact sequences from algebraic topology. We then understand the theorem as a consequence of applying the index theory of Dirac operators to complex manifolds. This leads to the Hirzebruch–Riemann–Roch theorem, which generalises the Riemann–Roch theorem to compact complex manifolds of arbitrary dimension.

1 Introduction

A Riemann surface is a geometric object locally modelled on the complex plane. These were introduced by Riemann in order to deal with the multi-valued behaviour of algebraic functions such as the square root. Riemann surfaces can be considered as the natural domains of holomorphic and meromorphic functions, and are used in the theory of analytic continuation. However, Riemann surfaces are also fundamental geometric objects in their own right, as the simplest examples of complex manifolds. Moreover, the technical tool of the cohomology of sheaves is more clearly understood in this special case, and we use this to prove the Riemann–Roch theorem (introduced in Section 3 and proved in Section 4). The Riemann–Roch formula allows for the computation of the dimension of a space of meromorphic functions with prescribed zeroes and poles on a compact Riemann surface. This is a natural problem to consider, since a common strategy to study the geometry of a surface is to study functions on it, and it is well-known that there are no nonconstant holomorphic functions on a compact Riemann surface.

In the second half of this report (Sections 5-7) we summarise important notions from the index theory of Dirac operators. The index is an integer expressed in terms of analytical quantities, namely a difference in the dimensions of certain kernels related to a Dirac operator. The Atiyah–Singer index theorem (Atiyah and Singer 1963) relates the index to topological quantities called characteristic classes. Atiyah, Bott, and Patodi (1973) gave a new proof of a local version of the index theorem by studying the heat equation associated to a Dirac operator. This report focuses on how this proof is applied to a particular Dirac operator called the Dolbeault–Dirac operator. This gives a generalisation of the Riemann–Roch theorem to compact complex manifolds of arbitrary dimension, which is known as the Hirzebruch–Riemann–Roch theorem. A version of the Hirzebruch–Riemann–Roch theorem had previously been known for projective algebraic varieties, but its generalisation to compact complex manifolds is due to index theory.

2 Statement of Authorship

The treatment of Riemann surfaces follows the textbook by Forster (1981). The material on the heat kernel proof of the index theorem primarily follows the definitions and overall structure of the exposition in the textbook by Roe (1998), with other sources used and appropriately cited. I summarised and explained the ideas and proofs, emphasising the connections between the Riemann–Roch theorem and index theory.

3 Meromorphic Functions on Riemann Surfaces

A Riemann surface is a connected one-dimensional complex manifold. This involves charts from the surface to the complex plane which induce an appropriate topology and notion of complex differentiability. We now expand upon what this definition means.

Definition 3.1 (Riemann surface)**.** Let X be a two-dimensional topological manifold, i.e. a Hausdorff topological space which is locally homeomorphic to $\mathbb{R}^2$. A complex chart is a homeomorphism $\varphi : U \to V$ of an open subset U of X to an open subset V of $\mathbb{C}$. A complex atlas is a collection $\mathcal{A} = (\varphi_i)_{i\in I}$ of complex charts $\varphi_i : U_i \to V_i$ such that $\bigcup_{i\in I} U_i = X$.

Two complex charts $\varphi_i : U_i \to V_i$, $i = 1, 2$, are holomorphically compatible if the sets $\varphi_i(U_1 \cap U_2)$, $i = 1, 2$, are open and the transition map

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

is a biholomorphic complex function, i.e. a bijective holomorphic function with a holomorphic inverse. We say that two complex atlases $\mathcal{A}$ and $\mathcal{A}'$ are analytically equivalent if every chart in $\mathcal{A}$ is holomorphically compatible with every chart in $\mathcal{A}'$. It is easily seen that analytical equivalence is an equivalence relation. Its equivalence classes are called complex structures.

Finally, a one-dimensional complex manifold is a pair (X, Σ) where X is a two dimensional topological manifold and Σ is a complex structure on X. If moreover X is connected, we say that it is a Riemann surface. We abbreviate (X, Σ) to X.

An atlas for a Riemann surface allows us to define what it means for a mapping $f : X \to Y$ between Riemann surfaces to be holomorphic: we ask that the corresponding map between subsets of the complex plane given by composing f with complex charts is holomorphic, for all charts in the atlas. The importance of the definition of analytical equivalence is that it implies that we can check the definition of a holomorphism for any atlas which induces the chosen complex structure on the Riemann surface. Some theorems about holomorphic functions on the complex plane immediately generalise to results about holomorphic maps between Riemann surfaces; notably, we have the identity theorem.

Theorem 3.2 (Identity theorem)**.** *Let X and Y be Riemann surfaces, and f and g be holomorphic maps from X to Y which coincide on a set $A \subset X$. If A contains a limit point of X, then $f = g$.*

Given the definition of holomorphic maps, we can then define meromorphic maps in the same way that meromorphic functions are defined from holomorphic functions on the complex plane.

Definition 3.3 (Meromorphic functions)**.** Let X be a Riemann surface and Y an open subset of X. A meromorphic function on Y is a holomorphic function $f : Y' \to \mathbb{C}$ for an open subset $Y' \subseteq Y$, such that $Y \setminus Y'$ only contains isolated points, and for every point $p \in Y \setminus Y'$, one has

$$\lim_{x \to p} |f(x)| = \infty.$$

These points are called the poles of f. The set of meromorphic functions on Y is denoted $\mathscr{M}(Y)$.

To study meromorphic functions on a Riemann surface, we introduce the notion of a divisor. We now work with compact Riemann surfaces for simplicity and since these are all we need for the Riemann–Roch theorem.

Definition 3.4 (Divisor, degree)**.** Let X be a compact Riemann surface. A divisor on X is a map $D : X \to \mathbb{Z}$ with finite support. The degree of D, denoted $\deg D$, is the sum $\sum_{x \in X} D(x)$.

We can associate divisors to meromorphic functions in the following way.

Definition 3.5. Let X be a compact Riemann surface and Y an open subset of X. Let X be a Riemann surface and Y an open subset of X. For all meromorphic functions $f \in \mathscr{M}(Y)$ and

$a \in Y$, define

$$\mathrm{ord}_a(f) = \begin{cases} 0, & \text{if } f \text{ is holomorphic and nonzero at } a \\ k, & \text{if } f \text{ has a zero of order } k \text{ at } a \\ -k, & \text{if } f \text{ has a pole of order } k \text{ at } a \\ \infty, & \text{if } f \text{ is identically zero in a neighbourhood of } a. \end{cases}$$

Proposition 3.6 (Divisor of a meromorphic function)**.** *Let f be a nonzero meromorphic function on a compact Riemann surface X. Then the map $x \mapsto \mathrm{ord}_x(f)$ is a divisor on f, denoted (f) and called the divisor of f.*

Proof. If $\mathrm{ord}_a(f) = \infty$ for some $a \in X$, then f is identically zero on a set for which a is an interior point, and hence a limit point. It follows by the identity theorem, Theorem 3.2, that $f = 0$ on X. Due to the assumption that f is nonzero, this shows that (f) is indeed a map to $\mathbb{Z}$.

Consider the set S of poles of f. It is by definition a discrete subset of the compact space X. Hence S is finite. By the identity theorem, the zeroes of a nonzero meromorphic function are also a discrete subset of X. Therefore (f) has finite support. Thus (f) is a divisor. □

The Riemann–Roch theorem gives a formula for the dimension of the vector space of meromorphic functions which are at least as well-behaved as a given divisor. In the next section we make this interpretation precise (see Remark 4.2) and provide a proof using cohomology.

4 Čech Cohomology and the Riemann–Roch Theorem

Our proof of the Riemann–Roch theorem uses exact sequences to compute the cohomology groups of sheaves. We first explain these technical terms and then present the proof.

A sheaf is a collection of abelian groups with elements called sections, which can be thought of as a generalisation of functions, with a way to restrict them, and such that the sections are determined locally and can be glued together. See Appendix A for a detailed definition. Here are examples which we will use later. For every open subset U of a Riemann surface X, let $\mathcal{O}(U)$ be the vector space of holomorphic functions on U. Together with the usual restriction maps, this gives the sheaf $\mathcal{O}$ of holomorphic functions on X.

Example 4.1 (The sheaf $\mathcal{O}_D$). Let D be a divisor on a compact Riemann surface X. For all open sets $U \subseteq X$, define $\mathcal{O}_D(U) = \{f \in \mathscr{M}(U) : \operatorname{ord}_x(f) \geq -D(x) \text{ for all } x \in U\}$. Together with the natural restriction maps, $\mathcal{O}_D$ is a sheaf, called the sheaf of multiples of the divisor $-D$.

Remark 4.2. Suppose $D(x) = m > 0$. Then $\operatorname{ord}_x(f) \geq -m$ if and only if f is holomorphic or has a pole of order less than or equal to m at x. If $D(x) = n \leq 0$, then $\operatorname{ord}_x(f) \geq -n$ if and only if f has a zero of at least order $-n$ at x. This is what is rigorously meant by the statement that D gives rise to a space of meromorphic functions with prescribed zeroes and permissible poles: this is referring to $\mathcal{O}_D(X)$. The Riemann–Roch formula computes the dimension of $\mathcal{O}_D(X)$.

Example 4.3 (Skyscraper sheaf). Let p be a point on a Riemann surface X. The skyscraper sheaf $\mathbb{C}_p$ is given by the abelian groups

$$\mathbb{C}_p(U) = \begin{cases} \mathbb{C} & \text{if } p \in U, \\ 0 & \text{if } p \notin U. \end{cases}$$

Cohomology gives a way to study a Riemann surface (or a more abstract topological space) by associating it with abelian groups. Čech cohomology does this by assign a section of a sheaf to intersections of a cover on the Riemann surface. These are called cochains.

Definition 4.4 (Cochain group). Let X be a topological space and $\mathscr{F}$ a sheaf of abelian groups on X. Let $\mathfrak{U} = (U_i)_{i \in I}$ be an open cover of X. For $q \geq 0$, define the q-th cochain group of $\mathscr{F}$ with respect to $\mathfrak{U}$ as the direct product

$$C^q(\mathfrak{U}, \mathscr{F}) := \prod_{(i_0, \dots, i_q) \in I^{q+1}} \mathscr{F}\Big(\bigcap_{k=0}^{q} U_{i_k}\Big).$$

Elements of $C^q(\mathfrak{U}, \mathscr{F})$ are called q-cochains.

We are mostly interested in q-cochains where q is from 0 to 2. For a toy example, consider an open cover $\mathfrak{U}$ consisting of two sets U_1 and U_2. Then a 0-cochain is a pair (f_1, f_2) where $f_1 \in \mathscr{F}(U_1)$ and $f_2 \in \mathscr{F}(U_2)$. A 1-cochain is a tuple $(f_{11}, f_{12}, f_{21}, f_{22})$ where $f_{11} \in \mathscr{F}(U_1)$, f_{12} and f_{21} are in $\mathscr{F}(U_1 \cap U_2)$, and $f_{22} \in \mathscr{F}(U_2)$.

The coboundary operator sends a q-cochain to a $(q+1)$-cochain.

Definition 4.5 (Coboundary operator). We define the coboundary operator from the 0-th cochain group to the first cochain group by

$$\delta : C^0(\mathfrak{U}, \mathscr{F}) \to C^1(\mathfrak{U}, \mathscr{F}), \qquad (f_i)_{i \in I} \mapsto (g_{ij})_{i,j \in I},$$

where $g_{ij} := f_j - f_i \in \mathscr{F}(U_i \cap U_j)$. From the first to the second cochain group we define

$$\delta : C^1(\mathfrak{U}, \mathscr{F}) \to C^2(\mathfrak{U}, \mathscr{F}), \qquad (f_{ij})_{i,j \in I} \mapsto (g_{ijk})_{i,j,k \in I}$$

where $g_{ijk} := f_{jk} - f_{ik} + f_{ij} \in \mathscr{F}(U_i \cap U_j \cap U_k)$. In the above definitions it is understood that we restrict to the appropriate intersection before adding the sections.

Definition 4.6 (Cocycle and coboundary)**.** Denote the kernel of $\delta : C^1(\mathfrak{U}, \mathscr{F}) \to C^2(\mathfrak{U}, \mathscr{F})$ by $Z^1(\mathfrak{U}, \mathscr{F})$, and the image of $\delta : C^0(\mathfrak{U}, \mathscr{F}) \to C^1(\mathfrak{U}, \mathscr{F})$ by $B^1(\mathfrak{U}, \mathscr{F})$. Elements of $Z^1(\mathfrak{U}, \mathscr{F})$ are called 1-cocycles. Elements of $B^1(\mathfrak{U}, \mathscr{F})$ are called 1-coboundaries.

An important fact is that 1-coboundaries are 1-cocycles, or symbolically $\delta^2 = 0$. If g_{ij}, g_{ik} and g_{jk} are elements of a 1-coboundary $(g_{ij})_{i,j \in I}$, then by definition there exists a zero cochain $(f_i)_{i \in I}$ such that $g_{ij} = f_j - f_i$ for all $i, j \in I$. Thus

$$g_{ij} + g_{jk} = f_j - f_i + f_k - f_j = f_k - f_i = g_{ik}.$$

This finishes the proof since $g_{ij} + g_{jk} = g_{ik}$ is precisely the condition for $(g_{ij})_{i,j \in I}$ to be a cocycle. Therefore we can consider 1-cocycles modulo 1-coboundaries, which is the definition of the first cohomology group.

Definition 4.7 (First cohomology group with respect to a cover)**.** Let X be a topological space, $\mathfrak{U}$ an open cover of X, and $\mathscr{F}$ a sheaf of abelian groups on X. The quotient group

$$H^1(\mathfrak{U}, \mathscr{F}) = Z^1(\mathfrak{U}, \mathscr{F})/B^1(\mathfrak{U}, \mathscr{F})$$

is called the first cohomology group with coefficients in $\mathscr{F}$ with respect to the cover $\mathfrak{U}$. Its elements are called cohomology classes. Note that if $\mathscr{F}$ is a sheaf of vector spaces, then $C^1(\mathfrak{U}, \mathscr{F})$ and $H^1(\mathfrak{U}, \mathscr{F})$ are vector spaces.

We can remove the dependence on the cover to form the first cohomology group $H^1(X, \mathscr{F})$ which only depends on the topological space X and the sheaf $\mathscr{F}$. This is done by taking the quotient by an equivalence relation where two cohomology classes are equivalent if there exists a common subcover on which they agree. A precise definition is in Section 12 of Lectures on Riemann Surfaces (Forster 1981), which also introduces the zeroth cohomology group $H^0(X, \mathscr{F})$. This is equal to the group of global sections on X, i.e. $H^0(X, \mathscr{F}) = \mathscr{F}(X)$.

Exact sequences provide a practical method for computations involving cohomology groups. A sequence of homomorphisms between abelian groups $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} A_n$, where $n \geq 3$, is called an exact sequence if $\operatorname{im} f_i = \ker f_{i+1}$ for all $i = 1, \dots n-2$. A sheaf homomorphism $\alpha : \mathscr{F} \to \mathscr{G}$ is a family of homomorphisms $\alpha(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ for all open sets U, which commute with the restriction homomorphisms. A sheaf homomorphism induces a homomorphisms $\mathscr{F}_x \to \mathscr{G}_x$ between the stalks of the sheaf; stalks are introduced in Appendix A. Thus we define a sequences of sheaves $\mathscr{F} \to \mathscr{G} \to \mathscr{H}$ to be exact if the corresponding sequences of the stalks $\mathscr{F}_x \to \mathscr{G}_x \to \mathscr{H}_x$, which is a sequence of abelian groups, is exact for all $x \in X$. The following important theorem relates exact sequences of sheaves with exact sequences of cohomology groups.

Theorem 4.8. *Suppose that X is a topological space and*

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$$

is an exact sequence of sheaves on X. Then there is an induced exact sequence of cohomology groups

$$\begin{aligned} 0 &\to H^0(X, \mathscr{F}) \to H^0(X, \mathscr{G}) \to H^0(X, \mathscr{H}) \\ &\to H^1(X, \mathscr{F}) \to H^1(X, \mathscr{G}) \to H^1(X, \mathscr{H}). \end{aligned}$$

This is proved as Theorem 15.12 of Lectures on Riemann Surfaces (Forster 1981). Finally, we need to quote the fact that for X a compact Riemann surface, the cohomology group $H^1(X, \mathcal{O})$ is finite dimensional. This is proved in Sections 13 and 14 of the same textbook, by analysing the inhomogenous Cauchy–Riemann equation and developing L^2-theory for holomorphic cochains. Therefore in this case we define the genus of X as $g = \dim H^1(X, \mathcal{O}) < \infty$.

We have now developed enough theory to state and prove the Riemann–Roch theorem.

Theorem 4.9 (Riemann–Roch)**.** *Let D be a divisor on a compact Riemann surface X of genus g. Then $H^0(X, \mathcal{O}_D)$ and $H^1(X, \mathcal{O}_D)$ are finite dimensional vector spaces and*

$$\dim H^0(X, \mathcal{O}_D) - \dim H^1(X, \mathcal{O}_D) = 1 - g + \deg D.$$

Recall that $H^0(X, \mathcal{O}_D) = \mathcal{O}_D(X)$. Therefore the Riemann–Roch theorem gives a formula for the complex dimension of this space of meromorphic functions which was interpreted in Remark 4.2.

Proof. The proof is by induction. Consider $D = 0$. From the definitions, $\mathcal{O}_0 = \mathcal{O}$, the sheaf of holomorphic functions. Then $H^0(X, \mathcal{O}) = \mathcal{O}(X)$ consists of only constant functions since X

is compact, so $\dim H^0(X,\mathcal{O}) = 1$. We have $H^1(X,\mathcal{O}) = g$ by definition and $\deg D = 0$ since $D = 0$. Therefore the Riemann–Roch formula holds.

Given a point $P \in X$, we also denote by P the divisor which takes the value 1 at the point P and zero elsewhere. Let $D' = D + P$. In Appendix B, we prove that there is an exact sequence $0 \to \mathcal{O}_D \to \mathcal{O}_{D'} \to \mathbb{C}_P \to 0$, and calculate that $H^0(X,\mathbb{C}_P) = \mathbb{C}$ and $H^1(X,\mathbb{C}_P) = 0$. Therefore Theorem 4.8 gives an exact sequence

$$\begin{aligned} 0 \to H^0(X,\mathcal{O}_D) \to H^0(X,\mathcal{O}_{D'}) \to \mathbb{C} \\ \to H^1(X,\mathcal{O}_D) \to H^1(X,\mathcal{O}_{D'}) \to 0. \end{aligned}$$

Let $V = \operatorname{im}(H^0(X,\mathcal{O}_{D'}) \to \mathbb{C})$ and $W = \mathbb{C}/V$. Clearly $\dim V + \dim W = 1 = \deg D' - \deg D$. From the definition of V and W, the sequences

$$\begin{aligned} 0 \to H^0(X,\mathcal{O}_D) \to H^0(X,\mathcal{O}_{D'}) \to V \to 0, \\ 0 \to W \to H^1(X,\mathcal{O}_D) \to H^1(X,\mathcal{O}_{D'}) \to 0 \end{aligned}$$

are exact. By the rank-nullity theorem and exactness, the alternating sum of the dimensions of these sequences are zero. Therefore $\dim H^0(X,\mathcal{O}_{D'}) = \dim H^0(X,\mathcal{O}_D) + \dim V$ and $\dim H^1(X,\mathcal{O}_D) = \dim H^1(X,\mathcal{O}_{D'}) + \dim W$. Note that if the vector spaces corresponding to D are finite dimensional then they are for D', and vice versa. Adding these equations gives

$$\begin{aligned} \dim H^0(X,\mathcal{O}_{D'}) - \dim H^1(X,\mathcal{O}_{D'}) &= \dim H^0(X,\mathcal{O}_D) - \dim H^1(X,\mathcal{O}_D) + \dim V + \dim W \\ &= \dim H^0(X,\mathcal{O}_D) - \dim H^1(X,\mathcal{O}_D) + \deg D' - \deg D. \end{aligned}$$

which shows that

$$\dim H^0(X,\mathcal{O}_{D'}) - \dim H^1(X,\mathcal{O}_{D'}) - \deg D' = \dim H^0(X,\mathcal{O}_D) - \dim H^1(X,\mathcal{O}_D) - \deg D.$$

It is now clear that if the Riemann–Roch theorem holds for one of D or D', then it holds for the other. If it holds for D it holds for $D+P$, and if it holds for $D = (D-P)+P$ it holds for $D-P$. An arbitrary divisor D on X can be written as a sum $D = P_1 + \cdots + P_m - P_{m+1} - \cdots - P_n$ from points in X. This proves the Riemann–Roch theorem by induction from $D = 0$. $\square$

5 Dirac Operators

In the remainder of this report we present a modern perspective on the Riemann–Roch theorem. Index theory allows this theorem to be approached by studying an invariant of an operator, called the index of a Dirac operator. In particular, applying index theory to complex manifolds gives a generalisation of the Riemann–Roch theorem to higher dimensions. Since index theory is a large field, we focus on giving an overview of the main ideas, rather than on proofs. Proofs can be found in Elliptic Operators, Topology and Asymptotic Methods (Roe 1998), which is our primary reference. Another comprehensive reference is Heat Kernels and Dirac Operators (Berline, Getzler, and Vergne 1992).

To define Dirac operators, we first need to introduce Clifford bundles. Let V be a real vector space with a symmetric bilinear form $(\cdot,\cdot)$. The Clifford algebra gives a way to multiply vectors, satisfying a certain rule. Namely, the Clifford algebra $\mathrm{Cl}(V)$ is heuristically the algebra generated by V subject to the rule that $v^2 = -(v,v)1$ for all $v \in V$, where 1 is the unit in $\mathrm{Cl}(V)$. The idea of generation can be made precise by enforcing a universal property.

Definition 5.1 (Clifford algebra)**.** Let V be as above. A Clifford algebra for V is an associative algebra A equipped with a map $\varphi : V \to A$ such that $\varphi(v)^2 = -(v,v)1$, and which is universal among such maps in the sense that if there there exists another such map $\varphi' : V \to A'$, then there exists a unique algebra homomorphism $A \to A'$ such that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[dr, "\varphi'"'] & A \arrow[d] \\ & A' \end{tikzcd}$$

A Clifford algebra exists for any V and, by the universal property, is unique up to unique isomorphism. We denote it by $\mathrm{Cl}(V)$. Given a basis $e_1, \dots, e_n$ for V, we can realise $\mathrm{Cl}(V)$ as the span of the 2^n possible products of $\varphi(e_1)^{k_1}, \dots, \varphi(e_n)^{k_n}$, where each k_i is 0 or 1, with multiplication satisfying the rule

$$\varphi(v_1)\varphi(v_2) + \varphi(v_2)\varphi(v_1) = -2(v_1, v_2).$$

Another realisation of $\mathrm{Cl}(V)$ is the quotient of the tensor algebra $T(V)$ by the ideal generated by

$$I = \{v \otimes w + w \otimes v + 2(v,w) \mid v, w \in V\}$$

(Berline, Getzler, and Vergne 1992, Proposition 3.2). In fact, the map $\varphi : V \to \mathrm{Cl}(V)$ is injective, so we identify $v \in V$ with its image $\varphi(v) \in \mathrm{Cl}(V)$.

We define a Clifford module as a left module over the complexified Clifford algebra $\mathrm{Cl}(V) \otimes_{\mathbb{R}} \mathbb{C}$. The tangent bundle TM of a Riemannian manifold M gives rise to a Clifford bundle S. This is a bundle of Clifford modules, meaning that it is a vector bundle with fibres S_p for $p \in M$ which are left modules over $\mathrm{Cl}(T_pM) \otimes \mathbb{C}$, satisfying additional compatibility assumptions. Importantly, S has a connection. We review connections, vector bundles, and sections in Appendix C.

The Clifford bundles we are interested in have the additional structure of a $\mathbb{Z}_2$-grading. This is a direct sum decomposition $S = S_+ \oplus S_-$. Elements in S_+ as considered even and elements in S_- are considered odd. We ask that the metric and connection are even, in that they send S_+ to S_+ and S_- to S_-, while the Clifford action is odd, i.e. it sends S_+ to S_- and vice versa.

Definition 5.2 (Dirac operator)**.** The Dirac operator D of a Clifford bundle S is the operator on $\Gamma(S)$, the space of smooth sections of S, given by the composition

$$\Gamma(S) \xrightarrow{\nabla} \Gamma(T^*M \otimes S) \xrightarrow{g} \Gamma(TM \otimes S) \rightarrow \Gamma(S)$$

where the first arrow is the connection ∇ on S, the second arrow is the metric, and the final arrow is the Clifford action.

In local coordinates, i.e. in a local orthonormal tangent frame $(e_i)_{i\in I}$, the Dirac operator is expressed for any section $s \in S$ by

$$Ds = \sum_i e_i \nabla_i s.$$

In terms of the $\mathbb{Z}_2$-grading, D is odd, since the connection and the metric are even while the action is odd. This is an important property and leads to cancellations (discussed in Section 6).

A motivation for Dirac operators comes from trying to find a square root of a Laplacian operator. The square of the Dirac operator of a Clifford bundle as defined here is not a Laplacian, but it is related to one by the Lichnerowicz–Weitzenböck formula

$$D^2 = \nabla^*\nabla + \mathrm{F}^{\mathrm{S}} + \frac{1}{4}\kappa \tag{5.1}$$

Here $\nabla^*\nabla$ is the Bochner Laplacian (* denotes formal adjoint), F^{S} is a quantity of the Clifford bundle called the Clifford contraction of the twisting curvature, and κ is the scalar curvature of the metric on M.

The Dirac operator relevant to the Riemann–Roch theorem is the Dolbeault–Dirac operator $\bar\partial + \bar\partial^*$. This is defined on differential forms on a complex manifold X. Recall that differential forms make sense of expressions such as $f\,dx\,dy$. Let $\dim_{\mathbb{C}}(X) = n$. Consider local coordinates z_k, for $k = 1, \dots, n$, on an open subset U of X. Let $z_k = x_k + iy_k$. Then the differential of a smooth (infinitely differentiable with respect to the real coordinates) function f is given by

$$df = \sum_k \frac{\partial f}{\partial x_k}\, dx_k + \sum_k \frac{\partial f}{\partial y_k}\, dy_k.$$

After defining the Wirtinger derivatives

$$\frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right), \quad \frac{\partial}{\partial \bar{z}_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\right),$$

one calculates that the differential of f can also be expressed as

$$df = \sum_k \frac{\partial f}{\partial z_k}\, dz_k + \sum_k \frac{\partial f}{\partial \bar{z}_k}\, d\bar{z}_k.$$

The second sum in this expression is defined to be $\bar{\partial} f$. A smooth $(0,q)$-form on X is given by $\sum_{k=1}^{q} f_k\, d\bar{z}_{k_1}\, d\bar{z}_{k_2} \cdots d\bar{z}_{k_q}$ for smooth functions f_k. For example, if $n = 3$, a $(0,2)$-form looks like $f\, d\bar{z}_1\, d\bar{z}_2 + g\, d\bar{z}_1\, d\bar{z}_3 + h\, d\bar{z}_2\, d\bar{z}_3$. By convention, a function is a $(0,0)$-form. Denote by $\Omega^{(0,q)}(X)$ the space of smooth $(0,q)$-forms on X. The $\bar{\partial}$ operator on a $(0,q)$-form is given locally by

$$\sum_{k=1}^{q} f_k\, d\bar{z}_{k_1}\, d\bar{z}_{k_2} \cdots d\bar{z}_{k_q} \mapsto \sum_{k=1}^{q} \bar{\partial} f_k\, d\bar{z}_{k_1}\, d\bar{z}_{k_2} \cdots d\bar{z}_{k_q}$$

Note that it sends a $(0,q)$ form to a $(0,q+1)$ form. In fact, this map is independent of the choice of local coordinates, and so this defines $\bar{\partial}$ as an operator $\Omega^{0,q}(X) \to \Omega^{0,q+1}(X)$. Its formal adjoint $\bar{\partial}^*$ is therefore a map $\Omega^{0,q+1}(X) \to \Omega^{0,q}(X)$, so it sends $(q+1)$-forms to q-forms. As mentioned above, the sum $\bar{\partial} + \bar{\partial}^*$ is called the Dolbeault–Dirac operator.

The following theorem, stated as Proposition 3.27 by Roe (1998), shows how the Dolbeault–Dirac operator relates to the Dirac operator of a particular Clifford bundle.

Theorem 5.3. *Let X be a complex manifold. This induces a Clifford bundle S, called a spin bundle, such that the space of smooth sections $\Gamma(S)$ is isomorphic to the direct sum $\bigoplus_{q=0}^{\infty} \Omega^{0,q}$. The Dirac operator of S is $\sqrt{2}(\bar{\partial} + \bar{\partial}^*) + A$, where A is an endomorphism of S.*

Although $\sqrt{2}(\bar{\partial} + \bar{\partial}^*)$ is not a Dirac operator, it is up to an endomorphism of S. Such an operator is called a generalised Dirac operator, which share many properties with Dirac operators.

6 Index and Supertrace

The Fredholm index of a bounded linear operator between $T : X \to Y$ between Banach spaces is defined as

$$\operatorname{ind} T = \dim \ker T - \dim \operatorname{coker} T,$$

provided these dimensions are finite, where $\operatorname{coker} T = Y/\operatorname{im} T$. Such an operator is called a Fredholm operator. For an invertible operator, $\dim\ker T = \dim\operatorname{coker} T = 0$, so $\operatorname{ind} T = 0$. Unlike $\dim\ker T$ or $\dim\operatorname{coker} T$ when considered individually, their difference $\operatorname{ind} T$ is homotopy invariant (Bleecker and Booß-Bavnbek 2013, Theorem 3.11). This motivates looking for a way to express the index in terms of topological quantities (see Section 7).

To study geometry, we consider the index of a graded Dirac operator on a $\mathbb{Z}_2$-graded Clifford bundle $S = S_+ \oplus S_-$. Let D_+ be the restriction of D to sections in S_+, and D_- the restriction of D to sections in S_-. In the case of Dolbeault–Dirac operator $D = \sqrt{2}(\bar\partial + \bar\partial^*)$, D_+ is the restriction of D to the space of sections $\bigoplus_{q>0 \text{ even}} \Omega^{(0,q)}$ and D_- is the restriction to the space $\bigoplus_{q>0 \text{ odd}} \Omega^{(0,q)}$.

Dirac operators are an important example of elliptic operators, which are a type of differential operators. The restriction D_+ is also an elliptic operator. Elliptic operators over compact manifolds, which is the case we consider, are Fredholm (Bleecker and Booß-Bavnbek 2013, Corollary 9.18). Therefore, on a compact manifold, D_+ is a Fredholm operator. It is conventional, for instance see (Roe 1998, Definition 11.7), to define the index of a Dirac operator D as the Fredholm index of D_+. That is,

$$\operatorname{ind} D = \dim\ker D_+ - \dim\operatorname{coker} D_+.$$

However, the formal adjoint of D_+ is D_-, so one can calculate that $\dim\operatorname{coker} D_+ = \dim\ker D_-$. This recovers the usual definition of the index of a Dirac operator, namely

$$\operatorname{ind} D = \dim\ker D_+ - \dim\ker D_-.$$

One of the reasons that the index of a Dirac operator is important is that it can be explicitly computed. The first step towards this is the McKean–Singer formula. This relates the index to the heat operator associated to D, which is notated e^{-tD^2} and can be defined using spectral theory: see the section 'The functional calculus' in (Roe 1998). Note that the heat operator solves the heat equation for D, which is

$$\frac{\partial s}{\partial t} + D^2 s = 0.$$

A fundamental property of the heat operator is that it has a smooth kernel, which means that it can be expressed by integrating against a smooth function on $M \times M$. So for all $s \in L^2(S)$,

$$e^{-tD^2} s(p) = \int_M k_t(p,q) s(q)\operatorname{vol}(q).$$

The smooth kernel $k_t(p,q)$ is called the heat kernel. It depends on the parameter t since the heat operator also does. The existence of the smooth kernel is due to the fact that $e^{-tx^2}:\mathbb{R}\to\mathbb{R}$ is a rapidly decaying function (Roe 1998, Proposition 5.31).

The trace of an operator can be defined more generally, but since the heat operator is a self-adjoint operator with a smooth kernel, it is simply the sum of its eigenvalues, counted with multiplicity (Roe 1998, Propositions 8.7 and 8.10). Define the grading operator ε which sends $s=s_++s_-\in S$, where $s_+\in S_+$ and S_-, to $\varepsilon(s)=s_+-s_-\in S$. Finally, the supertrace of the heat operator is defined as the trace of the composition of the heat operator with the grading operator: $\operatorname{Tr}_s(e^{-tD^2}):=\operatorname{Tr}(\varepsilon e^{-tD^2})$.

Theorem 6.1 (McKean–Singer)**.** *Let D be a graded Dirac operator on a $\mathbb{Z}_2$-graded Clifford bundle S over a compact manifold M. Then*

$$\operatorname{ind} D=\operatorname{Tr}_s e^{-tD^2}.$$

Proof. Note that μ is an eigenvalue for εe^{-tD^2} if and only if μ is an eigenvalue for e^{-tD^2} with an even eigensection or $-\mu$ is an eigenvalue for e^{-tD^2} with an odd eigensection. For all eigenvalues λ of D^2, let $n_+(\lambda)$ denote the dimension of the λ-eigenspace H_λ^+ of D restricted to S_+, i.e. of D_-D_+, and correspondingly define $n_-(\lambda)$. Then, summing the eigenvalues of εe^{-tD^2},

$$\operatorname{Tr}_s e^{-tD^2}=\sum_\lambda e^{-t\lambda}(n_+(\lambda)-n_-(\lambda)).$$

We claim that all terms with $\lambda\neq 0$ cancel. Let $\lambda\neq 0$ and $s\in H_\lambda^+$. Since D_+ commutes with D^2,

$$D^2(D_+s)=D_+(D^2s)=D_+\lambda s=\lambda(D_+s),$$

so $D_+s\in H_\lambda^-$. This gives a linear map $H_\lambda^+\xrightarrow{D_+}H_\lambda^-$. For $s\in H_\lambda^-$, by a similar argument

$$D^2(\lambda^{-1}D_-s)=\lambda^{-1}D_-(D^2s)=\lambda^{-1}D_-(\lambda s)=\lambda(\lambda^{-1}D_-s).$$

Therefore there is a linear map $H_\lambda^-\xrightarrow{\lambda^{-1}D_-}H_\lambda^+$ which is clearly a two-sided inverse to the map $H_\lambda^+\xrightarrow{D_+}H_\lambda^-$. This shows that H_λ^+ and H_λ^- are isomorphic, so $n_+(\lambda)-n_-(\lambda)=0$ for all $\lambda\neq 0$. Hence

$$\begin{aligned}\operatorname{Tr}_s e^{-tD^2}&=\sum_\lambda e^{-t\lambda}(n_+(\lambda)-n_-(\lambda))\\&=n_+(0)-n_-(0)=\dim\ker D_-D_+-\dim\ker D_+D_-\\&=\dim\ker D_+-\dim\ker D_-=\operatorname{ind} D.\end{aligned}\tag{6.1}$$

Equation (6.1) uses the fact that $\ker D^2 = \ker D$, which follows from the fact that D is formally self-adjoint. Indeed, if $x \in \ker D^2$, then

$$\langle D^2 x, x\rangle_2 = \langle Dx, Dx\rangle_2 = \|Dx\|_2^2 = 0$$

using the L^2 inner product, so $x \in \ker D$. $\square$

The cancellations in the proof also have an interpretation in terms of supersymmetric quantum mechanics, where the $\mathbb{Z}_2$-grading corresponds to bosons and fermions (Alvarez-Gaumé 1983).

By the McKean–Singer formula, the heat operator is an approach to calculating the index. On the other hand, since the index is a integer and independent of t, so is the supertrace of the heat operator. For an operator with a smooth kernel, such as the heat operator, its trace is given by integrating the kernel along the 'diagonal' $k(x, x)$. Adjusting this to give a characterisation of the supertrace gives

$$\operatorname{ind} D = \operatorname{Tr}_s e^{-tD^2} = \int_M \operatorname{tr}_s(k_t(x,x)) \operatorname{vol}(x),$$

where the local supertrace $\operatorname{tr}_s(a)$ is defined as $\operatorname{tr}(\varepsilon a)$ for all $a \in \operatorname{End}(S_x)$, where tr is the usual trace of an endomorphism (Roe 1998, Proposition 11.2).

This is a tractable expression since the heat kernel k_t has an asymptotic expansion. Namely, there is an asymptotic expansion of $\operatorname{ind} D$ near $t = 0$

$$\operatorname{ind} D = \operatorname{Tr}_s e^{-tD^2} \sim \frac{1}{(4\pi t)^{n/2}} \left(\int \operatorname{tr}_s \Theta_0 \operatorname{vol} + t \int \operatorname{tr}_s \Theta_1 \operatorname{vol} + \dots \right),$$

where the asymptotic expansion coefficients Θ_0, Θ_1 are given by algebraic expressions in terms of the metric, the connection coefficients, and their derivatives. It appears that taking $t \to 0^+$ causes this expression to diverge to infinity. However, we know that it is equal to the index of D, which is constant. Therefore, all the terms must be zero, except potentially for the constant term which appears when n is even. Hence for an even-dimensional real manifold, such as a complex manifold, we have

$$\operatorname{ind} D = \frac{1}{(4\pi t)^{n/2}} \int \operatorname{tr}_s \Theta_{n/2} \operatorname{vol}$$

(Roe 1998, Proposition 11.4). It only remains to evaluate the local supertrace. This can be computed using a symbolic calculus due to Getzler (1986).

7 The Hirzebruch–Riemann–Roch Theorem

During the final calculations, the square of the Dirac operator introduces terms related to curvature, by the Lichnerowicz–Weitzenböck formula, Equation (5.1). This is related to topological quantities called characteristic classes. The Chern–Weil theory of characteristic classes associates a de Rham cohomology class, i.e. a closed differential form, to a vector bundle by applying an invariant polynomial P to its curvature K, considered as a matrix in local coordinates. The invariance of P means that $P(XY) = P(YX)$ so $P(Y^{-1}XY) = P(X)$; this ensures that $P(K)$ is independent of the choice of local coordinates. This finally leads to the Atiyah–Singer index theorem, which expresses the index of a Dirac operator in terms of an integral involving characteristic classes.

Applied to the Dolbeault–Dirac operator, the Atiyah–Singer theorem leads to the Hirzebruch–Riemann–Roch theorem. For a holomorphic vector bundle V over a compact manifold X, let $H^{0,k}(V)$ be the de Rham cohomology group of $\Omega^{0,k}(V)$, the space of smooth sections of $(0,k)$-forms on V. These spaces are finite dimensional due to Hodge theory, and the Hirzebruch–Riemann–Roch theorem computes their alternating sum, called the Euler characteristic.

Theorem 7.1 (Hirzebruch–Riemann–Roch)**.** *Let V be a holomorphic vector bundle over a compact complex manifold X with complex dimension n. Then*

$$\chi(X, V) = \sum_k (-1)^k \dim H^{0,k}(V) = \int_X \mathrm{ch}(X)\,\mathrm{td}(X)$$

where $\mathrm{ch}(X)$ is the Chern character of X and $\mathrm{td}(X)$ is the Todd class of the tangent bundle TX.

8 Discussion and Conclusion

This report explored the Riemann–Roch theorem from two viewpoints. First we proved the classical statement in one dimension using cohomology. Then we gave an overview of index theory, which gives an alternative and modern perspective involving invariants of operators, allowing for the Riemann–Roch theorem to be generalised to arbitrary dimensions.

Generalisations and recent directions in index theory include studying families of operators rather than a single operator, equivariant index theory which involves additional symmetries, index theory on manifolds with boundary, and extending index theory to non-elliptic operators such as Bismut's hypoelliptic Laplacian. An alternative generalisation of the Hirzebruch–Riemann–Roch theorem is the Grothendieck–Riemann–Roch theorem in algebriac geometry.

Appendix A Sheaves

In order to define a sheaf, we first define a presheaf. This is a collection of data assigned to open sets in a topological space, with a way to restrict to smaller open subsets.

Definition A.1 (Presheaf)**.** Let $(X, \mathcal{T})$ be a topological space. A presheaf of abelian groups on X is a pair $(\mathscr{F}, \rho)$ where

i) $\mathscr{F} = (\mathscr{F}(U))_{U \in \mathcal{T}}$ is a family of abelian groups defined on the open sets of X

ii) $\rho = (\rho_V^U)_{U,V \in \mathcal{T}, V \subseteq U}$ is a family of group homomorphisms $\rho_V^U : \mathscr{F}(U) \to \mathscr{F}(V)$

such that $\rho_U^U = \mathrm{id}_{\mathscr{F}(U)}$ for all $U \in \mathcal{T}$ and $\rho_W^V \circ \rho_V^U = \rho_W^U$ for all $U, V, W \in \mathcal{T}$ with $W \subseteq V \subseteq U$.

We similarly define a presheaf of vector spaces. The homomorphisms are called restriction homomorphisms, and ρ_V^U should be read as the restriction from U to V. Elements of a presheaf are called sections. For a section $f \in \mathscr{F}(U)$, we use the simpler notation $f|_V = \rho_V^U(f)$ for the restriction of f to V. In all the examples needed in this report, the restriction homomorphisms are just the usual restriction maps.

Definition A.2 (Sheaf)**.** A sheaf is a presheaf $\mathscr{F}$ on a topological space X such that for every open set U and every collection of open sets $(U_i)_{i \in I}$ such that $\bigcup_i U_i = U$, the following conditions are satisfied:

(S1) If $f, g, \in \mathscr{F}(U)$ are sections such that $f|_{U_i} = g|_{U_i}$ for all $i \in I$, then $f = g$

(S2) Given sections $f_i \in \mathscr{F}(U_i), i \in I$, over U_i such that

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j} \qquad \text{for all } i, j \in I,$$

there exists a section $f \in \mathscr{F}(U)$ over U such that $f|_{U_i} = f_i$ for every $i \in I$.

The sheaf axiom (S1) states that sections are determined by their local behaviour, and (S2) states that compatible sections, namely sections which agree on their overlaps, can be glued together to give a section defined on a larger domain. Note that for presheaves of functions with their natural restriction maps, these conditions are trivially satisfied, so they are sheaves.

Sheaves are defined on open sets. It will also be useful, in particular when defining exact sequences of sheaves, to consider a particular point. This is the idea of the stalk of a sheaf at a point a, which is the quotient of the sheaf by local equivalence at a.

Definition A.3 (Stalk)**.** Let $\mathscr{F}$ be a presheaf of sets on a topological space X. Let $a \in X$ be a point and $N(a)$ be the set of open neighbourhoods of a. On the disjoint union

$$\bigcup_{U \in N(a)} \mathscr{F}(U),$$

let two sections $f \in \mathscr{F}(U)$ and $g \in \mathscr{F}(V)$ be equivalent, written $f \sim g$, if there exists an open set W with $a \in W \subseteq U \cap V$ such that $f|_W = g|_W$. The stalk of $\mathscr{F}$ at the point a is the quotient

$$\mathscr{F}_a := \Big(\bigcup_{U \in N(a)} \mathscr{F}(U) \Big) \Big/ \sim .$$

If $\mathscr{F}$ is a presheaf of abelian groups, then $\mathscr{F}_a$ is again an abelian group with addition defined via representatives.

Appendix B Proof of the Riemann–Roch Theorem

The exact sequence of sheaves used in the proof of the Riemann–Roch theorem is described as follows. Let D be an arbitrary divisor on a compact Riemann surface X and $D' = D + P$, as in the proof. There is an inclusion $\mathcal{O}_D \hookrightarrow \mathcal{O}_{D'}$ because the sheaf $\mathcal{O}_{D'}$ permits one more pole at the point P than $\mathcal{O}_D$ does. Let (V, z) be local coordinates on X with $z(P) = 0$. We define a sheaf homomorphism

$$\beta : \mathcal{O}_{D'} \to \mathbb{C}_P$$

for all open sets U in X as follows. If $P \notin U$, let β_U be the zero homomorphism. If $P \in U$ and $f \in \mathcal{O}_D(U)$, the function f admits a Laurent series expansion about P, given by

$$f = \sum_{n=-D(P)-1}^{\infty} c_n z^n$$

in the local coordinate z. Set $\beta_U(f) = c_{-D(P)-1} \in \mathbb{C} = \mathbb{C}_P(U)$.

Clearly β_U is surjective. Moreover, if $f \in \mathcal{O}_D$, then it cannot have a pole of order $D(P)+1$ at the point P. Hence $\beta_U(f) = 0$. Now suppose that $f \in \ker \beta \subset \mathcal{O}_{D'}$. Then $\mathrm{ord}_P(f) \geq -D(P)$ since $c_{-D(P)-1} = 0$. Hence $f \in \mathcal{O}_D$. This shows that the image of the inclusion $\mathcal{O}_D \hookrightarrow \mathcal{O}_{D'}$ equals $\ker \beta$. Therefore there is an exact sequence of sheaves

$$0 \to \mathcal{O}_D \hookrightarrow \mathcal{O}_{D'} \xrightarrow{\beta} \mathbb{C}_P \to 0.$$

We also compute cohomology groups of the skyscraper sheaf. By definition of the skyscraper sheaf, $H^0(X,\mathbb{C}_P)=\mathbb{C}_P(X)=\mathbb{C}$. To calculate $H^1(X,\mathbb{C}_P)$, let $\mathfrak{U}$ be an open cover of X, and let $\xi\in H^1(X,\mathbb{C}_P)$ be a cohomology class represented by a cocycle $(f_{ij})_{i,j\in I}$ in $Z^1(\mathfrak{U},\mathbb{C}_P)$. Then $\mathfrak{U}$ has a refinement $\mathfrak{V}=(V_i)_{i\in I}$ such that $P\in V_i$ for precisely one $i\in I$. Then $P\notin V_i\cap V_j$ for all $i,j\in I$ with $i\neq j$. By definition of $\mathbb{C}_P$, this means that $f_{ij}|_{V_i\cap V_j}=0$ for $i\neq j$. It is a general fact that $f_{ii}=0$ for cocycles; to see this, note that $f_{ik}=f_{ij}+f_{jk}$ and take $i=j=k$. Hence $Z^1(\mathfrak{V},\mathbb{C}_P)=0$ and so $\xi=0$. As ξ was arbitrary, we conclude that $H^1(X,\mathbb{C}_P)=0$.

Appendix C Vector Bundles and Connections

The aim of this appendix is to define connections on vector bundles. These definitions can be found in textbooks in differential geometry; we have referred to the textbook by Lee (2009).

An example of a vector bundle is the tangent bundle TM of a smooth manifold. This is a smooth manifold with a projection map $\pi : TM\to M$. For all $p\in M$, $\pi^{-1}(p)=T_pM$ is a real vector space.

Definition C.1 (Vector bundle)**.** Let E and M be smooth manifolds and $\pi : E\to M$ be a smooth map. Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, and let V be a finite-dimensional $\mathbb{K}$-vector space. The quadruple (E,π,M,V) is a smooth $\mathbb{K}$-vector bundle with typical fibre V if the following conditions are satisfied:

(i) Every point $p\in M$ has an open neighbourhood and a diffeomorphism $\phi:\pi^{-1}(U)\subset E\to U\times V$ such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & U\times V \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

where $\mathrm{pr}_1 : (u,v)\mapsto u$ projects onto the first factor. The pair (U,ϕ) is called a local trivialisation of the vector bundle. Due to the commutative diagram, ϕ must be of the form (π,Φ) for a map $\Phi:\pi^{-1}(U)\to V$.

(ii) For all $x\in M$, the set $E_x:=\pi^{-1}(x)$ is a $\mathbb{K}$-vector space, isomorphic to the typical fibre V.

(iii) Every point $p\in M$ is in the domain of some local trivialisation (U,ϕ) such that for all $x\in U$,

$$\Phi|_{E_x} : E_x\to V$$

is an isomorphism of vector spaces.

We call E the total space, π the vector bundle projection, and M the base space. For each $p \in M$, the set $E_p := \pi^{-1}(p)$ is called the fibre over p. Abusing notation, we may simply write the vector bundle as V.

Definition C.2 (Section of a vector bundle)**.** A smooth section of a vector bundle (E, π, M, V) is a smooth right inverse of the vector bundle projection map, i.e. a smooth map $\sigma : M \to E$ such that $\pi \circ \sigma = \mathrm{id}_M$. The set of smooth sections of the vector bundle V is denoted $\Gamma(V)$.

A section of a vector bundle is different to a section of a sheaf, although they can both be viewed as generalisations of functions. In the case of the tangent bundle TM, a section is a vector field on M. Connections give a way to differentiate sections, in analogy to how directional derivatives can differentiate vector fields.

Definition C.3 (Connection)**.** A connection on a smooth $\mathbb{K}$-vector bundle (E, π, M, V) is a bilinear map

$$\nabla : \Gamma(TM) \times \Gamma(V) \to \Gamma(V)$$

such that for all smooth functions f on M, $X \in \Gamma(TM)$ and $s \in \Gamma(V)$,

$$\nabla_{fX}(s) = f\nabla_X s, \quad \text{and} \quad \nabla_X(fs) = (Xf)s + f\nabla_X s,$$

where Xf is the Lie derivative of X along f.

All manifolds have connections. Riemannian manifolds have a connection called the Levi–Civita connection which is compatible with its metric in a sense that can be made precise.

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