Long-Term Behaviour of Ranking-Based Process

Kate Zhang

Supervised by Dr Nathan Ross

The University of Melbourne

Contents

Abstract

We study systems with items who's counts evolve over time. Specifically, the rule for evolution is random and depends on the ranking of the current counts of items. We model this kind of system by a random walk where the distribution of the random step depends on the ranking of its components. Our interest lies in the whether the ranking stabilises in the long term. We show that under a mild assumption, the ranking of the system stabilises almost surely and discuss a counterexample where stability might not be reached if the condition is not met.

Acknowledgements

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1 Introduction

In many systems with rich-get-richer dynamics, the rank of the object of interest plays a crucial role. For example, users of online shopping platforms are more likely to click on products with higher popularity rank that appear on the top of the screen, hence making these products more popular and more likely to be clicked on [6]. Rankings drives the rich-get-richer dynamics in these ranking-based system.

Despite its prevalence, the dynamics and long-term behaviour of ranking-based systems have not been extensively explored. There is one previous work [1] that models the ranking-based systems in the general case, and the models in other related work [7, 2, 5, 3] only allow binary increment of one component at each "step". The model introduced in [1] is a discrete-time Markov process $X_n \in \mathbb{R}^d$, where the ith component corresponds to the value of item i, and distribution of the future increments depends on the ranking of its components. This model allows increments to take continuous values and enables various component to change simultaneously. The value of the process at each discrete time n can be considered as a proxy for the ranking at time n. Assuming that no ties are allowed, [1] showed that if for any pairs of components in the system, the component that is ranked higher gets more increment on average than the other component in the pair, then the ranking will stabilise almost surely. With a weaker assumption that under any ranking, it's possible for any component to get the largest increment at any time, [1] showed that the ranking stabilises with positive probability. However, both assumptions, especially the one required for almost sure stability, are quite strong, and are not seen in many ranking-based system in reality. For example, in online shopping platforms, if the lower ranked product has very high quality, then it is possible for a lower ranked product to get more increment on average then a higher ranked product.

1.1 Model

In this work, we use the model introduced in [1] and we consider a non-homogeneous random walk $(X_n)_{n\geq 0} \in \mathbb{R}^d$, where each component is the count of an item in the system at time n. The distribution of the random step—the distribution of the increment—depends only on the ranking of its components. Specifically, if the current ranking of the walk is r, then the random step follows a probability distribution μ^r. The fact that the distribution of the step doesn't change as long as the ranking doesn't change, and that the ranking is fully determined by the differences of all possible pairs of components in the system allows us to model the dynamics of each ranking by another random walk. For every ranking r in the system, we can model its dynamics by $(Y_n^r)_{n\geq 0} \in \mathbb{R}^{\binom{d}{2}}$. The components of the $(Y_n^r)_{n\geq 0}$ are the differences between the count of all possible ordered pairs of components i, j in the system, where component i is ranked higher than component j. Specifically, we denote the random walk of this component as $(Y_n^{(i,j)^r})_{n\geq 0}$. For each pair i, j, the random steps of $(Y_n^{(i,j)^r})_{n\geq 0}$ are independent and identically distributed with the same distribution as $Z_i^r - Z_j^r$, where (Z_i^r, Z_j^r) are distributed as the components i and j of μ^r. The drift of the $(Y_n^{(i,j)^r})_{n\geq 0}$—the expectations of the random steps—tells us a lot about its behaviour. We let the drift of $(Y_n^r)_{n\geq 0}$ be $\mathbb{E}(Z_i^r - Z_j^r) = q_i^r - q_j^r$.

Given that the system enters a ranking r, it is still at r after n discrete time step if all components of $(Y_n^r)_{n\geq 0}$ never admits negative values up to the nth step. This fact allows us to study the dynamics of the system by studying $(Y_n^r)_{n\geq 0}$ for all possible rankings r in the system, up until the (possibly infinite) time they leave. A more detailed description of this random walk can be found in Section 2.3.

For convenience, we assume that there are no ties between the counts and that for all pairs of items i, j $q_i^r \neq q_j^r$. However, all our results can be easily extended to a model that allows ties by taking into account a few extra cases.

1.2 Stable pair and stable ranking

If the rankings of the system stabilises in the long term, there must be at least a ranking in the system that is relatively "stable" where the system can stay there forever. To explore which pair of items and which ranking in the systm are "stable", we introduce the concept of stable pair and stable ranking.

We classify all ordered pairs into two categories, either "stable" or "unstable". If component i is ranked higher than j and on average i has more increment than j (in other words, $q_i^r > q_j^r$), then we call (i, j) a *stable pair*, otherwise (i, j) is an *unstable pair*.

We say that a ranking is a *stable ranking* if all pairs under this ranking are stable pairs, otherwise it's an unstable ranking.

1.3 Simple example: $(X_n)_{n\geq 0} \in \mathbb{R}^2$

Consider a simple system with only 2 items a and b. In this section we explore the dynamics of this simple system.

Let's first consider the implication of a stable ranking. If ranking ab is a stable ranking, then $(Y_n^{ab})_{n\geq 0} \in \mathbb{R}$ is a random walk with random step following the same distribution as $Z_a^r - Z_b^r$. $(Y_n^{ab})_{n\geq 0}$ has drift $q_a^r - q_b^r > 0$, hence it has positive probability of never admitting negative values (see Theorem 10.1. in [4]). This implies that once the system enters a stable ranking, it has positive probability of staying there forever. If ab is unstable,

$(Y_n^{ab})_{n\geq 0} \in \mathbb{R}$ is a random walk with drift $q_a^r - q_b^r < 0$, hence it eventually becomes negative (and thus the ranking changes) almost surely (see Theorem 8.2. in [4]). This implies that the system leaves any unstable rankings almost surely.

First consider the dynamics of the system when both ab and ba are stable. In this case, entering any ranking in the system, we will have uniformly bounded below by 0 probability of staying in that ranking forever. It is easy to see that in this case the system reaches stability almost surely. Next, assume ab is stable and ba is unstable. In this case if the system enters ba, we will go to ab almost surely following the drift of $(Y_n^{ba})_{n\geq 0}$. We say ranking ba *drifts* to ab if from ba the system can naturally enter ab following the drift of $(Y_n^{ba})_{n\geq 0}$. Since ba drifts to ab and ab is stable, it follows that the system also staying in that ranking forever almost surely in this case. Lastly, if both ab and ba are unstable, then ab drifts to ba and ba drifts to ab. In this case the system will keep oscillating between ab and ba and will never reach stability. We show in our work that given there is at least one stable ranking in the system, this kind of endless oscillation between unstable rankings is the only way the for the system to not stabilise almost surely.

1.4 Main results

In higher dimension the situation is much more complicated than the simple scenario above, since the random walks are multidimensional and the components are correlated. To deal with this complication, we make Assumption 3.4, which ensures that for any ranking r in the system, if we only consider the random walk with components that are the differences of stable pairs in $(Y_n^r)_{n\geq 0}$, then this random walk has positive probability of entering the positive orthant from the origin without leaving the closed positive orthant. With this assumption, we prove that for any ranking r, the stable pairs have positive probability of retaining their relative order until the system leaves ranking r (Lemma 3.6). With this Lemma, we show that even in higher dimension, once the system enters a stable ranking, it has positive probability of staying there forever (Proposition 3.7).

In higher dimension, for every unstable ranking r we specifies the *drift set* of r, D_r, such that under Assumption 3.4, if the system leaves r, it has uniformly bounded away from 0 probability of entering one of the rankings in D_r (Lemma 4.4). This is one of the key Lemmas in this project as it gives a positive probability path of transitions between rankings.

In higher dimension, the oscillatory behaviour that we saw in Section 1.3 becomes oscillatory behaviour at the level of drift sets. (See Figure 3 for an example of this oscillatory behaviour in higher dimension, in this diagram bac and bca are two unstable rankings. We draw an arrow from a ranking r to another ranking r' if r' is in the drift set of r). Hence to ensure that the ranking will stabilise almost surely, we construct an extra condition, which helps eliminates the oscillatory behaviour. Under Assumption 3.4 and this extra condition we show that the rankings will stabilise almost surely. We characterise a more general condition for the rankings to stabilise almost surely than the one in [1].

The organisation of the document is as follows: in Section 2 we introduce the notations; in Section 3 we introduce the concept of stable and unstable objects and prove their properties; in Section 4 we introduce the idea of drift and use it to show the possible transitions that the system can make between rankings; finally in Section 5 we state the condition required for the rankings to stabilise almost surely; in Section 6 we provide the proof for our key Lemma.

1.5 Statement of authorship

The results presented in this report is the work of the author under the supervision of Dr Nathan Ross.

2 Preliminaries

2.1 Rankings

Let X be a vector in $\mathbb{R}^d$ with unique elements. The ranking r of X is the permutation of the components of X satisfying $r(a) > r(b)$ whenever $X_a > X_b$. We denote the ranking of X by $\mathrm{rk}(X)$.

Given that $\mathrm{rk}(X) = r$, and a, b are components of X, we say that $r(a) < r(b)$ if a is ranked higher than b.

We denote $\mathcal{R} = \mathcal{R}(d)$ as the set of all possible rankings of a count vector with d components.

2.2 Ranking-based process

Let ν be a probability distribution on $\mathbb{R}^d$ with finite second moments. For each $r \in \mathcal{R}$, let μ^r be a probability distribution on $\mathbb{R}^d$, also with finite second moments. We consider a random walk $(X_n)_{n\geq 0}$, where $X_0 \sim \nu$ and the random step at time t, $\Delta X_t = X_{t+1} - X_t \mid X_t \sim \mu^{\mathrm{rk}(X_t)}$.

For each $r \in \mathcal{R}$, we denote by $Z^r = (Z_r^1, ..., Z_r^d)$ a random variable with distribution μ^r and $q^r = \mathbb{E}(Z^r)$.

2.3 Ranking and random walk

The fact that the distribution of the increment of each component purely depends on the current ranking allows us to consider the dynamics of each ranking separately. Specifically, we can model the dynamics of each ranking with a multidimensional random walk.

For an arbitrary ranking r let $T^r = \inf\{n \in \mathbb{Z}_0^+ : \mathrm{rk}(X_n) = r\}$. We denote the set of ordered indices as $\mathcal{I}_r = \{(i,j) \in [d] \times [d] : r(i) < r(j)\}$. For each $(i,j) \in \mathcal{I}_r$, we let

$$Y_0^{(i,j)^r} = 0,$$

and

$$Y_0^r = (Y_0^{(r^{-1}(1),r^{-1}(2))^r}, \dots, Y_0^{(r^{-1}(d-1),r^{-1}(d))^r}) \in \mathbb{R}^{\binom{d}{2}}.$$

For each (i,j), we let $\Delta Y_0^{(i,j)^r}, \Delta Y_1^{(i,j)^r}, \dots$ be independent and all have the same distribution as $Z_i^r - Z_j^r$. For $n \in \mathbb{Z}^+$, let

$$\Delta Y_n^r = (\Delta Y_n^{(r^{-1}(1),r^{-1}(2))^r}, \dots, \Delta Y_n^{(r^{-1}(d-1),r^{-1}(d))^r}) \in \mathbb{R}^{\binom{d}{2}}.$$

We define the random walk of r as the sequence $(Y_n^r)_{n\geq 0}$ where

$$Y_{n+1}^r = Y_n^r + \Delta Y_n^r.$$

When it comes to the dynamics of the entire system, we can model what happens to the system once it enters ranking r with $(Y_n^r)_{n\geq 0}$. Since $(Y_n^r)_{n\geq 0}$ starts at the origin, the random vector of the walk at each discrete time n represents $X_{T_r+n} - X_{T_r}$. Specifically, if we have $t = \inf\{n \in \mathbb{Z}^+ : \exists(i,j) \in \mathcal{I}_r, Y_n^{(i,j)^r} < (X_{T_r}^j - X_{T_r}^i)\} < \infty$, then this corresponds to $\mathrm{rk}(X_{T^r+t}) \neq r$.

3 Stable pair and stable ranking

As mentioned in the introduction, if the ranking stabilises, there has to be at least one ranking in the system that is relatively "stable" so there is positive probability for the system to stay there forever. In this section, we specify the definition of stable pair and stable ranking and show why they are relatively "stable".

Definition 3.1. For any pairs of components i, j from a ranking-based process and $r \in \mathcal{R}$ with $r(i) < r(j)$. We say that (i,j) is a *stable pair* under r if $q_i^r > q_j^r$. We denote the set of stable pairs of r as S_r.

Definition 3.2. For any $r \in \mathcal{R}$, we say that r is a *stable ranking* if $|S_r| = \binom{d}{2}$, otherwise we say that r is an *unstable ranking*. We denote the set of all stable rankings in the system as S and the set of all unstable rankings in the system as U.

In the next proposition, we show that the process leaves every unstable ranking with probability 1.

Proposition 3.3. *Assuming T_0 is a stopping time with $rk(X_{T_0}) = r \in U$ on the event that $T_0 < \infty$. Let $T' = \inf\{n > T_0 : rk(X_n) \neq r\}$. Then*

$$\mathbb{P}(T' < \infty | \, rk(X_{T_0}) = r) = 1. \tag{1}$$

Proof. On the event that $T_0 < \infty$, it's enough to show that the stopping time $\inf\{n : Y_n^{(i,j)^r} < -M\}$ for some $(i,j) \in \mathcal{I}_r$ if finite for all $M > 0$.

By definition, there exist at least a pair of components (i,j) such that $r(i) < r(j)$ and $q_j^r > q_i^r$. This means there exists $(i,j) \in \mathcal{I}_r$ such that $\mathbb{E}[\Delta Y_n^{(i,j)^r}] < 0$.

By the well-known result that a random walk with negative drift is transient and goes to $-\infty$ almost surely (see Theorem 8.2. in [4]), we get

$$\begin{aligned}\mathbb{P}(T_1 < \infty | \, \mathrm{rk}(X_{T_0}) = r) &\geq \mathbb{P}\Big(\bigcup_{(i,j)\in\mathcal{I}_r} \bigcup_{t>0} \{Y_t^{(i,j)^r} < (X_{T_r}^j - X_{T_r}^i)\}\Big) \\ &= 1.\end{aligned} \tag{2}$$

□

Before we move our focus to the stable pair and stable ranking, we need to introduce the following assumption, which is necessary to ensure the stable property of the stable pair and stable ranking.

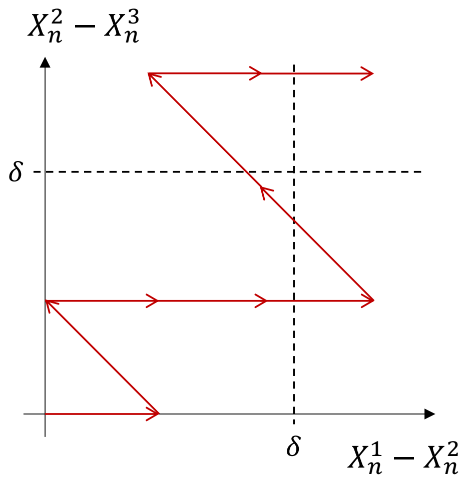

Figure 1: Random walk of pairs $(1, 2)$ and $(2, 3)$ entering the δ-positive orthant. Time axis is hidden.

Assumption 3.4. For any $r \in \mathcal{R}$, if we consider the random walk with components that are the differences of stable pairs in $(Y_n^r)_{n\geq 0}$, then for some $\epsilon, \delta > 0$, this random walk has greater than ϵ probability of entering the δ-positive orthant without leaving the positive orthant. In mathematical symbols, we have

$$\mathbb{P}\Big(\bigcup_{t>0}\big(\bigcap_{(i,j)\in S_r}\{Y_t^{(i,j)^r} > \delta\} \quad \bigcap \quad \bigcap_{s=1}^{t-1}\bigcap_{(i,j)\in S_r}\{Y_s^{(i,j)^r} \geq 0\}\big)\Big) > \epsilon.$$

Remark 3.5. One might think this assumption looks rather complex but it is easy to verify that it's a weak assumption. For example this alternative assumption below implies Assumption 3.4, but is stricter.

For any $r \in \mathcal{R}$ and $(i, j) \in S_r$

$$\mu^r\Big(\{X_i > X_j\} \quad \cap \quad \forall (k,l) \in S_r \setminus (i,j) \quad \{X_k \geq X_l\}\Big) > 0.$$

This alternative assumption ensures that for the random walk with components that are the differences of stable pairs in $(Y_n^r)_{n\geq 0}$, every component has positive probability of taking a step in the positive direction while all other components don't take steps in the negative direction. However, it turns out that the alternative assumption is stricter than Assumption 3.4. See Figure 1 for an example. The random walk in Figure 1 can either only increased by $(0, w)$ or $(-w, w)$ at each step, where w is some constant. This random walk has positive probability of entering the δ-positive orthant following the steps in the Figure 1. Hence this walk satisfies Assumption 3.4 but doesn't satisfy the alternative assumption since there's no way to increase in the vertical direction without decreasing in the horizontal direction.

Under Assumption 3.4, the next Lemma and Proposition shows that the stable pairs and stable rankings are indeed stable. Moreover, Lemma 3.6 shows that under Assumption 3.4, for any ranking $r \in \mathcal{R}$, the stable pairs under r have positive probability of retaining their relative order until the system leaves ranking r; Proposition 3.7 shows that under Assumption 3.4, once the system enters a stable ranking, it stays in that ranking forever with positive probability.

Lemma 3.6. *Let X_n be a ranking-based process with d items satisfying Assumption 3.4, then for any $r \in \mathcal{R}$ and some $\epsilon > 0$,*

$$\mathbb{P}\Big(\bigcap_{t>0}\bigcap_{(i,j)\in S_r}\{Y_t^{(i,j)^r} \geq 0\}\Big) > \epsilon. \tag{3}$$

Proof. We know that for all $(i,j) \in S_r$, $(Y_n^{(i,j)^r})_{n\geq 0}$ is a random walk with positive drift, hence as $n \to \infty$, $Y_n^{(i,j)^r} \to \infty$ (Theorem 8.2. in [4]). Hence, there exists some $M > 0$ such that, for all $n_0 > 0$,

$$\mathbb{P}\,(\bigcup_{n>0} Y_{n_0+n}^{(i,j)^r} - Y_{n_0}^{(i,j)^r} \leq -M) < \frac{1}{d^2}. \tag{4}$$

Let

$$A_{n_0} = \bigcap_{(i,j)\in S_r}(\{Y_{n_0}^{(i,j)^r} > M\}\bigcap_{n=0}^{n_0-1}\{Y_n^{(i,j)^r} \geq 0\}).$$

For $(i,j) \in S_r$, let

$$B_{n_0}^{(i,j)} = \bigcap_{n>0}\{(Y_{n_0+n}^{(i,j)^r} - Y_{n_0}^{(i,j)^r}) > -M\}.$$

It is clear that the Lemma follows if $A_{n_0} \cap \bigcap_{(i,j)\in S_r} B_{n_0}^{(i,j)}$ happens with uniformly bounded away from 0 probability. Note that A_{n_0} is independent of $\bigcap_{(i,j)\in S_r} B_{n_0}^{(i,j)}$.

Let $\delta > 0$ be as in Assumption 3.4 and let $\beta = \lceil \frac{M}{\delta} \rceil$. It follows that for some $\epsilon_1 > 0$ and some $n_0 > 0$,

$$\mathbb{P}(A_{n_0}) > \epsilon_1^\beta. \tag{5}$$

Now for $\bigcap_{(i,j)\in S_r} B_{n_0}^{(i,j)}$, we have that for some $\epsilon_2 > 0$

$$\begin{aligned}
\mathbb{P}\,(\bigcap_{(i,j)\in S_r} B_{n_0}^{(i,j)}) &= 1 - \mathbb{P}\,(\bigcup_{(i,j)\in S_r}\bigcup_{n>n_0}\{(Y_{n_0+n}^{(i,j)^r} - Y_{n_0}^{(i,j)^r}) \leq -M\}) \\
&\geq 1 - \sum_{(i,j)\in S_r}\mathbb{P}\,(\bigcup_{n>n_0}\{(Y_{n_0+n}^{(i,j)^r} - Y_{n_0}^{(i,j)^r}) \leq -M\}) \\
&\geq 1 - \frac{1}{d} \\
&> \epsilon_2,
\end{aligned} \tag{6}$$

where the first equality follows by taking complements; the first inequality follows from Boole's inequality; the second inequality follows from the fact that $|S_r| \leq \binom{d}{2}$.

Hence,

$$\begin{aligned}
\mathbb{P}\Big(\bigcap_{t>0}\bigcap_{(i,j)\in S_r}\{Y_t^{(i,j)^r} \geq 0\}\Big) &= \mathbb{P}\Big(A_{n_0} \cap \bigcap_{(i,j)\in S_r} B_{n_0}^{(i,j)}\Big) \\
&= \mathbb{P}\Big(A_{n_0}\Big)\cdot\mathbb{P}\Big(\bigcap_{(i,j)\in S_r} B_{n_0}^{(i,j)}\Big) \\
&> \epsilon_1^\beta \cdot \epsilon_2,
\end{aligned} \tag{7}$$

where the first equality follows from the definition of $A_{n_0} \cap \bigcap_{(i,j)\in S_r} B_{n_0}^{(i,j)}$; the second equality follows from the fact that A_{n_0} and $\bigcap_{(i,j)\in S_r} B_{n_0}^{(i,j)}$ are independent; and the inequality follows from (5) and (6). □

Proposition 3.7. *Let X_n be a ranking-based process satisfying Assumption 3.4, then for any initial distribution ν and some $\epsilon > 0$,*

$$\mathbb{P}_\nu(\bigcap_{t>n} \mathrm{rk}(X_t) \in S | \mathrm{rk}(X_n) \in S) > \epsilon \tag{8}$$

This Proposition directly follows from Lemma 3.6 since all pairs under a stable ranking are stable pairs.

4 Drift

In the last section, we have shown that the system leaves every unstable ranking almost surely. In this section, we explore the ways the process leaves a given unstable ranking. For a given unstable ranking r, we specify the *drift set* of r, denoted D_r, which are the set of rankings which the random walk enters by "respecting" the relative drifts. We will show in this section that under Assumption 3.4, the system has uniformly bounded away from 0 probability of arriving at a ranking in D_r after it leaves r.

Definition 4.1. Given a ranking r, we say that D_r is the *drift set* of r if $D_r = \{r' \in \mathcal{R} \setminus r : \forall (i,j) \in S_r, r'(i) < r'(j)\}$.

Given $\mathrm{rk}(X_0) = r$ and $T' = \inf\{n > 0 : \mathrm{rk}(X_n) \neq r\}$. We say that r *drifts* to r' if $\mathrm{rk}(X_{T'}) = r'$ and $r' \in D_r$. The probability of r drifting to r' from x is

$$P_x(r') = \mathbb{P}\left(\mathrm{rk}(X_{T'}) = r' \middle| X_0 = x \right). \tag{9}$$

Remark 4.2. Given the process leaves r to another ranking r'. All stable pairs under r remain their relative order if and only if r drifts to r'. This is clear from the definition of drift.

Example 4.3. Let X_n be a ranking-based process with 3 components a, b and c. If $r = abc$, then depending on q^r, D_r consists of the following rankings:

q^r	D_r
$q_a^r > q_b^r > q_c^r$	$\emptyset$
$q_a^r > q_c^r > q_b^r$	$\{acb\}$
$q_b^r > q_a^r > q_c^r$	$\{bac\}$
$q_b^r > q_c^r > q_a^r$	$\{bac, bca\}$
$q_c^r > q_a^r > q_b^r$	$\{(acb, cab\}$
$q_c^r > q_b^r > q_a^r$	$\{acb, bac, cab, bac, cba\}$

Now we prove our claim that under Assumption 3.4, the probability the we leave an unstable ranking to one of the rankings that it drifts to is bounded away from 0.

Lemma 4.4. *Let X_n be a ranking-based process that satisfies Assumption 3.4 with d components and $rk(x) = r \in U$. Assume $rk(X_0) = r$. Then there exist some $\epsilon > 0$ such that,*

$$\sum_{r' \in D_r} P_x(r') > \epsilon. \tag{10}$$

Proof of Lemma 4.4. Assume that the process satisfies Assumption 3.4 and let

$$E^r = \bigcap_{(i,j)\in S_r} \bigcap_{n>0} \{Y_n^{(i,j)^r} > 0\}$$

$$F^r = \bigcup_{(i,j)\in \mathcal{I}_r \setminus S_r} \bigcup_{n>0} \{Y_n^{(i,j)^r} < (X_0^j - X_0^i)\}$$

$$G^r = \bigcap_{(i,j)\in S_r} \bigcap_{n>0} \{Y_n^{(i,j)^r} > (X_0^j - X_0^i)\}.$$

Here E^r is the event that in $(Y_n^r)_{n\geq 0}$ all components of stable pairs under r always stay in the positive quadrant. F^r is the event that an unstable pair under r change their relative order. G^r is the event that in $(Y_n^r)_{n\geq 0}$, all stable pairs under r always remain their relative order. It is clear that the Lemma follows if event $F^r \cap G^r$ happens with bounded away from zero probability. And since G^r is a subset of E^r, we see that the Lemma also follows if event $F^r \cap E^r$ happens with bounded away from zero probability.

By Proposition 3.3, we have

$$\mathbb{P}(\bigcup_{(i,j)\in \mathcal{I}_r} \bigcup_{t>0} \{Y_t^{(i,j)^r} < (X_0^j - X_0^i)\}) = 1.$$

This means if E^r occurs, the only way for the process leave ranking r at time t is for F^r to occur. Hence when E^r occurs, F^r must occur as will, this gives us the result

$$E^r \cap F^r = E^r. \tag{11}$$

By Lemma 3.6, there exists some $\epsilon > 0$ such that,

$$\mathbb{P}(E^r) > \epsilon. \tag{12}$$

Hence

$$\begin{aligned}\sum_{r'\in D_r} P_{X_0}(r') &= \mathbb{P}(G^r \cap F^r) \\ &\geq \mathbb{P}(E^r \cap F^r) \\ &= \mathbb{P}(E^r) > \epsilon\end{aligned} \tag{13}$$

where the first equality follows from the definition of $G^r \cap F^r$; the first inequality follows from the relationship between G^r and E^r; the second equality follows from (11); and the last inequality follows from (12). ☐

With the concept of drift and drift set, we can describe all the drift relations within the system by a *drift diagram.*

Definition 4.5. Let X_n be a ranking-based process with d components. We use a *ranking transition diagram* with $d!$ nodes to visualise the relationship between rankings in the system. Each node in the diagram is labelled as one of the $d!$ ranking. For any pairs of rankings r,r' in the system, we draw an arrow from node r to node r' if $r' \in D_r$.

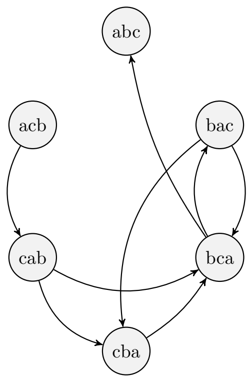

Figure 2: An example of a drift diagram for a system with three items a, b and c.

Note if a node in the drift has degree d, then by Lemma 4.4, the system has uniformly bounded away from 0 probability of following one of the d arrows out and it might also have positive probability of going to one of the nodes that the arrows don't point to. This is very different from the commonly seen transition diagram of a Markov Chain.

Example 4.6. Let X_n be a ranking-based process with 3 components a, b and c with the following q^r for $r \in \mathcal{R}$, then Figure 2 is a drift diagram for this system.

r	q^r	D_r
abc	$q_a^r > q_b^r > q_c^r$	$\emptyset$
acb	$q_c^r > q_a^r > q_b^r$	$\{cab\}$
bac	$q_c^r > q_b^r > q_a^r$	$\{bca, cba\}$
bca	$q_a^r > q_b^r > q_c^r$	$\{abc, bac\}$
cab	$q_b^r > q_c^r > q_a^r$	$\{(bca, cba\}$
cba	$q_b^r > q_c^r > q_a^r$	$\{bca\}$

5 Convergence of ranking

In this section we state the condition for almost sure convergence of rankings. We start this section with an illustrative concerning example where the ranking might not converge with positive probability, which leads us to finding the condition required for almost sure convergence.

Example 5.1. The system with the drift diagram in Figure 3 seems to have positive probability of never reaching stability.

Suppose the process starts with ranking bca. Assume that with μ^{bca}, $q_b^{bca} > q_a^{bca} > q_c^{bca}$. Specifically, item b is likely to gain a very large increment and item a c are likely to gain small increments. In this case, the system drifts to bac. Suppose $q_c^{bac} > q_b^{bac} > q_a^{bac} = 0$, where item a doesn't gain any increment almost surely and the increments that item b and c gains are likely to be close. In this case, bac is much more likely to follow the

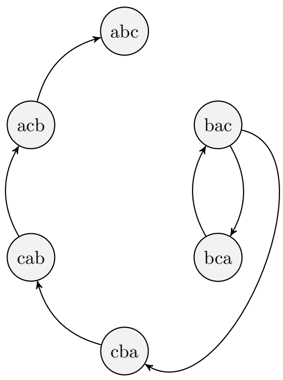

Figure 3: An example of a drift diagram with a loop (*bac*-*bca* loop).

arrow to transition back to *bca* than going to *cba* since item b is likely to be taking the lead by a significant count coming attribute to the boost from ranking *bca*. Every time the system transitions back to *bca*, b's leading position will be reinforced even more, which might eventually result the system has close to 0 probability of drifting from *bac* to *cba*. Hence we see that the system seems to have positive probability of oscillating between rankings *bac* and *bca* forever without ever reaching stability.

If we want the system to stabilise almost surely, it is clear that we have to eliminate the behaviour in Example 5.1. To deal with this concerning behaviour, in the proof relating convergence of rankings we set an extra condition that the drift diagram doesn't contain any cycles, which eliminates the oscillating behaviour that we see in this example.

We do not have a proof here to show that if there exits a loop in the drift diagram, the system will get stuck with oscillating between unstable rankings with positive probability. Hence this extra condition might not be optimal, which can be a problem to investigate for future work.

With the above condition, we now state the Lemma and main Theorem of the section, which brings together the implication of Lemma 4.4 and drift diagram when it comes to convergence of ranking.

Lemma 5.2. *Let X_n be a ranking-based process satisfying assumption 3.4. Let S be the set of all stable rankings in the system. If S is non-empty and the drift diagram has no cycles, then for any v and some $\epsilon > 0$,*

$$\mathbb{P}_v(\bigcup_{n>0}\bigcap_{t>n} \mathrm{rk}(X_t) \in S) > \epsilon. \tag{14}$$

We now come to our main result for this project:

Theorem 5.3. *Let X_n be a ranking-based process satisfying assumption 3.4. If S is non-empty and the drift diagram has no cycles, then*

$$\mathbb{P}_v(\bigcup_{n>0}\bigcap_{t>n} \mathrm{rk}(X_t) \in S) = 1. \tag{15}$$

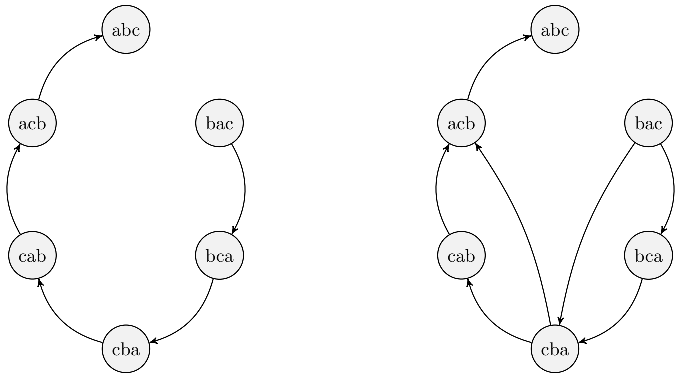

Figure 4: Two drift diagrams. All unstable ranking have degree 1 in the left diagram. Some unstable ranking have degree > 1 in the right diagram.

Proof. By Lemma 5.2, starting from any ranking and some $\epsilon > 0$, there's greater than ϵ probability of entering a stable ranking and staying there forever. Hence this is a Bernoulli experiment with greater ϵ chance of winning. Hence the ranking will converge eventually with probability 1. □

6 Proof for Lemma 5.2

Example 6.1. In this example we explain some intuition behing the proof for Lemma 5.2. Given that the drift diagram has no loop, starting from each unstable ranking, we will be able to follow the arrow in the diagram and arrive at the stable ranking. Therefore, to prove Lemma 5.2, we need to show that for some $\epsilon > 0$, the probability of following path to the stable ranking starting from any $r \in \mathcal{R}$ is greater than ϵ.

If all unstable rankings in the diagram has degree 1, then the result will follow nicely. If the node for $r \in U$ has degree 1, this means $|D_r| = 1$. From Lemma 4.4, r will follow the arrow with greater than ϵ_1 probability for some $\epsilon_1 > 0$. Hence it's easy to see that from any $r \in \mathcal{R}$, the probability of following path to the stable ranking is greater than ϵ.

On the other hand if some unstable rankings have degree > 1, the situation is more complicated. Take the left diagram in Figure 4 as an example. For some $\epsilon_1, \epsilon_2, \epsilon_3 > 0$, starting at ranking *cba*, the probability of following the path $cba - cab - acb - abc$ is $\epsilon_{1.1}(X) \cdot \epsilon_2 \cdot \epsilon_3$ and the probability of following the paths $cba - acb - abc$ is $\epsilon_{1.2}(X) \cdot \epsilon_3$. Here $\epsilon_{1.d}(X)$ is the probability of choosing to leave raking *cba* with the dth arrow when the random walk is at X. Hence it's by Lemma 4.4 $\epsilon_{1.1}(X) + \epsilon_{1.2}(X) = \epsilon_1$. In this case, the probability of going down the path $cba - cab - acb - abc$ might no longer be uniformly bounded away from 0 since the value $\epsilon_1(X)$ is random. The same problem goes with the probability of going down the path $cba - acb - abc$. However, notice that the probability of going down either of these two paths is $> \epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3$.

We solve the kind of concern we see with the left diagram in Figure 4 by joining probability of following the paths that bifurcates through induction.

In order for the induction to work, we define the ranking path, which is a sequence of ranking that represents a path in the drift diagram. In the proof for Lemma 5.2, we use the ranking paths to identify paths that bifurcates.

Definition 6.2. A *ranking path* is a sequence

$$(w_i)_{i=0}^d = (w_0, w_1, \dots, w_d)$$

of $d+1$ rankings, where for $i \in \{0,\dots,d\}$, each w_i is a ranking in the system and $w_{i+1} \in D_{w_i}$ for $i \in \{0,\dots,d-1\}$. Either $d < \infty$, in which case $w_d \in S$ and for all $i \in \{0,\dots,d-1\}$ $w_i \in U$, or $d = \infty$.

Given $(w_i)_{i=0}^d$, let $T_0 = 0$ and for $i \in \{1,\dots,d\}$, $T_i = T_i(w_{i-1}) = \inf\{n > T_{i-1} : \mathrm{rk}(X_n) \neq w_{i-1}\} \leq \infty$. To shorten the notation in the proof, we denote the probability of traversing the path $(w_i)_{i=0}^d$ with initial distribution v as $R_{w,v}$ and specifically,

$$\begin{aligned} R_{w,v} &= \mathbb{P}_v(\mathrm{rk}(X_0) = w_0) \cdot \prod_{i=1}^d \mathbb{P}\left(\mathrm{rk}(X_{T_i}) = w_i \middle| \bigcap_{k=0}^{i-1} \{\mathrm{rk}(X_{T_k}) = w_k\} \right) \\ &= \mathbb{P}_v(\mathrm{rk}(X_0) = w_0) \cdot \prod_{i=1}^d \int \mathbb{P}_x\left(\mathrm{rk}(X_{T_1}) = w_i \right) \mu_{i-1}(dx), \end{aligned} \tag{16}$$

where $\mu_{i-1} = \mathcal{L}\Big(X_{T_{i-1}} \Big| \bigcap_{k=0}^{i-1} \bigcap_{j=T_k}^{T_{k+1}-1} \{\mathrm{rk}(X_j) = w_k\}\Big)$. Using the definition of drift, this can be simplified to,

$$R_{w,v} = \mathbb{P}_v(\mathrm{rk}(X_0) = w_0) \cdot \prod_{i=1}^d \int P_x(w_i)\, \mu_{i-1}(dx). \tag{17}$$

Remark 6.3. Note the condition "all ranking paths consist of distinct element" is equivalent to the condition that the drift diagram has no cycles. Since we are using ranking paths to formulate the induction, in the proof for 5.2 we are going to state the condition as all ranking paths consist of distinct element instead.

Proof. Let X_n be a ranking based process with k elements and let $d = k!$. Let

$$P_v(S) = \mathbb{P}_v(\bigcup_{n>0} \mathrm{rk}(X_t) \in S)$$

Let W be the collection of all ranking paths in the system. Since all ranking paths consist of distinct rankings, they have length at most d and the last element is a stable ranking. For notation convenience, for a ranking path $(w_i)_{i=0}^h$, then for all $j > h$, we let

$$\int P_x(w_j)\, \mu_{j-1}(dx) = 1.$$

Hence,

$$\begin{aligned} P_v(S) &\geq \sum_{(w_n)_{i\geq 0} \in W} R_{w,v} \\ &= \sum_{(w_n)_{i\geq 0} \in W} \mathbb{P}_v(\mathrm{rk}(X_0) = w_0) \cdot \prod_{i=1}^d \int P_x(w_i)\, \mu_{i-1}(dx). \end{aligned} \tag{18}$$

Now, we claim that for $u \in \{0,\dots,d\}$ and $\epsilon_1 > 0$,

$$P_v(S) \geq \epsilon_1^u \sum_{(w_n)_{i\geq 0} \in W} \mathbb{P}_v(\mathrm{rk}(X_0) = w_0) \cdot \prod_{i=1}^{d-u} \int P_x(w_i)\, \mu_{i-1}(dx) \tag{19}$$

We prove by induction that for $u \in \{0, \dots, d\}$, the claim is true.

We already know the claim is true for $u = 0$. Assume that (19) is true for some $v \in \{0, \dots d-1\}$.

For all $(w_i)_{i\geq 0} \in W$ such that $l(w) \geq d - v + 1$, we consider the partitions $W^{(1)}_{d-v}, \dots, W^{(m)}_{d-v}$ and sequences of rankings of length $d-v$, $\sigma^{(1)}_{d-v}, \dots, \sigma^{(m)}_{d-v}$. For $j \in \{1, \dots, m\}$, the rankings paths in $W^{(j)}_{d-v}$ has its first $d-v$ elements equal to $\sigma^{(j)}_{d-v}$, specifically,

$$W^{(1)}_{d-v} = \{(w_i)_{i\geq 0} \in W : (w_i)^{d-v-1}_{i=0} = \sigma^{(1)}_{d-v}\}$$
$$\vdots$$
$$W^{(m)}_{d-v} = \{(w_i)_{i\geq 0} \in W : (w_i)^{d-v-1}_{i=0} = \sigma^{(m)}_{d-v}\}.$$

Note $m < \infty$ since there can only be finitely many sequences of rankings of length $d - v$.

We denote

$$W^{(m+1)}_{d-v} = \{(w_i)_{i\geq 0} : l(w) < d - v + 1\}$$

and note that $W^{(1)}_{d-v}, \dots, W^{(m+1)}_{d-v}$ partitions W.

Hence,

$$\begin{aligned}
P_v(S) &= \sum_{1\leq j\leq m+1} \sum_{(w_i)_{i\geq 0}\in W^{(j)}_{d-v}} R_{w,v} \\
&\geq \epsilon_1^v \Big(\sum_{1\leq j\leq m} \mathbb{P}_v(X_0 = w^{(j)}_0) \cdot \prod_{i=1}^{d-v} \int P_x(w^{(j)}_{i+1})\, \mu^{(j)}_i(dx) \\
&\qquad + \sum_{(w_i)_{i\geq 0}\in W^{(m+1)}_{d-v}} \mathbb{P}_v(X_0 = w_0) \cdot \prod_{i=1}^{d-v} \int P_x(w_{i+1})\, \mu_i(dx) \Big) \\
&\geq \epsilon_1^{(v+1)} \Big(\sum_{1\leq j\leq m} \mathbb{P}_v(X_0 = w^{(j)}_0) \cdot \prod_{i=1}^{d-(v+1)} \int P_x(w^{(j)}_{i+1})\, \mu^{(j)}_i(dx) \\
&\qquad + \sum_{(w_i)_{i\geq 0}\in W^{(m+1)}_{d-v}} \mathbb{P}_v(X_0 = w_0) \cdot \prod_{i=1}^{d-(v+1)} \int P_x(w_{i+1})\, \mu_i(dx) \Big) \\
&= \epsilon_1^{(v+1)} \cdot \sum_{(w_i)_{i\geq 0}\in W} \mathbb{P}_v(\mathrm{rk}(X_0) = w_0) \cdot \prod_{i=1}^{d-(v+1)} \int P_x(w_i)\, \mu_{i-1}(dx),
\end{aligned} \tag{20}$$

where the first equality follows from the partition; the first inequality follows from the induction premise; the second inequality follows from the fact that for all $j \in \{1, \dots, m\}$, $\{w_{d-v} : (w^{(j)}_n)_{i\geq 0} \in W^{(j)}_{d-v}\}$, is the set of all rankings that $w^{(j)}_{d-v-1}$ drifts to and that all ranking paths in $W^{(m+1)}_{d-v}$ has length less than $d - v + 1$; the last equality follows from the partition.

By mathematical induction , we get

$$P_v(S) > \epsilon_1^d \cdot \sum_{(w_i)_{i\geq 0}\in W} \mathbb{P}_v(\mathrm{rk}(X_0) = w_0) \geq \epsilon_1^d. \tag{21}$$

From Proposition 3.7, it follows that for $\epsilon_2 > 0$

$$\mathbb{P}(\bigcap_{t>n} \mathrm{rk}(X_t) \in S | \, \mathrm{rk}(X_n) \in S) > \epsilon_2, \tag{22}$$

since all stable pairs under a stable ranking have positive probability of retaining their relative other, which means if the process enters a stable ranking, it will stay there forever with positive probability.

Hence

$$\begin{aligned} \mathbb{P}_v(\bigcup_{n>0} \bigcap_{t>n} \mathrm{rk}(X_t) \in S) &= \mathbb{P}(\bigcap_{t>n} \mathrm{rk}(X_t) \in S | \, \mathrm{rk}(X_n) \in S) \cdot P_v(S) \\ &> \epsilon_1^d \cdot \epsilon_2 \\ &= \epsilon \end{aligned} \tag{23}$$

□

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