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Dirichlet problem in light of capacity theory

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Contents

1 Prelude

1.1 Abstract

We first introduce the Dirichlet problem and Dirichlet condition along with the Laplace operator and the capacity theory. We then talk about the how we encounter the Laplace operator in the capacity theory and under what circumstance a capacity not defined. We then proceed to show a coincidence between the mathematical notion of capacity and the concept of capacitance in physics. Then we finish with introducing another operator called the Fractional Laplacian and make a comparison with the Laplace operator to explain why it is a non-local operator.

1.2 Introduction

Dirichlet problems are a class of PDE problems that arose in early 19 century, the idea is to look for a function which solves a PDE in the interior region while the value of the function at the boundary of the region is given.

The solvability of the Dirichlet problem is linked to the theory of capacities which was initiated by the French mathematician Gustave Choquet in 1950s. This theory introduced a mathematical analogue of a set's ability to hold electrical charge with a given potential energy with respect to an idealized "flat" ground.

1.3 Statement of authorship

None of the results or ideas in this report are original, it is simply a recollection of my readings of the work of innumerably many other authors. I benefited greatly and want to thank my supervisor, Serena Dipierro, who helped and guided my investigation and explain the topics for me.

2 Dirichlet problem and the Laplace operator

A Dirichlet problem is the problem of looking for a function that solves a specific PDE inside a given region that takes prescribed values on the boundary of that region. The Dirichlet problem can be solved for a lot of PDEs with respect to all kinds of operators, but for the main part of this report, we are considering the Dirichlet problem with respect to Laplace's equation and thus, the Laplace operator. The Laplace operator is also called *Laplacian* and it is defined as the divergence of the gradient of a function f in the n-dimensional Euclidean space, denoted by Δ:

$$\Delta f = div\nabla f = \nabla \cdot \nabla f = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} x \tag{1}$$

In this report, the *Dirichlet problem* can be put as the following:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial(\overline{\Omega}) \end{cases} \tag{2}$$

Given a function g that has values everywhere on the boundary of a region, is there a unique continuous function 'u' twice continuously differentiable in the interior and continuous on the boundary, such that 'u' is harmonic in the interior and $u = g$ on the boundary? This type of condition is also called the Dirichlet condition. (Dipierro and Valdinoci, 2021).

3 Capacitance and capacity

The mathematical concept of capacity was originally introduced by the French mathematician Gustave Choquet. It states that, given a bounded and open set $\Omega \subset \mathbb{R}^n$ with boundary of class $C^{2,\alpha}$, for some $\alpha \in (0,1)$, we define the *capacity* of Ω as (Choquet, 1954):

$$Cap(\Omega) = \inf_{v \in C_0^\infty(\mathbb{R}^n), v=1 \text{ in } \Omega} \int_{\mathbb{R}^n \setminus \overline{\Omega}} |\nabla v(x)|^2 \, dx \tag{3}$$

By the density result of Sobolev spaces (Leoni, 2017a), we can put this definition into the form:

$$Cap(\Omega) = \inf_{v \in D^{1,2}(\mathbb{R}^n), v=1 \text{ in } \Omega} \int_{\mathbb{R}^n \setminus \overline{\Omega}} |\nabla v(x)|^2 \, dx \tag{4}$$

3.1 The link between Laplacian and capacity theory

Consider a functional:

$$F(u) = \int |\nabla u|^2 \tag{5}$$

If we want to find a specific function u that makes this functional reaches minimum, we need to know the derivative, just like when we learned the derivative via the first principle in high school, we can mimic the process of defining the derivative of single variable functions by assuming a test function v which has all the continuous and differentiable properties we need. And in the end, we will obtain an expression consisting of the test function v and the Laplacian of the function u. This shows the importance of the Laplacian in our topic.

$$\lim_{\epsilon \to 0} \frac{F(u+\epsilon v) - F(u)}{\epsilon} \tag{6}$$

$$= \lim_{\epsilon \to 0} \frac{\int |\nabla u|^2 + 2\epsilon \int \nabla u \cdot \nabla v + \epsilon^2 \int |\nabla v|^2 - \int |\nabla u|^2}{\epsilon} \text{(Substitute the functional expression)} \tag{7}$$

$$= 2 \int_{\mathbb{R}^n} \nabla u \cdot \nabla v = 0 \tag{8}$$

$$\Longrightarrow \int_{\mathbb{R}^n} [div(\nabla u v) - \Delta u v] \tag{9}$$

$$= - \int_{\mathbb{R}^n} \Delta u v = 0 \tag{10}$$

3.2 When n = 1 or 2

We next show that the definition of capacity is meaningful only when $n \geq 3$ by showing that $Cap(\Omega) = 0$ for $n = 1, 2$.

Lemma 3.1. *Let $n \in$ 1,2. Then $Cap(\Omega) = 0$ for all bounded and open set $\Omega \subset \mathbb{R}^n$ with boundary of class $C^{2,\alpha}$ for some $\alpha \in$ (0,1).*

Proof. Let $n = 1$ and $\epsilon \Subset (-R, R)$ for some $R \geq 0$ and we have the function $v_\epsilon(x)$ to be

$$v_\epsilon = \begin{cases} 1 & \text{if x} \in \text{[-R,R]} \\ \epsilon(R + \frac{1}{\epsilon} - |x|) & \text{if } |x| \in (R, R + \frac{1}{\epsilon}) \\ 0 & \text{if } |x| \in [R + \frac{1}{\epsilon}, \infty), \end{cases} \tag{11}$$

We can calculate that

$$\int_{\mathbb{R} \setminus \overline{\Omega}} |v'_\epsilon(x)|^2 dx = 2\epsilon \tag{12}$$

and by the definition of capacity in equation (4), we should take the infimum by sending the ϵ to zero:

$$Cap(\Omega) = \lim_{\epsilon \to 0} \int_{\mathbb{R} \setminus \overline{\Omega}} |v'_\epsilon(x)|^2 dx \tag{13}$$

$$= \lim_{\epsilon \to 0} 2\epsilon = 0 \tag{14}$$

For the case of n=2, we take ϵ>0 small enough such that $\Omega \Subset B_{1/\epsilon}$ and we let $w_\epsilon(x)$ to be

$$w_\epsilon = \begin{cases} 1 & \text{if } |x| \in [0, \frac{1}{\epsilon} \\ \frac{2\ln\epsilon - \ln x}{\ln \epsilon} & \text{if } |x| \in (\frac{1}{\epsilon}, \frac{1}{\epsilon^2}) \\ 0 & \text{if } |x| \in [\frac{1}{\epsilon^2}, \infty) \end{cases} \tag{15}$$

Similar to the case of n=1, we can find the capacity by making ϵ as small as we want which results that

$$Cap(\Omega) = \lim_{\epsilon \to 0} \int_{\mathbb{R}\setminus\overline{\Omega}} |v_\epsilon'(x)|^2 dx = \lim_{\epsilon \to 0} \frac{2\pi}{|\ln \epsilon|} = 0 \tag{16}$$

Therefore, we can conclude that the $Cap(\Omega) = 0$ for $n = 1$ and $n = 2$. □

3.3 An essential coincidence between math and physics

Proposition 3.2. *Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ be a bounded and open set with boundary of class $C^{2,\alpha}$ for some α. Then there exists a unique function $u \in C^2(\mathbb{R}^n \setminus \Omega) \cap D^{1,2}(\mathbb{R}^n \setminus \Omega)$ such that*

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega} \\ u = 1 & \text{on } \delta\Omega \end{cases} \tag{17}$$

Additionally, such function should be that $0 \leq u(x) \leq 1$ for all $x \in \mathbb{R}^n \setminus \overline{\Omega}$, $\lim_{|x|\to\infty} u(x) = 0$ and

$$Cap(\Omega) = \int_{\mathbb{R}^n\setminus\overline{\Omega}} |\nabla u(x)|^2 dx. \tag{18}$$

We may assume that the function u in this proposition is defined in all $\mathbb{R}^n$ just by setting u to be constantly equal to 1 in Ω.

Proof. Let $k_0 \in \mathbb{N}$ be such that $\Omega \Subset B_{k_0}$. Let $v \in C_0^\infty(B_{k_0})$ such that $v = 1$ in Ω. For all $k \in \mathbb{N} \cap [l_0, \infty)$ we consider the Sobolev space X_k of the functions $w \in D^{1,2}(\mathbb{R}^n$ such that $w = v$ in $\Omega \cup (\mathbb{R}^n \setminus B_k)$. We let u_k to be the minimiser in X_k of the functional:

$$X_k \ni w \mapsto \int_{\mathbb{R}^n\setminus\overline{\Omega}} |\nabla w(x)|^2 dx. \tag{19}$$

Please note that such a minimiser exists, by the Direct Method of the Calculus of Variations (Leoni, 2017*b*). If we let $u_k^\star(x)$ to be:

$$u_k^\star(x) = \begin{cases} 1 & \text{if } u_k(x) > 1 \\ u_k(x) & \text{if } u_k(x) \in [0,1] \\ 0 & \text{if } u_k(x) < 0 \end{cases} \tag{20}$$

then

$$\int_{\mathbb{R}^n\setminus\overline{\Omega}} |\nabla u_k^\star(x)|^2 dx = \int_{(\mathbb{R}^n\setminus\overline{\Omega})\cap\{0\leq u_k\leq 1\}} |\nabla u_k(x)|^2 dx \leq \int_{\mathbb{R}^n\setminus\overline{\Omega}} |\nabla u_k(x)|^2 dx. \tag{21}$$

We then replace u_k by $u_k^\star$ to obtain:

$$0 \le u_k(x) \le 1 \text{ for all } x \in \mathbb{R}^n \setminus \overline{\Omega}. \tag{22}$$

Also, since $X_{k_0} \subseteq X_k$ for all $k \in \mathbb{N} \cap [k_0, \infty)$, the minimization property in X_k gives that

$$\int_{\mathbb{R}^n \setminus \overline{\Omega}} |\nabla u_k(x)|^2 dx \le \int_{\mathbb{R}^n \setminus \overline{\Omega}} |\nabla u_{k_0}(x)|^2 dx = C_0 \in [0, \infty). \tag{23}$$

Thus, we have a sub-sequence that u_k converges to some function u in $L^2_{loc}(\mathbb{R}^n)$ and ∇u_k converges to ∇u weakly in $L^2(\mathbb{R}^n)$. The minimiser u_k is harmonic in $\mathbb{R}^n \setminus \overline{\Omega}$ from the conclusion of section 3.1, so is u. Also, since $u_k = v = 1$ in Ω, we have $u = 1$ in Ω. This gives that u is an admissible function for the infimum procedure in the right hand side of equation (4), by the weak convergence we mentioned we obtain:

$$0 \le \lim_{k\to\infty} inf \int_{\mathbb{R}^n \setminus \overline{\Omega}} |\nabla u_k(x) - \nabla u(x)|^2 dx = -\int_{\mathbb{R}^n \setminus \overline{\Omega}} |\nabla u(x)|^2 dx + \lim_{k\to\infty} inf \int_{\mathbb{R}^n \setminus \overline{\Omega}} |u_k(x)|^2 dx. \tag{24}$$

And by equation (23):

$$\int_{\mathbb{R}^n \setminus \overline{\Omega}} |\nabla u(x)|^2 dx \le C_0. \tag{25}$$

We then need to show such an $u(x)$ is indeed the infimum we wanted:

$$\int_{\mathbb{R}^n \setminus \overline{\Omega}} |\nabla u(x)|^2 dx = \inf_{\zeta \in D^{1,2}(\mathbb{R}^n), \zeta = 1 \text{ in } \Omega} \int_{\mathbb{R}^n \setminus \overline{\Omega}} |\nabla \zeta(x)|^2 \, dx. \tag{26}$$

We begin by supposing it's not, then there exist $b > 0$ and $\zeta = 1$ in Ω such that

$$\int_{\mathbb{R}^n \setminus \overline{\Omega}} |\nabla \zeta(x)|^2 dx + b \le \int_{\mathbb{R}^n \setminus \overline{\Omega}} |\nabla u(x)|^2 dx. \tag{27}$$

By the density results in Sobolev spaces, we can find $w \in C_0^\infty(\mathbb{R}^n)$ with $w = 1$ in Ω such that

$$\int_{\mathbb{R}^n \setminus \overline{\Omega}} |\nabla w(x)|^2 dx \le \frac{b}{2} + \int_{\mathbb{R}^n \setminus \overline{\Omega}} |\nabla \zeta(x)|^2 dx. \tag{28}$$

We take $k_\star \in \mathbb{N} \cap [k_0, \infty)$ large enough to make the support of w is contained in $B_{k_\star}$, as a consequence, we have $w \in X_k$ for all $k \ge k_\star$ and thus

$$\int_{\mathbb{R}^n \setminus \overline{\Omega}} |\nabla u_k(x)|^2 dx \le \int_{\mathbb{R}^n \setminus \overline{\Omega}} |\nabla w(x)|^2 dx. \tag{29}$$

By combining all of these information along with equation (24) we have a contradiction which proves equation (26):

$$\begin{aligned} \frac{b}{2} = b - \frac{b}{2} &\le \Big(\int_{\mathbb{R}^n \setminus \overline{\Omega}} |\nabla u(x)|^2 dx - \int_{\mathbb{R}^n \setminus \overline{\Omega}} |\nabla \zeta(x)|^2 dx \Big) + \Big(\int_{\mathbb{R}^n \setminus \overline{\Omega}} |\nabla \zeta(x)|^2 dx - \int_{\mathbb{R}^n \setminus \overline{\Omega}} |\nabla w(x)|^2 dx \Big) \\ &\le \lim_{k\to\infty} inf \int_{\mathbb{R}^n \setminus \overline{\Omega}} |\nabla u(x)|^2 dx - \int_{\mathbb{R}^n \setminus \overline{\Omega}} |\nabla w(x)|^2 dx \le 0. \end{aligned} \tag{30}$$

Therefore, we have proved equation (18) by combining the equation (4) and equation (26).

We also need to prove that $u \in C(\mathbb{R}^n \setminus \Omega)$: Let$z \in \partial\Omega$ and use the regularity of Ω to find a ball $B_\rho(z_0)$ such that $B_{\rho_0} \subseteq \Omega$ and $z \in (\partial\Omega) \cap (\partial B_{\rho_0}))$, we define:

$$\Phi(x) = \begin{cases} u(x) - \frac{\rho_0^{n-2}}{u(x)-1} & \text{if } x \in \mathbb{R}^n \setminus B_{\rho_0}(z_0) \\ u(x) - 1 & \text{if } x \in B_{\rho_0}(z_0) \end{cases} \tag{31}$$

and we observe that $\Psi \in D^{1,2}(\mathbb{R}^n \setminus \Omega)$ and if $x \in \partial\Omega$ then $|x - z_0| \geq \rho_0$, so in the Sobolev trace space, we have that $\Psi \geq u - 1 = 0$ along $\partial\Omega$ and Ψ is harmonic in $\mathbb{R}^n \setminus \overline{\Omega}$, this leads to

$$\liminf_{x \to z} u(x) \geq \lim_{x \to z} inf \frac{\rho_0^{n-2}}{|x - z_0|^{n-2}} = 1, \tag{32}$$

since we constructed $u \leq 1$, we get

$$\lim_{x \to z} u(x) = 1, \tag{33}$$

which proves that $u \in C(\mathbb{R}^n \setminus \Omega)$. Now we are going to show that u(x) is vanishing at infinity:

$$\lim_{|x| \to \infty} = 0. \tag{34}$$

If we use equation (25) and the Gagliardo-Nirenberg-Sobolev Inequality and $n \geq 3$, we can have that

$$\|u\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n \setminus \Omega)} \leq C_1, \tag{35}$$

where $C_1 > 0$. Also, since we constructed that $0 \leq u_k(x) \leq 1$ for all $x \in \mathbb{R}^n \setminus \overline{\Omega}$, we have that $\|u\|_{L^\infty(\mathbb{R}^n) \setminus \overline{\Omega}}$. From this and Cauchy's estimate, we obtain:

$$\forall x_0 \in \mathbb{R}^n \text{ such that } B_2(x_0) \cap \Omega = \phi, |\nabla u(x_0)| \leq C, \text{ for some constant } C > 0 \text{ depending only on } n. \tag{36}$$

Recalling that

$$\text{If a uniformly continuous function } \phi \text{ is in } L^q(\mathbb{R}^n) \text{ for some } q \in [1, \infty), \text{ then } \lim_{|x| \to \infty} \phi(x) = 0. \tag{37}$$

Using this with equation (35) and equation (36), we can infer equation (34) via proof by contradiction, assuming that there exists a sequence $p_j \in \mathbb{R}^n$ such that $\lim_{j \to \infty} |p_j| = \infty$ and such that $|\phi(p_j)| \geq a$, for some $a > 0$. Then we take $\rho > 0$ such that $|\phi(x) - \phi(p_j)| \leq \frac{a}{2}$ for all $x \in B_\rho(p_j)$. Since up to a sub-sequence, we can assume that the balls $B_\rho(p_j)$ are disjoint to each other and we obtain:

$$\|\phi\|^q_{L^q(\mathbb{R}^n)} \geq \sum_{j=0}^{\infty} \int_{B_\rho(p_j)} |\phi(x)|^q dx \geq \sum_{j=0}^{\infty} \int_{B_\rho(p_j)} (|\phi(p_j)| - |\phi(x) - \phi(p_j)|)^q dx \geq \sum_{j=0}^{\infty} \int_{B_\rho(p_j)} (\frac{a}{2})^q dx = \infty. \tag{38}$$

Which is a contradiction. In the next part of this proof, we need to show such a function u is unique. We suppose that u and v satisfy the conditions in proposition 3.2, and let $w = \frac{u+v}{2}$. Since $w = 1$ on $\partial\Omega$, we can get

$$\begin{aligned}\int\limits_{\mathbb{R}^n\setminus\overline{\Omega}} |\nabla u(x) - \nabla v(x)|^2 dx &= \int\limits_{\mathbb{R}^n\setminus\overline{\Omega}} (|\nabla u(x)|^2 + |\nabla v(x)|^2 - 2\nabla u(x)\cdot\nabla v(x))dx \\ &= 2Cap(\Omega) - 2\int\limits_{\mathbb{R}^n\setminus\overline{\Omega}} \nabla u(x)\cdot\nabla v(x)dx \leq \int\limits_{\mathbb{R}^n\setminus\overline{\Omega}} |\nabla w(x)|^2 dx - 2\int\limits_{\mathbb{R}^n\setminus\overline{\Omega}} \nabla u(x)\cdot\nabla v(x)dx \\ &= \frac{1}{2}\int\limits_{\mathbb{R}^n\setminus\overline{\Omega}} |\nabla u(x) - \nabla v(x)|^2 dx.\end{aligned} \tag{39}$$

This gives us that $\int\limits_{\mathbb{R}^n\setminus\overline{\Omega}} |\nabla u(x) - \nabla v(x)|^2 dx = 0$ which means the function $u - v$ is constant, by looking at the values at infinity, we know that $u - v$ is constantly equal to zero, thus we have the uniqueness we desired.

For the last part of this proof, we need to prove that $u \in C^2(\mathbb{R}^n \setminus \Omega)$. For this, we consider the problem:

$$\begin{cases}\Delta v = 0 & \text{in } B_{k_0}\setminus\overline{\Omega} \\ v = 1 & \text{on } \partial\Omega \\ v = u & \text{on } \partial B_{k_0}\end{cases} \tag{40}$$

We can use Theorem 4.4.4 in (Dipierro and Valdinoci, 2021) to say that it has a unique solution $v \in C^2(B_{k_0} \setminus \overline{\Omega}) \cap C(\overline{B_{k_0}} \setminus \Omega)$. by the uniqueness statement and $u \in C(\mathbb{R}^n \setminus \Omega)$, we deduce $v = u$, as a result of this and estimate in Theorem 4.4.4, we get $u \in C^2(\mathbb{R}^n \setminus \Omega)$ as needed. □

A more precise version of the Proposition 3.2 can be made and used later for finding an essential coincidence between the mathematical notion of capacity and the self-capacitance in physics.

Proposition 3.3. *Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ be a bounded and open set with boundary of class $C^{2,\alpha}$ for some $\alpha \in (0,1)$. Let u be the function given by Proposition 3.2, assume that $\Omega \subset B_{R_0}$, for some $R_0 > 0$. Then for all $x \in \mathbb{R}^n \setminus B_{R_0}$, $u(x) \leq \frac{R_0^{n-2}}{|x|^{n-2}}$.*

Proof. For all $\epsilon \in (0,1)$, we set $v_\epsilon(x)$ to be:

$$v_\epsilon(x) = \frac{R_0^{n-2}}{|x|^{n-2}} + \epsilon. \tag{41}$$

From the equation (34), there exits $R_\epsilon \geq \frac{1}{\epsilon}$ such that $|u(x)| \leq \frac{\epsilon}{2}$ for every $x \in \mathbb{R}^n \setminus B_{R_\epsilon}$. We notice that

$$v_\epsilon(x) = 1 + \epsilon > 1 \geq u(x) \text{ on } \partial B_{R_0}$$
$$\text{and } v_\epsilon(x) \geq \epsilon > \frac{\epsilon}{2} \geq u(x) \text{ on } \partial B_\epsilon$$

Both u and v_ϵ are harmonic in $B_{R_\epsilon} \setminus B_{R_0}$, by the Weak Maximum Principle, we obtain that

$$u(x) \leq v_\epsilon(x) = \frac{R_0^{n-2}}{|x|^{n-2}} + \epsilon \text{ for every } x \in B_{R_\epsilon} \setminus B_{R_0}.$$

We can get the desired result by sending $\epsilon \to 0$. □

The results in Proposition 3.2 highlights the strong connection between the mathematical definition of capacity and the physical notion of capacitance for an isolated conductor, which is the amount of electric charge that must be added to the conductor to raise its electric potential by one unit. To make this connection clearer, we make the following observation:

Proposition 3.4. *Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ be a bounded and open set with boundary of class $C^{2,\alpha}$ for some $\alpha \in (0,1)$. Let u be the function given by Proposition 3.2, then*

$$Cap(\Omega) = \int\limits_{\mathbb{R}^n \setminus \overline{\Omega}} |\nabla u(x)|^2 \, dx = -\int\limits_{\partial\Omega} \frac{\partial u}{\partial v} d\mathcal{H}_x^{n-1}. \tag{42}$$

Proof. Since Ω is bounded, we can consider a radius $R_0 > 0$ such that $\Omega \subset B_{R_0}$. We take $R > 2R_0$ and we observe that, by the first Green's Identity, we obtain:

$$\begin{aligned} \int\limits_{\mathbb{R}^n \setminus \overline{\Omega}} |\nabla v(x)|^2 \, dx &= \int\limits_{\partial(B_R \setminus \overline{\Omega})} u(x) \frac{\partial u}{\partial v}(x) d\mathcal{H}_x^{n-1} \\ &= \int\limits_{\partial B_R} u(x) \frac{\partial u}{\partial v}(x) d\mathcal{H}_x^{n-1} - \int\limits_{\partial\Omega} \frac{\partial u}{\partial v}(x) d\mathcal{H}_x^{n-1}. \end{aligned} \tag{43}$$

We observe that

$$\lim_{R \to \infty} \int\limits_{B_R \setminus \overline{\Omega}} |\nabla u(x)|^2 dx = \int\limits_{\mathbb{R}^n \setminus \overline{\Omega}} |\nabla u(x)|^2 dx, \tag{44}$$

by the Dominated Convergence Theorem. We claim that

$$\lim_{R \to \infty} \int\limits_{\partial B_R} u(x) \frac{\partial u}{\partial v}(x) d\mathcal{H}_x^{n-1} = 0. \tag{45}$$

To prove it, we let $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$ such that $|x_0| \geq 2R$. From the previous proposition, we have that for all $x \in \mathbb{R}^n \setminus B_R$,

$$|u(x)| \leq \frac{C}{|x|^{n-2}} \leq \frac{C}{R^{n-2}}. \tag{46}$$

and $\|u\|_{L^1(B_{R/4}(x_0))} \leq CR^2$, by renaming the $C > 0$. Then by using the Cauchy's Estimates, we have that

$$|\nabla u(x_0)| \leq \frac{C\|u\|_{L^1(B_{R/4}(x_0))}}{R^{n+1}} \leq \frac{CR^2}{R^{n+1}} = CR^{1-n}, \tag{47}$$

where $C > 0$. From this and equation (46), we are able to obtain:

$$|\int\limits_{\partial B_R} u(x) \frac{\partial u}{\partial v}(x) d\mathcal{H}_x^{n-1}| \leq CR^{2-n},$$

plugging equation (43), (44) and (45), we obtain:

$$\int\limits_{\mathbb{R}^n \setminus \overline{\Omega}} |\nabla u(x)|^2 \, dx = -\int\limits_{\partial\Omega} \frac{\partial u}{\partial v} d\mathcal{H}_x^{n-1}.$$

As desired since we have proved that $Cap(\Omega) = \int\limits_{\mathbb{R}^n \setminus \overline{\Omega}} |\nabla u(x)|^2 dx$. in Proposition 3.2 already. □

Retaking the physical motivation discussed before, we observe that at the equilibrium, the boundary of the conductor Ω is an equipotential energy which can be denoted by $u : \mathbb{R}^n \setminus \Omega \to \mathbb{R}$. The potential generated by the conductor, up to a normalization we can set $u = 1$ on $\partial\Omega$. The corresponding electric field is $-\nabla u$ and, by Gauss's law, the total charge of the conductor can be calculated by taking a large ball such that $\Omega \Subset B_R$ and, ignoring some physical constants, it is equal to the flow of the electric field through ∂B_R:

$$\begin{aligned} Q &= -\int\limits_{\partial B_R} \frac{\partial u}{\partial v}(x) d\mathcal{H}^{n-1} \\ &= -\int\limits_{\partial(B_R \setminus \Omega)} \frac{\partial u}{\partial v}(x) d\mathcal{H}^{n-1} - \int\limits_{\partial\Omega} \frac{\partial u}{\partial v}(x) d\mathcal{H}^{n-1} \\ &= -\int\limits_{B_R \setminus \Omega} div(\nabla u(x)) d\mathcal{H}^{n-1} - \int\limits_{\partial\Omega} \frac{\partial u}{\partial v}(x) d\mathcal{H}^{n-1} \\ &= -\int\limits_{\partial\Omega} \frac{\partial u}{\partial v}(x) d\mathcal{H}^{n-1}. \end{aligned} \tag{48}$$

Which is equal to $Cap(\Omega)$ by Proposition 3.4 we just proved which is the mathematical definition of the capacity we defined as the begging of this section. Plus, since the potential V along the surface of the conductor is normalized to 1, this total charge corresponds to the *self-capacitance* in physics which is $\frac{Q}{V}$. That is, it is confirming the essential coincidence between the mathematical notion of capacity and the concept of self-capacitance in physics. In this spirit, in the physical jargon the function u in Proposition 3.2 is often referred to 'conductor potential'.

4 Fractional Laplacian

As mentioned during the prelude part of this report, there are many other operators, one of them is called Fractional Laplacian. We are going to compare it with the Laplacian operator to gain an intuitive idea of what is a non-local operator. An alternative definition of Laplacian operator is that let $x_0 \in \mathbb{R}^n$ and $r > 0$, suppose that $u \in C^2(B_r(x_0))$, then,

$$\lim_{\rho \to 0} \frac{1}{\rho^2} \left(⨍_{\partial B_\rho(x_0)} u(x) d\mathcal{H}^{n-1} - u(x_0) \right) = 1/2n \Delta u(x_0). \tag{49}$$

The *fractional Laplacian* for a function $u(x)$ and $s \in (0,1)$ is defined as (Di Nezza et al., 2011):

$$(-\Delta)^s u(x) = -C(n,s) P.V. \int\limits_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} dy, \tag{50}$$

where $C(n,s)$ is a dimensional constant only depends on n and s and P.V. means 'in the principal value sense'.

By inspection of the definition of Laplacian, we can see that when finding the Laplacian of a function at a certain point, we are comparing the difference between the value of $u(x_0)$ and its neighbourhood ball then taking the average of them before shrinking the radius of the ball to zero. This gives us an insight that the Laplacian of a function at a particular point can be obtained by knowing the difference between this point and its neighbourhood.

However, this is not the case when we are dealing with the fractional Laplacian operator. In the numerator part of the kernel of the integral, it is doing something very similar to Laplacian operator, it is comparing the values of function. However, in the denominator of the kernel, the index is no longer just n but it contains the a $s \in (0,1)$ and we are integrating over y. This means that it is not possible to just use some neighbourhood to find the fractional Laplacian around a point. Instead, to find the fractional Laplacian, we need to consider all the points over the whole space. In other words, two points far from each other will still be able to affect each other with respect to the fractional Laplacian, and this makes the fractional Laplacian a non-local operator.

5 Discussion and Conclusion

We can easily calculate the capacity of a set in mathematical notion once the definition is given, but there is an essential coincidence between the mathematical definition of capacity and the notion of capacitance in our physical world, this coincidence to some extend confirms Gauss's law even though there were about a gap of 100 years between the math and physics. Since the definition of capacity is developed, we might be able to extend it to more general notion of energy and related to other operators such as the Fractional Laplacian.

6 Reference

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