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A New Weyl Multiplet for 6D Conformal Supergravity

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1 Prelude

1.1 Abstract

In this report we construct a multiplet of 6 spacetime dimensional fields that are invariant (close) under local transformations generated by the superconformal algebra. This multiplet is referred to in literature as a Weyl multiplet. By using the superspace formalism, we construct this full set of fields, and determine their equations of motion which are imposed by the constraints of the algebra. This is the first step of the superconformal method to find the 6D Hyper-Dilaton-Weyl multiplet, analogous to the 4D construction by Kitchin [1], which can then be used to create a 6D Poincare supergravity theory.

1.2 Statement of Authorship

The computations provided in this paper are my own. All gamma matrix identities and relations used in these computations are appropriately referenced in the Appendix. The 6D superconformal algebra is well-known, and it's structure is given in the Appendix and used throughout.

2 Introduction

Supersymmetry, or the unification of bosons and fermions, is widely considered to be part of the resolution to the issues between the Standard Model of Particle Physics, and General Relativity (GR). Currently, these theories must be incomplete, as naive quantisation of GR yields a non-renormalisable theory of gravity. These non-renormalisable theories have divergent integrals in the description of the theory, which cannot be recitfied by usual renormalisation techniques such as the energy dependent rescaling of "constants". Furthermore, neither theory has an explanation for the dark matter which is believed to make up about 85% of the matter in the universe, nor can provide a prediction for the value of the cosmological constant which determines the rate of expansion of the universe. When supersymmetry is realised in a theory as a local symmetry, gravity is automatically incorporated due to the components in the connection term of the covariant derivative. Such theories are called "supergravity" (SUGRA) theories.

Field theories defined on Minkowski spacetimes (that treat space and time as oppositely-signed metric components) are in agreement with Special Relativity if they possess invariance under the Poincare group. This is the group of rotations, translations, and the so called "boosts" which are generalised changes in velocity. If a field theory is invariant under the group, this means that the laws of physics are unchanged under the transformations of the group. This is what should be expected from a valid physical theory, as a fundamental axiom of physics is that the laws of physics are the same regardless of your coordinates or your velocity. The Poincare group can be described in terms of the Poincare algebra, given by the following Lie brackets:

$$[P^\mu, P^\nu] = 0, \tag{1}$$

$$[M^{\mu\nu}, P^{\sigma}] = i(P^{\mu}g^{\nu\sigma} - P^{\nu}g^{\mu\sigma}), \tag{2}$$

and

$$\begin{aligned}&[M^{\mu\nu}, M^{\rho\sigma}] \\ &\quad = i\left(M^{\mu\sigma}g^{\nu\rho} + M^{\nu\rho}g^{\mu\sigma} - M^{\mu\rho}g^{\nu\sigma} - M^{\nu\sigma}g^{\mu\rho}\right),\end{aligned} \tag{3}$$

where P is the generator of translations, g is the metric, and M is the generator of rotations and Lorentz boosts. Examining the symmetries of a theory often yields interesting information about the physics of the theory. In particular in physics, we are interested in primarily continuous symmetries of Lie Groups. As an example, continuous symmetries give rise to conserved quantities by Noether's theorem. In order to discover new physics, one pathway is to examine the theory to determine if there are additional symmetries. Alternatively, one can extend the underlying symmetry group, and see if there is a corresponding physical theory. Naturally, one question arises: "how can one extend a symmetry group, or algebra, and have a physically realisable theory?".

According to the Coleman Mandula Theorem, the Poincare algebra is the largest Lie algebra that corresponds to a physically realisable theory, up to the addition of arbitrarily many generators that commute with the entire algebra (for example, in the Standard Model, this "internal" symmetry group is $SU(3) \otimes SU(2) \otimes U(1)$, which is enough to unify 3 out of 4 fundamental forces). Thus, there is no non-trivial way of extending the Poincare Lie algebra. Instead, however, one can generalise the Lie algebra to a graded Lie algebra, in this case a ($\mathbb{Z}_2$-graded) "super" Lie algebra. This bypasses the Coleman Mandula Theorem, and, in fact, is the only way of doing so (by the Haag, Lopuszanski-Sohnius Theorem). One can then extend the superalgebra non-trivially by adding anticommuting "fermionic" elements to the algebra. These fermionic generators are known as the supercharges, denoted by Q. These elements are spinorial in nature, and are often denoted Q_{α} and $\overline{Q}_{\dot{\alpha}}$, where α is the spinor index.

In the simplest case, there is one such pair of generators. This is the case that we will focus on in this project ($N = 1$). When these generators act on bosonic states, they are transformed into fermions, and vice versa. This yields the supersymmetric extension of the Poincaré algebra, with additional relations

$$[Q_{\alpha}, M^{\mu\nu}] = (\sigma^{\mu\nu})_{\alpha}{}^{\beta} Q_{\beta}, \tag{4}$$

$$\left[\bar{Q}^{\dot{\alpha}}, M^{\mu\nu}\right] = (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{Q}^{\dot{\beta}}, \tag{5}$$

$$[Q_{\alpha}, P^{\mu}] = \left[\bar{Q}^{\dot{\alpha}}, P^{\mu}\right] = \{Q_{\alpha}, Q_{\beta}\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0, \tag{6}$$

and

$$\left\{Q_{\alpha}, \bar{Q}_{\dot{\beta}}\right\} = 2\left(\sigma^{\mu}\right)_{\alpha\dot{\beta}} P_{\mu}, \tag{7}$$

where σ^{μ} is the 4-vector of Pauli matrices, and $\sigma^{\mu\nu}$ are the spin generators of the Lorentz group.

This yields the unification of bosons (force-carrying particles) and fermions (matter particles) via adding a "supersymmetric partner" particle of opposite type for each particle in the standard model. Again, when the fields of a theory are invariant under local transformations involving these generators, gravity is automatically incorporated in the theory via the connection. Supergravity theories are thus an exciting avenue for research, as they unify the fundamental particles of the standard model with gravity.

In order to study a theory of supergravity, a set of fields must be constructed that are closed "off-shell". This refers to a closed set of fields that transform into one another under the algebra, with the need to apply the equations of motion of the fields as additional constraints ("on-shell"). Constructing an off-shell multiplet of fields is far more powerful than constructing an on-shell set, as an off-shell multiplet is independent of the particular model used (for example, if a particle is massless or massive).

To then determine the physics of the theory, fields can then be used to build an action, the mechanics principle of least action can be applied to determine the dynamics of the theory. However, building these actions are in practice quite difficult. One way of constructing these actions in a model independent manner is to use the superconformal method. In essence, the Poincare superalgebra is extended to the superconformal algebra by adding unphysical symmetries that violate the Coleman Mandula theorem. A theory can then be constructed, and the unphysical symmetries can be removed via gauge fixing, thus yielding a physical theory. The superconformal algebra extends the Poincare super algebra by adding the generators $\mathbb{D}$ for dilations, K_a for the special conformal transformations, Y for chiral transformations, J_i^j for the so-called "R-symmetry", and S_α^i for special superconformal transformations. The full algebra structure is given in the Appendix. Each of these generators has a corresponding connection, which gives the field content of the theory. These fields must contain the same number of fermionic and bosonic degrees of freedom due to supersymmetry. This is often not the case, and thus must be coupled to additional fields. Such representations of conformal supergravity are known as Weyl multiplets. A superconformally invariant action can then be constructed. However, the theory remains unphysical. In order to reduce to Poincare supergravity, one must introduce one or more matter multiplets, known as compensating multiplets. This eliminates the extra unphysical degrees of freedom after imposing some constraints. The result is an action invariant under Poincare supergravity, containing the gauge fields coupled to the remaining matter fields.

In the 1980's, Muller [2] constructed two (minimal) sets of fields which close under 4D $N = 2$ supergravity transformations. These two irreducible representation are dubbed the vector and hyper Muller multiplets, named after their field content. In order to construct an action for these Muller multiplets, it is far easier to start with a Weyl multiplet that reduces to a Muller multiplet under the superconformal method. In 2017, Butter et al [3] did this for the vector Muller multiplet, by developing the so-called 4D dilaton Weyl multiplet. In 2021, Kitchin [1] recovered the Muller hypermultiplet by coupling the standard Weyl multiplet to an on-shell hypermultiplet, yielding the 4D hyper-dilaton Weyl multiplet.

The paper will set out the groundwork for creating the analogous hyper-dilaton Weyl multiplet for the 6D case. To construct their respective multiplets, both authors used the conformal superspace formalism. In this formalism, the usually spacetime coordinates are extended by the addition of spinorial Grassman coordinates. For example, in 6D, coordinates are given by

$$z^M = (x^m, \theta_i^\mu), \tag{8}$$

where there are 6 regular spacetime coordinates ($m = 0, 1, ..., 5$), and 4 (2-component) fermionic coordinates ($\mu = 1, ..., 4$ and $i = 1, 2$). Instead of using fields written in terms of spacetime coordinates, superfields are used written in terms of superspace coordinates. This effectively allows multiple fields to be combined (as Taylor series coefficients) as a single superfield, and makes transformation laws simpler. For example, supersymmetry translations are simply translations in superspace. In conformal superspace, the conformal generators and connections are written in terms of superspace coordinates.

In this project, we take the first steps in recovering the 6D analogue of the Muller hypermultiplet. More specifically, we build the superconformal on-shell hypermultiplet analogously to Kitchin [1], starting with the superfield

$$q^{i\bar{i}} = \begin{pmatrix} \bar{q}^i \\ q^i \end{pmatrix} \tag{9}$$

This multiplet is the supersymmetric extension of the electron. We apply the constraints given by the 6D algebra to determine the field content of the multiplet. We then begin the construction of the equations of motion of the multiplet fields. This lays the groundwork for the reduction to a Poincare supergravity theory in 6D.

6D $N = 1$ theories are of interest due to their similarity of field content to the 4D $N = 2$ theories worked on by Butter and Kitchin. 6D supergravity models (with two small space dimensions) have in fact been used in the past to search for phenomenological extensions of GR and the standard model. In a string theory context, there are proposals to study the entropy of Black Holes [4]. This task is highly simplified for largely supersymmetric systems. Supersymmetric black holes in string theory work great as a toy model to understand the quantum behaviour of black holes. Macroscopic black hole entropy arises naturally from supergravity theories. These are very challenging to be constructed, and our techniques might be used in this direction. From a pure mathematical point of view, it is also an interesting representation theory problem to classify the representation of local superconformal and Poincare superalgebras.

3 Actions of the Algebra on Superfield q

3.1 Basic Constraints

We begin by imposing the constraints (hypermultiplet)

$$\nabla_\alpha^{(i} q^{j)\underline{k}} = 0, \tag{10}$$

and the superconformally primary constraint

$$S_i^\alpha q^{j\underline{k}} = 0 \tag{11}$$

$$K^a q^{i\underline{j}} = 0 \tag{12}$$

Together, these two constraints imply that

$$\{S_i^\alpha, \nabla_\beta^{(j}\} q^{k)\underline{r}} = 0 \tag{13}$$

Further, $q^{i\underline{j}}$ is a Lorentz scalar, and thus

$$M_{\alpha\beta}q^{i\underline{j}} = 0. \tag{14}$$

3.2 Action of J on Superfield q

Then, the action of J^{lp} on $q^{k\underline{r}}$ can be found. Indeed, beginning with the commutation relation

$$[J_{ij}, \nabla^k_\alpha]q^{n\underline{m}} = -\delta^k_{(i}\nabla_{\alpha j)}q^{n\underline{m}} \tag{15}$$

Then,

$$[J^{lp}, \nabla^k_\alpha]q^{n\underline{m}} = -\epsilon^{li}\epsilon^{pj}\delta^k_{(i}\nabla_{\alpha j)}q^{n\underline{m}} \tag{16}$$

Expanding out the permuation of indices

$$= -\frac{1}{2}\epsilon^{li}\epsilon^{pj}(\delta^k_i\nabla_{\alpha j} + \delta^k_j\nabla_{\alpha i})q^{n\underline{m}} \tag{17}$$

$$= -\frac{1}{2}(\epsilon^{lk}\epsilon^{pj}\nabla_{\alpha j} + \epsilon^{li}\epsilon^{pk}\nabla_{\alpha i})q^{n\underline{m}} \tag{18}$$

$$= -\frac{1}{2}(\epsilon^{lk}\nabla^p_\alpha + \epsilon^{pk}\nabla^l_\alpha)q^{n\underline{m}} \tag{19}$$

$$= (\epsilon^{k(l}\nabla^{p)}_\alpha)q^{n\underline{m}}. \tag{20}$$

Thus,

$$J^{lp}\nabla^k_\alpha q^{n\underline{m}} - \nabla^k_\alpha J^{lp}q^{n\underline{m}} = (\epsilon^{k(l}\nabla^{p)}_\alpha)q^{n\underline{m}} \tag{21}$$

Note that from (10), one has

$$\nabla^k_\alpha q^{n\underline{m}} = -\nabla^n_\alpha q^{k\underline{m}} \tag{22}$$

Furthermore, this antisymmetry in indices means

$$\nabla^k_\alpha q^{n\underline{m}} = -\frac{1}{2}\epsilon^{kn}\nabla^r_\alpha q^{\underline{m}}_r \tag{23}$$

Thus, (21) becomes

$$-\frac{1}{2}\epsilon^{kn}J^{lp}\nabla^r_\alpha q^{\underline{m}}_r - \nabla^k_\alpha J^{lp}q^{n\underline{m}} = (\epsilon^{k(l}\nabla^{p)}_\alpha)q^{n\underline{m}} \tag{24}$$

$$= \frac{1}{2}(\epsilon^{kl}\nabla^p_\alpha q^{n\underline{m}} + \epsilon^{kp}\nabla^l_\alpha q^{n\underline{m}}) = -\frac{1}{2}(\epsilon^{kl}\nabla^n_\alpha q^{p\underline{m}} + \epsilon^{kp}\nabla^n_\alpha q^{l\underline{m}} = -\nabla^n_\alpha\epsilon^{k(l}q^{p)\underline{m}} \tag{25}$$

$$\implies -\frac{1}{2}\epsilon^{kn}J^{lp}\nabla^r_\alpha q^{\underline{m}}_r - \nabla^k_\alpha J^{lp}q^{n\underline{m}} = -\nabla^n_\alpha\epsilon^{k(l}q^{p)\underline{m}} \tag{26}$$

Now, setting $k = n$, this reduces to

$$-\nabla^n_\alpha J^{lp}q^{n\underline{m}} = -\nabla^n_\alpha\epsilon^{n(l}q^{p)\underline{m}}. \tag{27}$$

This implies

$$J^{lp}q^{n\underline{m}} = \epsilon^{n(l}q^{p)\underline{m}} \tag{28}$$

3.3 Computing the Dilaton Weight of q

Now, we aim to compute the dilaton weight,

$$\mathbb{D}q^{i\underline{j}} = \lambda_q q^{i\underline{j}} \tag{29}$$

To do so, note that in the algebra, it is known that

$$\{S_i^\alpha, \nabla_\beta^j\} = 2\delta_\beta^\alpha \delta_j^i \mathbb{D} - 4\delta_i^j M_\beta^\alpha + 8\delta_\beta^\alpha J_i^j \tag{30}$$

Then, from (13), one has

$$(2\delta_\beta^\alpha \delta_{(j}^i \mathbb{D} - 4\delta_i^{(j} M_\beta^\alpha + 8\delta_\beta^\alpha J_i^{(j})q^{k)\underline{h}} = 0 \tag{31}$$

$$\implies (2\delta_\beta^\alpha \delta_{(j}^i \mathbb{D} + 8\delta_\beta^\alpha J_i^{(j})q^{k)\underline{h}} = 0 \tag{32}$$

by the Lorentz scalar property of $q^{i\underline{i}}$.

Expanding:

$$\implies \delta_\beta^\alpha(2\delta_{(j}^i \lambda_q q^{k)\underline{h}} + 8J_i^{(j} q^{k)\underline{h}}) = 0 \tag{33}$$

$$\implies \delta_\beta^\alpha((\delta_j^i \lambda_q q^{k\underline{h}} + \delta_k^i \lambda_q q^{j\underline{h}}) + 4(J_i^j q^{k\underline{h}} + J_i^k q^{j\underline{h}})) = 0 \tag{34}$$

$$\implies \delta_\beta^\alpha((\delta_j^i \lambda_q q^{k\underline{h}} + \delta_k^i \lambda_q q^{j\underline{h}}) + 4\epsilon_{im}(J^{mj} q^{k\underline{h}} + J^{mk} q^{j\underline{h}})) = 0 \tag{35}$$

Applying (28),

$$\implies \delta_\beta^\alpha((\delta_j^i \lambda_q q^{k\underline{h}} + \delta_k^i \lambda_q q^{j\underline{h}}) + 4\epsilon_{im}(\epsilon^{k(m} q^{j)\underline{h}} + \epsilon^{j(m} q^{k)\underline{h}})) = 0 \tag{36}$$

Expanding permuation of indices

$$\implies \delta_\beta^\alpha((\delta_j^i \lambda_q q^{k\underline{h}} + \delta_k^i \lambda_q q^{j\underline{h}}) + 2\epsilon_{im}((\epsilon^{km} q^{j\underline{h}} + \epsilon^{kj} q^{m\underline{h}}) + (\epsilon^{jm} q^{k\underline{h}} + \epsilon^{jk} q^{m\underline{h}}))) = 0 \tag{37}$$

$$\implies \delta_\beta^\alpha((\delta_j^i \lambda_q q^{k\underline{h}} + \delta_k^i \lambda_q q^{j\underline{h}}) + 2((-\delta_k^i q^{j\underline{h}} + \epsilon^{kj} q^{m\underline{h}}) + (-\delta_i^j q^{k\underline{h}} + \epsilon^{jk} q^{m\underline{h}}))) = 0 \tag{38}$$

$$\implies \delta_\beta^\alpha((\delta_j^i \lambda_q q^{k\underline{h}} + \delta_k^i \lambda_q q^{j\underline{h}}) + 2((-\delta_k^i q^{j\underline{h}} - \delta_i^j q^{k\underline{h}})) = 0 \tag{39}$$

$$\implies (\lambda_q - 2)\delta_\beta^\alpha((\delta_j^i q^{k\underline{h}} + \delta_k^i q^{j\underline{h}}) = 0 \tag{40}$$

This implies that

$$\lambda_q = 2 \tag{41}$$

And thus,

$$\mathbb{D}q^{i\underline{j}} = 2q^{i\underline{j}}. \tag{42}$$

4 Constructing the Multiplet

4.1 First Spinorial Derivative of q

We will now compute the repeated action of the derivative ∇^i_α on $q^{j\underline{j}}$. Firstly, note that

$$\nabla^i_\alpha q^{j\underline{j}} = \nabla^{(i}_\alpha q^{j)\underline{j}} + \nabla^{[i}_\alpha q^{j]\underline{j}} \tag{43}$$

From (10),

$$\nabla^i_\alpha q^{j\underline{j}} = \nabla^{[i}_\alpha q^{j]\underline{j}} \tag{44}$$

Note that for a tensor M, $M_{[ab]} = \frac{1}{2!}\delta^{cd}_{ab}M_{cd} = -\frac{1}{2}\epsilon_{ab}\epsilon^{cd}M_{cd}$. Thus,

$$\nabla^i_\alpha q^{j\underline{j}} = -\frac{1}{2}\epsilon^{ij}\epsilon_{nm}\nabla^n_\alpha q^{m\underline{j}} \tag{45}$$

$$\implies \nabla^i_\alpha q^{j\underline{j}} = -\frac{1}{2}\epsilon^{ij}\nabla^n_\alpha q^{\underline{j}}_n \tag{46}$$

Then, defining the field $\psi_\alpha = \nabla^n_\alpha q^{\underline{j}}_n$,

$$\nabla^i_\alpha q^{j\underline{j}} = -\frac{1}{2}\epsilon^{ij}\psi_\alpha \tag{47}$$

4.2 Second Spinorial Derivative of q

Now, we aim to compute the 2nd spinorial derivative of q.

Note that

$$\nabla^i_\alpha \nabla^j_\beta q^{k\underline{r}} = \frac{1}{2}[\nabla^i_\alpha, \nabla^j_\beta]q^{k\underline{r}} + \frac{1}{2}\{\nabla^i_\alpha, \nabla^j_\beta\}q^{k\underline{r}} \tag{48}$$

Then considering the first term of (48)

$$\frac{1}{2}[\nabla^i_\alpha, \nabla^j_\beta]q^{k\underline{r}} = \frac{1}{2}(\nabla^i_\alpha\nabla^j_\beta - \nabla^j_\beta\nabla^i_\alpha)q^{k\underline{r}} \tag{49}$$

$$= \frac{1}{2}(\nabla^i_\alpha\nabla^j_\beta - \nabla^j_\beta\nabla^i_\alpha + \nabla^j_\alpha\nabla^i_\beta - \nabla^j_\alpha\nabla^i_\beta)q^{k\underline{r}} \tag{50}$$

$$= (\nabla^{[i}_\alpha\nabla^{j]}_\beta + \nabla^j_{[\alpha}\nabla^i_{\beta]})q^{k\underline{r}} \tag{51}$$

$$= (\nabla^{[i}_{(\alpha}\nabla^{j]}_{\beta)} + \nabla^{[i}_{[\alpha}\nabla^{j]}_{\beta]} + \nabla^{(j}_{[\alpha}\nabla^{i)}_{\beta]} + \nabla^{[j}_{[\alpha}\nabla^{i]}_{\beta]})q^{k\underline{r}} \tag{52}$$

$$= (\nabla^{[i}_{(\alpha}\nabla^{j]}_{\beta)} + \nabla^{(j}_{[\alpha}\nabla^{i)}_{\beta]})q^{k\underline{r}} \tag{53}$$

Considering the second term of (53)

$$\nabla^{(i}_{[\alpha}\nabla^{j)}_{\beta]}q^{k\underline{r}} = -\frac{1}{2}\nabla^{(i}_{[\alpha}\epsilon^{j)k}\psi^{\underline{r}}_{\beta]} = \frac{1}{2}\epsilon^{k(i}\nabla^{j)}_{[\alpha}\psi^{\underline{r}}_{\beta]} = \frac{1}{2}\epsilon^{k(i}\nabla^{j)}_{[\alpha}\nabla^p_{\beta]}q^{\underline{r}}_p \tag{54}$$

$$= \frac{1}{2}\epsilon^{k(i}\left(\{\nabla^{j)}_{[\alpha}, \nabla^p_{\beta]}\} - \nabla^p_{[\beta}\nabla^{j)}_{\alpha]}\right)q^{\underline{r}}_p = \frac{1}{2}\epsilon^{k(i}\left(\{\nabla^{j)}_{[\alpha}, \nabla^p_{\beta]}\} + \nabla^p_{[\alpha}\nabla^{j)}_{\beta]}\right)q^{\underline{r}}_p \tag{55}$$

$$= \frac{1}{2}\epsilon^{k(i}\left(\{\nabla^{j)}_{[\alpha}, \nabla^{p}_{\beta]}\}q^{\underline{r}}_{p} + \epsilon_{ps}\nabla^{p}_{[\alpha}\nabla^{j)}_{\beta]}q^{s\underline{r}}\right) = \frac{1}{2}\epsilon^{k(i}\left(\{\nabla^{j)}_{[\alpha}, \nabla^{p}_{\beta]}\}q^{\underline{r}}_{p} - \frac{1}{2}\epsilon_{ps}\epsilon^{j)s}\nabla^{p}_{[\alpha}\psi^{\underline{r}}_{\beta]}\right) \quad (56)$$

$$= \frac{1}{2}\epsilon^{k(i}\left(\{\nabla^{j)}_{[\alpha}, \nabla^{p}_{\beta]}\}q^{\underline{r}}_{p} + \frac{1}{2}\delta^{pj)}\nabla^{p}_{[\alpha}\psi^{\underline{r}}_{\beta]}\right) = \frac{1}{2}\epsilon^{k(i}\left(\{\nabla^{j)}_{[\alpha}, \nabla^{p}_{\beta]}\}q^{\underline{r}}_{p} + \frac{1}{2}\nabla^{j)}_{[\alpha}\psi^{\underline{r}}_{\beta]}\right) \quad (57)$$

Thus,

$$\frac{1}{2}\epsilon^{k(i}\nabla^{j)}_{[\alpha}\psi^{\underline{r}}_{\beta]} = \frac{1}{2}\epsilon^{k(i}\left(\{\nabla^{j)}_{[\alpha}, \nabla^{p}_{\beta]}\}q^{\underline{r}}_{p} + \frac{1}{2}\nabla^{j)}_{[\alpha}\psi^{\underline{r}}_{\beta]}\right) \quad (58)$$

So

$$\nabla^{(j}_{[\alpha}\nabla^{i)}_{\beta]}q^{k\underline{r}} = \frac{1}{2}\epsilon^{k(i}\nabla^{j)}_{[\alpha}\psi^{\underline{r}}_{\beta]} = \epsilon^{k(i}\{\nabla^{j)}_{[\alpha}, \nabla^{p}_{\beta]}\}q^{\underline{r}}_{p} \quad (59)$$

Substituting back into (53),

$$(\nabla^{[i}_{(\alpha}\nabla^{j]}_{\beta)})q^{k\underline{r}} + \epsilon^{k(i}\{\nabla^{j)}_{[\alpha}, \nabla^{p}_{\beta]}\}q^{\underline{r}}_{p} \quad (60)$$

$$= (-\frac{1}{2}\epsilon^{ij}\epsilon_{st}\nabla^{s}_{(\alpha}\nabla^{t}_{\beta)})q^{k\underline{r}} + \epsilon^{k(i}\{\nabla^{j)}_{[\alpha}, \nabla^{p}_{\beta]}\}q^{\underline{r}}_{p} \quad (61)$$

$$= (-\frac{1}{2}\epsilon^{ij}\nabla^{s}_{(\alpha}\nabla_{\beta)s})q^{k\underline{r}} + \epsilon^{k(i}\{\nabla^{j)}_{[\alpha}, \nabla^{p}_{\beta]}\}q^{\underline{r}}_{p} \quad (62)$$

We define $\nabla_{\alpha\beta} = \nabla^{s}_{(\alpha}\nabla_{\beta)s}$. Now, determine the action of $\nabla_{\alpha\beta}$ on $q^{j\underline{k}}$.

$$\nabla_{\alpha\beta}q^{j\underline{k}} = \nabla^{s}_{(\alpha}\nabla_{\beta)s}q^{j\underline{k}} = -\nabla_{s(\alpha}\nabla^{s}_{\beta)}q^{j\underline{k}} = \frac{1}{2}\epsilon^{si}\nabla_{s(\alpha}\psi_{\beta)} = -\frac{1}{2}\nabla^{i}_{(\alpha}\psi_{\beta)} \quad (63)$$

Now, note that

$$\nabla^{i}_{(\alpha}\psi_{\beta)} = \nabla^{i}_{(\alpha}\nabla^{s}_{\beta)}q^{\underline{j}}_{s} = \frac{1}{2}(\nabla^{i}_{\alpha}\nabla^{s}_{\beta} + \nabla^{i}_{\beta}\nabla^{s}_{\alpha})q^{\underline{j}}_{s} \quad (64)$$

$$= \frac{1}{2}(\{\nabla^{i}_{\alpha}, \nabla^{s}_{\beta}\} - \nabla^{s}_{\beta}\nabla^{i}_{\alpha} + \{\nabla^{i}_{\beta}\nabla^{s}_{\alpha}\} - \nabla^{s}_{\alpha}\nabla^{i}_{\beta})q^{\underline{j}}_{s} \quad (65)$$

$$= (\{\nabla^{i}_{(\alpha}, \nabla^{s}_{\beta)}\} - \nabla^{s}_{(\beta}\nabla^{i}_{\alpha)})q^{\underline{j}}_{s} \quad (66)$$

$$= (\{\nabla^{i}_{(\alpha}, \nabla^{s}_{\beta)}\} - \nabla^{s}_{(\beta}\nabla^{i}_{\alpha)})q^{\underline{j}}_{s} \quad (67)$$

By definition, this is

$$= \{\nabla^{i}_{(\alpha}, \nabla^{s}_{\beta)}\}q^{\underline{j}}_{s} + \nabla^{s}_{(\beta}\frac{1}{2}\epsilon^{in}\epsilon_{sn}\psi_{\alpha)} \quad (68)$$

$$= \{\nabla^{i}_{(\alpha}, \nabla^{s}_{\beta)}\}q^{\underline{j}}_{s} - \frac{1}{2}\delta_{si}\nabla^{s}_{(\beta}\psi_{\alpha)} \quad (69)$$

$$= \{\nabla^{i}_{(\alpha}, \nabla^{s}_{\beta)}\}q^{\underline{j}}_{s} - \frac{1}{2}\nabla^{i}_{(\alpha}\psi_{\beta)} \quad (70)$$

Thus,

$$\nabla^i_{(\alpha}\psi_{\beta)} = \frac{2}{3}\{\nabla^i_{(\alpha}, \nabla^s_{\beta)}\}q_s \tag{71}$$

So, from (64),

$$\nabla_{\alpha\beta}q^{i\underline{k}} = -\frac{1}{3}\{\nabla^i_{(\alpha}, \nabla^s_{\beta)}\}q^{\underline{k}}_s \tag{72}$$

This gives the expression

$$\frac{1}{2}[\nabla^i_\alpha, \nabla^j_\beta]q^{k\underline{r}} = (-\frac{1}{2}\epsilon^{ij}(-\frac{1}{3}\{\nabla^k_{(\alpha}, \nabla^s_{\beta)}\}q^{\underline{r}}_s) + \epsilon^{k(i}\{\nabla^{j)}_{[\alpha}, \nabla^p_{\beta]}\}q^{\underline{r}}_p)) \tag{73}$$

Thus, from (48),

$$\nabla^i_\alpha\nabla^j_\beta q^{k\underline{r}} = \frac{1}{2}\Big(\epsilon^{ij}\frac{1}{3}\{\nabla^k_{(\alpha}, \nabla^s_{\beta)}\}q^{\underline{r}}_s + \epsilon_{\alpha\beta}\epsilon^{k(j}\{\nabla^{i)\delta}, \nabla^s_\delta\}q^{\underline{r}}_s\Big) + \frac{1}{2}\{\nabla^i_\alpha, \nabla^j_\beta\}q^{k\underline{r}} \tag{74}$$

Writing in terms of ψ_α,

$$\epsilon^{jk}\nabla^i_\alpha\psi^{\underline{r}}_\beta = -\Big(\frac{1}{3}\epsilon^{ij}\{\nabla^k_{(\alpha}, \nabla^s_{\beta)}\}q^{\underline{r}}_s + 2\epsilon^{k(i}\{\nabla^{j)}_{[\alpha}, \nabla^p_{\beta]}\}q^{\underline{r}}_p\Big) - \{\nabla^i_\alpha, \nabla^j_\beta\}q^{k\underline{r}} \tag{75}$$

$$\epsilon_{kj}\epsilon^{jk}\nabla^i_\alpha\psi^{\underline{r}}_\beta = -\Big(\frac{1}{3}\epsilon_{kj}\epsilon^{ij}\{\nabla^k_{(\alpha}, \nabla^s_{\beta)}\}q^{\underline{r}}_s - \epsilon_{jk}2\epsilon^{k(i}\{\nabla^{j)}_{[\alpha}, \nabla^p_{\beta]}\}q^{\underline{r}}_p\Big) - \epsilon_{kj}\{\nabla^i_\alpha, \nabla^j_\beta\}q^{k\underline{r}} \tag{76}$$

$$2\nabla^i_\alpha\psi^{\underline{r}}_\beta = -\Big(-\frac{1}{3}\delta^k_i\{\nabla^k_{(\alpha}, \nabla^s_{\beta)}\}q^{\underline{r}}_s - \Bigg(\epsilon_{jk}\epsilon^{ki}\{\nabla^j_{[\alpha}, \nabla^p_{\beta]}\} + \epsilon_{jk}\epsilon^{kj}\{\nabla^i_{[\alpha}, \nabla^p_{\beta]}\}\Bigg)q^{\underline{r}}_p\Big) - \{\nabla^i_\alpha, \nabla_{\beta k}\}q^{k\underline{r}} \tag{77}$$

$$2\nabla^i_\alpha\psi^{\underline{r}}_\beta = -\Big(-\frac{1}{3}\delta^k_i\{\nabla^k_{(\alpha}, \nabla^s_{\beta)}\}q^{\underline{r}}_s - \Bigg(\delta^{ij}\{\nabla^j_{[\alpha}, \nabla^p_{\beta]}\} + 2\{\nabla^i_{[\alpha}, \nabla^p_{\beta]}\}\Bigg)q^{\underline{r}}_p\Big) - \{\nabla^i_\alpha, \nabla_{\beta k}\}q^{k\underline{r}} \tag{78}$$

$$\nabla^i_\alpha\psi^{\underline{r}}_\beta = \frac{1}{2}\Big(-\Big(-\frac{1}{3}\{\nabla^i_{(\alpha}, \nabla^s_{\beta)}\}q^{\underline{r}}_s - 3\{\nabla^i_{[\alpha}, \nabla^p_{\beta]}\}q^{\underline{r}}_p\Big) - \{\nabla^i_\alpha, \nabla_{\beta k}\}q^{k\underline{r}}\Big) \tag{79}$$

And thus

$$\nabla^i_\alpha\psi^{\underline{r}}_\beta = \frac{1}{6}\{\nabla^i_{(\alpha}, \nabla^s_{\beta)}\}q^{\underline{r}}_s + \frac{3}{2}\{\nabla^i_{[\alpha}, \nabla^p_{\beta]}\}q^{\underline{r}}_p - \frac{1}{2}\{\nabla^i_\alpha, \nabla_{\beta k}\}q^{k\underline{r}} \tag{80}$$

Note that the algebra of covariant derivatives is known to be

$$\begin{aligned}\{\nabla^i_\alpha, \nabla^j_\beta\} = &-(2i\epsilon^{ij}(\gamma^c)_{\alpha\beta}\nabla_c) - \frac{1}{2}(4i\epsilon^{ij}(\gamma_a)_{\alpha_\beta}W^{acd}M_{cd}) - (-\frac{3}{2}\epsilon^{ij}\epsilon_{\alpha\beta\gamma\delta}X^{\delta k}S^\gamma_k) \\ &- (i\epsilon^{ij}(\gamma^a)_{\alpha\beta}(\frac{1}{4}\eta_{ac}Y - \nabla^b W_{abc} + W^{ef}_a W_{cef})K^c)\end{aligned} \tag{81}$$

It is thus clear that due to antisymmetry of the Levi-Civita symbol that

$$\{\nabla^{(i}_\alpha, \nabla^{j)}_\beta\} = 0 \tag{82}$$

Furthermore, $(\gamma^c)_{\alpha\beta}$ is antisymmetric, and thus

$$\{\nabla^i_{(\alpha}, \nabla^j_{\beta)}\} = 0 \tag{83}$$

And

$$\{\nabla^i_{[\alpha}, \nabla^j_{\beta]}\} = \{\nabla^i_\alpha, \nabla^j_\beta\} \tag{84}$$

Thus, the derivative (80) becomes

$$\nabla^i_\alpha \psi^{\underline{r}}_\beta = \frac{3}{2}\{\nabla^i_\alpha, \nabla^p_\beta\}q^{\underline{r}}_p - \frac{1}{2}\{\nabla^i_\alpha, \nabla_{\beta p}\}q^{p\underline{r}} \tag{85}$$

$$= \frac{3}{2}\{\nabla^i_\alpha, \nabla^p_\beta\}q^{\underline{r}}_p + \frac{1}{2}\{\nabla^i_\alpha, \nabla^p_\beta\}q^{\underline{r}}_p \tag{86}$$

$$= 2\{\nabla^i_\alpha, \nabla^p_\beta\}q^{\underline{r}}_p \tag{87}$$

Appyling this anticommutator to q, and using the fact that q is superconformally primary (11) (12), and a Lorentz scalar (14), one sees that

$$\nabla^i_\alpha \psi^{\underline{r}}_\beta = -2\bigg((2i\epsilon^{is}(\gamma^c)_{\alpha\beta}\nabla_c)\bigg)q^{\underline{r}}_s \tag{88}$$

$$\implies \nabla^i_\alpha \psi^{\underline{r}}_\beta = -4i(\gamma^c)_{\alpha\beta}\nabla_c q^{i\underline{r}} = -4i\nabla_{\alpha\beta}q^{i\underline{r}} \tag{89}$$

4.3 Derivative of ψ

Now, we aim to compute the double spinor index derivative of ψ.

$$\nabla^{\alpha\beta}\psi^{\underline{i}}_\beta = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\nabla_{\gamma\delta}\psi^{\underline{i}}_\beta \tag{90}$$

Now, recall the algebra of covariant derivatives is given by

$$\begin{aligned}\{\nabla^i_\alpha, \nabla^p_\beta\} = &-(2i\epsilon^{ip}(\gamma^a)_{\alpha\beta}\nabla_a) - \frac{1}{2}(4i\epsilon^{ip}(\gamma_a)_{\alpha_\beta}W^{acd}M_{cd}) - (-\frac{3}{2}\epsilon^{ip}\epsilon_{\alpha\beta\gamma\delta}X^{\delta k}S^\gamma_k) \\ &- (i\epsilon^{ip}(\gamma^a)_{\alpha\beta}(\frac{1}{4}\eta_{ac}Y - \nabla^b W_{abc} + W^{ef}_a W_{cef})K^c)\end{aligned} \tag{91}$$

$$\begin{aligned}&\implies (\gamma^a)_{\alpha\beta}\nabla_a \\ &= \frac{i}{4}\epsilon_{pi}\bigg(- \{\nabla^i_\alpha, \nabla^p_\beta\} - \frac{1}{2}(4i\epsilon^{ip}(\gamma_a)_{\alpha_\beta}W^{acd}M_{cd}) \\ &\quad + (\frac{3}{2}\epsilon^{ip}\epsilon_{\alpha\beta\gamma\delta}X^{\delta k}S^\gamma_k) - (i\epsilon^{ip}(\gamma^a)_{\alpha\beta}(\frac{1}{4}\eta_{ac}Y - \nabla^b W_{abc} + W^{ef}_a W_{cef})K^c)\bigg)\end{aligned} \tag{92}$$

Thus,

$$\begin{aligned}\nabla^{\alpha\beta}\psi^{\underline{i}}_\beta &= \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\nabla_{\gamma\delta}\psi^{\underline{i}}_\beta \\ &= \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}(\gamma_a)_{\gamma\delta}\nabla_a\psi^{\underline{i}}_\beta \\ &= \frac{i}{8}\epsilon^{\alpha\beta\gamma\delta}\bigg(- \{\nabla_{\gamma p}, \nabla^p_\delta\} + (4i(\gamma_a)_{\gamma_\delta}W^{acd}M_{cd}) - (3\epsilon_{\gamma\delta\rho\tau}X^{\tau k}S^\rho_k) \\ &\quad + (2i(\gamma^a)_{\gamma\delta}(\frac{1}{4}\eta_{ac}Y - \nabla^b W_{abc} + W^{ef}_a W_{cef})K^c)\bigg)\psi^{\underline{i}}_\beta\end{aligned} \tag{93}$$

Now, expanding the first term of (93),

$$-\frac{i}{8}\epsilon^{\alpha\beta\gamma\delta}\{\nabla_{\gamma p},\nabla^{p}_{\delta}\}\psi^{\underline{i}}_{\beta}=-\frac{i}{8}\epsilon^{\alpha\beta\gamma\delta}(\nabla_{\gamma p}\nabla^{p}_{\delta}+\nabla^{p}_{\delta}\nabla_{\gamma p})\psi^{\underline{i}}_{\beta} \tag{94}$$

From (89)

$$=-\frac{i}{8}\epsilon^{\alpha\beta\gamma\delta}(\nabla_{\gamma p}(-4i\nabla_{\delta\beta})q^{p\underline{i}}+\epsilon_{pk}\nabla^{p}_{\delta}(-4i\nabla_{\gamma\beta})q^{k\underline{i}}) \tag{95}$$

$$=-\frac{i}{8}\epsilon^{\alpha\beta\gamma\delta}(\nabla_{\gamma p}(-4i\nabla_{\delta\beta})q^{p\underline{i}}-\nabla_{\delta p}(-4i\nabla_{\gamma\beta})q^{p\underline{i}}) \tag{96}$$

$$=-\frac{i}{8}(\epsilon^{\alpha\gamma\delta\beta}\nabla_{\gamma p}(-4i\nabla_{\delta\beta})q^{p\underline{i}}+\epsilon^{\alpha\delta\gamma\beta}\nabla_{\delta p}(-4i\nabla_{\gamma\beta})q^{p\underline{i}}) \tag{97}$$

$$=-\frac{i}{8}(\nabla_{\gamma p}(-4i\nabla^{\alpha\gamma})q^{p\underline{i}}+\nabla_{\delta p}(-4i\nabla^{\alpha\delta})q^{p\underline{i}}) \tag{98}$$

$$=-\frac{i}{4}(\nabla_{\beta p}(-4i\nabla^{\alpha\beta})q^{p\underline{i}}) \tag{99}$$

$$=-([\nabla_{\beta p},\nabla^{\alpha\beta}]+\nabla^{\alpha\beta}\nabla_{\beta p})q^{p\underline{i}}) \tag{100}$$

$$=-([\nabla_{\beta p},\nabla^{\alpha\beta}]q^{p\underline{i}}+\nabla^{\alpha\beta}\psi^{\underline{i}}_{\beta}) \tag{101}$$

Now, the commutator in the first term of (101) is given by

$$[\nabla_{\beta p},\nabla^{\alpha\beta}]=-(\gamma^{a})^{\alpha\beta}\epsilon_{pj}[\nabla_{a},\nabla^{j}_{\beta}] \tag{102}$$

$$\begin{aligned}&=-(\gamma^{a})^{\alpha\beta}\epsilon_{pj}\Bigg(-(-\frac{1}{2}(\gamma_{a})_{\beta\delta}\\ &W^{\delta\gamma}\delta^{j}_{k})\nabla^{k}_{\gamma}-(-\frac{\mathrm{i}}{2}(\gamma_{a})_{\beta\gamma}\\ &X^{\gamma j})\mathbb{D}-\frac{1}{2}(R(M)^{jcd}_{a_{\beta}})M_{cd}-(2\mathrm{i}\,(\gamma_{a})_{\beta\gamma}X^{\gamma(k}\varepsilon^{l)j})J_{kl}-(R(S)^{jk}_{\alpha\beta\gamma})S^{\gamma}_{k}-(R(K)^{j}_{\alpha\beta c})K^{c}\Bigg)\end{aligned} \tag{103}$$

Thus,

$$[\nabla_{\beta p},\nabla^{\alpha\beta}]q^{p\underline{i}}=(\gamma^{a})^{\alpha\beta}\epsilon_{pj}\Bigg((-\frac{1}{2}(\gamma_{a})_{\beta\delta}W^{\delta\gamma}\delta^{j}_{k})\nabla^{k}_{\gamma}+(-\frac{\mathrm{i}}{2}(\gamma_{a})_{\beta\gamma}X^{\gamma j})\mathbb{D}+(2\mathrm{i}\,(\gamma_{a})_{\beta\gamma}X^{\gamma(k}\varepsilon^{l)j})J_{kl}\Bigg)q^{p\underline{i}} \tag{104}$$

$$=(\gamma^{a})^{\alpha\beta}\Bigg(\epsilon_{pj}(-\frac{1}{2}(\gamma_{a})_{\beta\delta}W^{\delta\gamma}\delta^{j}_{k})\nabla^{k}_{\gamma}q^{p\underline{i}}-\epsilon_{pj}i\,(\gamma_{a})_{\beta\gamma}X^{\gamma j}q^{p\underline{i}}+\epsilon_{pj}2\mathrm{i}\,(\gamma_{a})_{\beta\gamma}X^{\gamma(k}\varepsilon^{l)j}\epsilon_{kn}\epsilon_{ml}\epsilon^{p(n}q^{m)\underline{i}}\Bigg) \tag{105}$$

Consider the last term of (105),

$$\begin{aligned}&\epsilon_{pj}2\mathrm{i}\,(\gamma_{a})_{\beta\gamma}X^{\gamma(k}\varepsilon^{l)j}\epsilon_{kn}\epsilon_{ml}\epsilon^{p(n}q^{m)\underline{i}}\\ &=\epsilon_{pj}\frac{1}{2}i\,(\gamma_{a})_{\beta\gamma}(X^{\gamma k}\epsilon^{lj}\epsilon_{kn}\epsilon_{ml}\epsilon^{pn}q^{m\underline{i}}+X^{\gamma k}\epsilon^{lj}\epsilon_{kn}\epsilon_{ml}\epsilon^{pm}q^{n\underline{i}}\\ &\qquad+X^{\gamma l}\epsilon^{kj}\epsilon_{kn}\epsilon_{ml}\epsilon^{pn}q^{m\underline{i}}+X^{\gamma l}\epsilon^{kj}\epsilon_{kn}\epsilon_{ml}\epsilon^{pm}q^{n\underline{i}})\end{aligned} \tag{106}$$

$$= \frac{1}{2} i \left(\gamma_a\right)_{\beta\gamma} \left(X^{\gamma k} \epsilon^{lj} \epsilon_{pj} \delta^p_k \epsilon_{ml} q^{m\underline{i}} + X^{\gamma k} \epsilon_{jl} \epsilon_{pj} \epsilon_{kn} \epsilon_{ml} \epsilon^{pm} q^{n\underline{i}} + X^{\gamma l} \delta^n_j \epsilon_{pj} \epsilon_{ml} \epsilon^{pn} q^{m\underline{i}} + X^{\gamma l} \delta^j_n \epsilon_{pj} \epsilon_{ml} \epsilon^{pm} q^{n\underline{i}}\right) \tag{107}$$

$$= \frac{1}{2} i \left(\gamma_a\right)_{\beta\gamma} \left(X^{\gamma k} \delta^k_l \epsilon_{ml} q^{m\underline{i}} + X^{\gamma k} \delta^l_p \epsilon_{kn} \delta^l_p q^{n\underline{i}} - 2X^{\gamma l} \epsilon_{ml} q^{m\underline{i}} + X^{\gamma l} \epsilon_{nl} q^{n\underline{i}}\right) \tag{108}$$

$$= \frac{1}{2} i \left(\gamma_a\right)_{\beta\gamma} \left(-X^\gamma_m q^{m\underline{i}} + X^\gamma_n q^{n\underline{i}} + 2X^\gamma_m q^{m\underline{i}} - X^\gamma_n q^{n\underline{i}}\right) \tag{109}$$

$$= \frac{i}{2} \left(\gamma_a\right)_{\beta\gamma} X^{\gamma k} q^{k\underline{i}} \tag{110}$$

Substituting into (105),

$$= (\gamma^a)^{\alpha\beta} \left(\epsilon_{pj} (-\frac{1}{2} \left(\gamma_a\right)_{\beta\delta} W^{\delta\gamma} \delta^j_k) \nabla^k_\gamma q^{p\underline{i}} + i \left(\gamma_a\right)_{\beta\gamma} X^\gamma_p q^{p\underline{i}} + \frac{i}{2} \left(\gamma_a\right)_{\beta\gamma} X^\gamma_p q^{p\underline{i}} \right) \tag{111}$$

$$= (\gamma^a)^{\alpha\beta} \left(\epsilon_{pj} (-\frac{1}{2} \left(\gamma_a\right)_{\beta\delta} W^{\delta\gamma} \delta^j_k) \nabla^k_\gamma q^{p\underline{i}} + \frac{3i}{2} \left(\gamma_a\right)_{\beta\gamma} X^\gamma_p q^{p\underline{i}} \right) \tag{112}$$

Using this result in (93),

$$\begin{aligned} &\nabla^{\alpha\beta} \psi^{\underline{i}}_\beta \\ &= (\gamma^a)^{\alpha\beta} \Bigg((\frac{1}{2} \left(\gamma_a\right)_{\beta\delta} \\ &W^{\delta\gamma}) \nabla_{p\gamma} q^{p\underline{i}} - \frac{3i}{2} \left(\gamma_a\right)_{\beta\gamma} X^\gamma_p q^{p\underline{i}} \Bigg) - \nabla^{\alpha\beta} \psi^{\underline{i}}_\beta + \frac{i}{8} \epsilon^{\alpha\beta\gamma\delta} \Bigg(- (4i(\gamma_a)_{\gamma_\delta} W^{acd} M_{cd}) \\ &+ (3\epsilon_{\gamma\delta\rho\tau} X^{\tau k} S^\rho_k) - (2i(\gamma^a)_{\gamma\delta} (\frac{1}{4} \eta_{ac} Y - \nabla^b W_{abc} + W^{ef}_a W_{cef}) K^c) \Bigg) \psi^{\underline{i}}_\beta \end{aligned} \tag{113}$$

$$\begin{aligned} &\nabla^{\alpha\beta} \psi^{\underline{i}}_\beta \\ &= \frac{1}{4} (\gamma^a)^{\alpha\beta} \Bigg(((\gamma_a)_{\beta\delta} \\ &W^{\delta\gamma}) \nabla_{p\gamma} - 3i \left(\gamma_a\right)_{\beta\gamma} X^\gamma_p \Bigg) q^{p\underline{i}} + \frac{i}{16} \epsilon^{\alpha\beta\gamma\delta} \Bigg(- (4i(\gamma_a)_{\gamma_\delta} W^{acd} M_{cd}) \\ &+ (3\epsilon_{\gamma\delta\rho\tau} X^{\tau k} S^\rho_k) - (2i(\gamma^a)_{\gamma\delta} (\frac{1}{4} \eta_{ac} Y - \nabla^b W_{abc} + W^{ef}_a W_{cef}) K^c) \Bigg) \psi^{\underline{i}}_\beta \end{aligned} \tag{114}$$

Now, the action of each of the generators on ψ must be computed.

Firstly, note that

$$M_{cd} \psi^{\underline{i}}_\beta = M_{cd} \nabla^j_\beta q^{\underline{i}}_j = [M_{cd}, \nabla^j_\beta] q^{\underline{i}}_j + \nabla^j_\beta M_{cd} q^{\underline{i}}_j = [M_{cd}, \nabla^j_\beta] q^{\underline{i}}_j \tag{115}$$

$$= \frac{1}{2}(\gamma_{cd})_\gamma^\alpha [M_\alpha^\gamma, \nabla_\beta^j] q_j^{\underline{i}} = \frac{1}{2}(\gamma_{cd})_\gamma^\alpha \left(-\delta_\beta^\gamma \nabla_\alpha^j + \frac{1}{4}\delta_\gamma^\alpha \nabla_\beta^j \right) q_j^{\underline{i}} \tag{116}$$

$$= \frac{1}{2}(\gamma_{cd})_\gamma^\alpha (-\delta_\beta^\gamma \psi_\alpha^{\underline{i}} + \frac{1}{4}\delta_\gamma^\alpha \psi_\beta^{\underline{i}}) = \frac{1}{2}(-(\gamma_{cd})_\beta^\alpha \psi_\alpha^{\underline{i}} + (\gamma_{cd})_\alpha^\alpha \frac{1}{4}\psi_\beta^{\underline{i}}) = -\frac{1}{2}(\gamma_{cd})_\beta^\alpha \psi_\alpha^{\underline{i}} \tag{117}$$

Where the tracelessness of γ_{cd} has been used. Now, in similar fashion,

$$S_k^\rho \psi_\beta^{\underline{i}} = S_k^\rho \nabla_\beta^j q_j^{\underline{i}} = \{S_k^\rho, \nabla_\beta^j\} q_j^{\underline{i}} - \nabla_\beta^j S_k^\rho q_j^{\underline{i}} = \{S_k^\rho, \nabla_\beta^j\} q_j^{\underline{i}} \tag{118}$$

Then,

$$\{S_k^\rho, \nabla_\beta^j\} q_j^{\underline{i}} = \left(2\delta_\beta^\rho \delta_k^j \mathbb{D} - 4\delta_k^j M_\beta^\rho + 8\delta_\beta^\rho J_k^j \right) q_j^{\underline{i}} = 4\delta_\beta^\rho \delta_k^j q_j^{\underline{i}} + 8\delta_\beta^\rho J_k^j q_j^{\underline{i}} \tag{119}$$

$$= 4\delta_\beta^\rho \delta_k^j q_j^{\underline{i}} + 8\delta_\beta^\rho \epsilon_{kp}\epsilon_{jn} J^{pk} q^{in} = 4\delta_\beta^\rho \delta_k^j q_j^{\underline{i}} + 8\delta_\beta^\rho \epsilon_{kp}\epsilon_{jn}\epsilon^{n(p} q^{k)\underline{i}} = 4\delta_\beta^\rho \delta_k^j q_j^{\underline{i}} + 4\delta_\beta^\rho (\delta_n^k \epsilon_{jn} q^{k\underline{i}} - \epsilon_{kp}\delta_j^k q^{p\underline{i}}) \tag{120}$$

$$= 4\delta_\beta^\rho \delta_k^j q_j^{\underline{i}} + 4\delta_\beta^\rho (\epsilon_{jk} q^{k\underline{i}} - \epsilon_{jp} q^{p\underline{i}}) = 4\delta_\beta^\rho q_k^{\underline{i}} \tag{121}$$

And

$$K^c \psi_\beta^{\underline{i}} = K^c \nabla_\beta^j q_j^{\underline{i}} = [K^c, \nabla_\beta^j] q_j^{\underline{i}} - \nabla_\beta^j K^c q_j^{\underline{i}} = [K^c, \nabla_\beta^j] q_j^{\underline{i}} \tag{122}$$

$$= -i(\gamma^c)_{\beta\gamma} S^{\gamma k} q_j^{\underline{i}} = 0 \tag{123}$$

Applying these results to (114),

$$\nabla^{\alpha\beta}\psi_\beta^{\underline{i}} = \frac{1}{4}(\gamma^a)^{\alpha\beta} \Bigg(((\gamma_a)_{\beta\delta} W^{\delta\gamma})\nabla_{p\gamma} - 3i\,(\gamma_a)_{\beta\gamma} X_p^\gamma \Bigg) q^{p\underline{i}} + \frac{i}{16}\epsilon^{\alpha\beta\gamma\delta} \Bigg(-(4i(\gamma_a)_{\gamma\delta} W^{acd}(-\frac{1}{2}(\gamma_{cd})_\beta^\alpha \psi_\alpha^{\underline{i}})) + (3\epsilon_{\gamma\delta\rho\tau} X^{\tau k}(4\delta_\beta^\rho q_k^{\underline{i}}) \Bigg) \tag{124}$$

$$\nabla^{\alpha\beta}\psi_\beta^{\underline{i}} = \frac{1}{4}(\gamma^a)^{\alpha\beta} \Bigg(((\gamma_a)_{\beta\delta} W^{\delta\gamma})\nabla_{p\gamma} - 3i\,(\gamma_a)_{\beta\gamma} X_p^\gamma \Bigg) q^{p\underline{i}} + \frac{i}{8}\epsilon^{\alpha\beta\gamma\delta} \Bigg(i(\gamma_a)_{\gamma\delta} W^{acd}(\gamma_{cd})_\beta^\alpha \psi_\alpha^{\underline{i}} + 6\epsilon_{\gamma\delta\beta\tau} X^{\tau k} q_k^{\underline{i}} \Bigg) \tag{125}$$

$$\nabla^{\alpha\beta}\psi_\beta^{\underline{i}} = \frac{1}{4}(\gamma^a)^{\alpha\beta} \Bigg(((\gamma_a)_{\beta\delta} W^{\delta\gamma})\nabla_{p\gamma} - 3i\,(\gamma_a)_{\beta\gamma} X_p^\gamma \Bigg) q^{p\underline{i}} - \frac{1}{8}\epsilon^{\alpha\beta\gamma\delta} \Bigg((\gamma_a)_{\gamma\delta} W^{acd}(\gamma_{cd})_\beta^\alpha \psi_\alpha^{\underline{i}} \Bigg) - \frac{36}{8} \Bigg(X^{\alpha k} q_k^{\underline{i}} \Bigg) \tag{126}$$

Now, note the identity

$$(\gamma^a)^{\alpha\beta}(\gamma_a)_{\gamma\delta} = 2\epsilon^{\alpha\beta\kappa\lambda}\epsilon_{\kappa\lambda\gamma\delta} = 2(\delta^\alpha_\delta\delta^\beta_\gamma - \delta^\alpha_\gamma\delta^\beta_\delta) \tag{127}$$

$$\implies \nabla^{\alpha\beta}\psi^{\underline{i}}_\beta$$
$$= \frac{1}{2}(\delta^\alpha_\delta\delta^\beta_\beta - \delta^\alpha_\beta\delta^\beta_\delta)W^{\delta\gamma}\nabla_{p\gamma}q^{p\underline{i}} - 6i(\delta^\alpha_\gamma\delta^\beta_\beta - \delta^\alpha_\beta\delta^\beta_\gamma)X^\gamma_p q^{p\underline{i}} - \frac{1}{8}\epsilon^{\alpha\beta\gamma\delta}\left((\gamma_a)_{\gamma\delta}W^{acd}(\gamma_{cd})^\alpha_\beta\psi^{\underline{i}}_\alpha\right) - \frac{36}{8}\left(X^{\alpha k}q^{\underline{i}}_k\right) \tag{128}$$

$$\implies \nabla^{\alpha\beta}\psi^{\underline{i}}_\beta = -\frac{1}{8}\epsilon^{\alpha\beta\gamma\delta}\left((\gamma_a)_{\gamma\delta}W^{acd}(\gamma_{cd})^\alpha_\beta\psi^{\underline{i}}_\alpha\right) - \frac{36}{8}\left(X^{\alpha k}q^{\underline{i}}_k\right) \tag{129}$$

Now, take the first term of (129),

$$-\frac{1}{8}\epsilon^{\alpha\beta\gamma\delta}(\gamma_a)_{\gamma\delta}W^{acd}(\gamma_{cd})^\alpha_\beta\psi^{\underline{i}}_\alpha \tag{130}$$

Note that

$$W^{acd} = -\frac{1}{3!}\epsilon^{acdefg}W_{efg} \tag{131}$$

and

$$W_{efg} = \frac{1}{8}(\gamma_{efg})_{\kappa\lambda}W^{\kappa\lambda} \tag{132}$$

Thus, the term (129) becomes

$$\frac{1}{8}\epsilon^{\alpha\beta\gamma\delta}(\gamma_a)_{\gamma\delta}\frac{1}{3!}\epsilon^{acdefg}\frac{1}{8}(\gamma_{efg})_{\kappa\lambda}W^{\kappa\lambda}(\gamma_{cd})^\alpha_\beta\psi^{\underline{i}}_\alpha = \frac{1}{8\cdot 48}\epsilon^{\alpha\beta\gamma\delta}(\gamma_a)_{\gamma\delta}\epsilon^{acdefg}(\gamma_{efg})_{\kappa\lambda}W^{\kappa\lambda}(\gamma_{cd})^\alpha_\beta\psi^{\underline{i}}_\alpha \tag{133}$$

Now, note that

$$\gamma_{def} = -\frac{1}{3!}\epsilon_{abcdef}\gamma^{abc} \tag{134}$$

$$\implies \epsilon^{abcdef}\gamma_{def} = -\frac{1}{3!}\epsilon^{abcdef}\epsilon_{abcdef}\gamma^{abc} = -\frac{6!}{3!}\gamma^{abc} \tag{135}$$

Thus,

$$\gamma^{abc} = -\frac{1}{5!}\epsilon^{abcdef}\gamma_{def} \tag{136}$$

Using this in (133),

$$\implies -\frac{5!}{8\cdot 48}\epsilon^{\alpha\beta\gamma\delta}(\gamma_a)_{\gamma\delta}(\gamma^{acd})_{\kappa\lambda}W^{\kappa\lambda}(\gamma_{cd})^\alpha_\beta\psi^{\underline{i}}_\alpha \tag{137}$$

Now, note that

$$(\gamma_a)_{\alpha\beta}(\gamma^{abc})_{\gamma\delta} = -2\epsilon_{\alpha\beta\gamma\epsilon}(\gamma^{bc})^\epsilon_\delta + 2(\gamma^{[b})_{\alpha\beta}(\gamma^{c]})_{\gamma\delta} \tag{138}$$

Thus, (137) becomes

$$\implies -\frac{5!}{8\cdot 48}\epsilon^{\alpha\beta\gamma\delta}\left[-2\epsilon_{\gamma\delta\kappa\epsilon}(\gamma^{cd})^\epsilon_\lambda + 2(\gamma^{[c})_{\gamma\delta}(\gamma^{d]})_{\kappa\lambda}\right]W^{\kappa\lambda}(\gamma_{cd})^\alpha_\beta\psi^{\underline{i}}_\alpha \tag{139}$$

Also, note

$$(\gamma^{cd})_\lambda^\epsilon(\gamma_{cd})_\beta^\alpha = -8\delta_\lambda^\alpha\delta_\beta^\epsilon + 2\delta_\lambda^\epsilon\delta_\beta^\alpha \tag{140}$$

And,

$$(\gamma^d)_{\kappa\lambda}(\gamma_{cd})_\beta^\alpha = -(\gamma^d)_{\kappa\lambda}(\gamma_{dc})_\beta^\alpha = -\eta_{ca}(\gamma_d)_{\kappa\lambda}(\gamma^{da})_\beta^\alpha = \eta_{ca}(-2\epsilon_{\kappa\lambda\beta\epsilon}(\tilde{\gamma}^a)^{\epsilon\alpha} - (\gamma^a)_{\kappa\lambda}{}^\delta{}_\beta^\alpha) \tag{141}$$

And, similarly,

$$(\gamma^c)_{\kappa\lambda}(\gamma_{cd})_\beta^\alpha = \eta_{da}(2\epsilon_{\kappa\lambda\beta\epsilon}(\tilde{\gamma}^a)^{\epsilon\alpha} + (\gamma^a)_{\kappa\lambda}\delta_\beta^\alpha) \tag{142}$$

Substituting this into (139)

$$\begin{aligned}\implies -\frac{5!}{8\cdot 48}\epsilon^{\alpha\beta\gamma\delta}\Bigg[&-2\epsilon_{\gamma\delta\kappa\epsilon}(-8\delta_\lambda^\alpha\delta_\beta^\epsilon + 2\delta_\lambda^\epsilon\delta_\beta^\alpha) + (\gamma^c)_{\gamma\delta}(-2\eta_{ca}\epsilon_{\kappa\lambda\beta\epsilon}(\tilde{\gamma}^a)^{\epsilon\alpha} - (\gamma^a)_{\kappa\lambda}\delta_\beta^\alpha)\\ &-(\gamma^d)_{\gamma\delta}(\eta_{da}(2\epsilon_{\kappa\lambda\beta\epsilon}(\tilde{\gamma}^a)^{\epsilon\alpha} - (\gamma^a)_{\kappa\lambda}\delta_\beta^\alpha))\Bigg]W^{\kappa\lambda}\psi_\alpha^{\underline{i}}\end{aligned} \tag{143}$$

There are also the relations

$$(\gamma^d)_{\gamma\delta}(\tilde{\gamma}_d)^{\epsilon\alpha} = 2(\delta_\gamma^\epsilon\delta_\delta^\alpha - \delta_\delta^\epsilon\delta_\gamma^\alpha) \tag{144}$$

And,

$$(\gamma^d)_{\gamma\delta}(\gamma_d)_{\kappa\lambda} = 2\epsilon_{\gamma\delta\kappa\lambda} \tag{145}$$

Using these two relations in (143)

$$\implies -\frac{5!}{8\cdot 48}\epsilon^{\alpha\beta\gamma\delta}\Bigg[-2\epsilon_{\gamma\delta\kappa\epsilon}(-8\delta_\lambda^\alpha\delta_\beta^\epsilon + 2\delta_\lambda^\epsilon\delta_\beta^\alpha) + [-8\epsilon_{\kappa\lambda\beta\epsilon}(\delta_\gamma^\epsilon\delta_\delta^\alpha - \delta_\delta^\epsilon\delta_\gamma^\alpha) - 4\epsilon_{\gamma\delta\kappa\lambda}\delta_\beta^\alpha]\Bigg]W^{\kappa\lambda}\psi_\alpha^{\underline{i}} \tag{146}$$

$$= -\frac{5!}{8\cdot 48}\Bigg[-2\epsilon_{\gamma\delta\kappa\epsilon}(-8\epsilon^{\alpha\beta\gamma\delta}\delta_\lambda^\alpha\delta_\beta^\epsilon + 2\epsilon^{\alpha\beta\gamma\delta}\delta_\lambda^\epsilon\delta_\beta^\alpha) + [-8\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\kappa\lambda\beta\epsilon}(\delta_\gamma^\epsilon\delta_\delta^\alpha - \delta_\delta^\epsilon\delta_\gamma^\alpha) - 4\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\gamma\delta\kappa\lambda}\delta_\beta^\alpha]\Bigg]W^{\kappa\lambda}\psi_\alpha^{\underline{i}} \tag{147}$$

$$= -\frac{5!}{8\cdot 48}\Bigg[-2\epsilon_{\gamma\delta\kappa\epsilon}(-8\epsilon^{\lambda\epsilon\gamma\delta} + 2\epsilon^{\beta\beta\gamma\delta}\delta_\lambda^\epsilon) + [-8\epsilon_{\kappa\lambda\beta\epsilon}(\epsilon^{\alpha\beta\epsilon\alpha} - \epsilon^{\gamma\beta\gamma\epsilon}) - 4\epsilon^{\beta\beta\gamma\delta}\epsilon_{\gamma\delta\kappa\lambda}]\Bigg]W^{\kappa\lambda}\psi_\alpha^{\underline{i}} \tag{148}$$

$$= -\frac{16\cdot 5!}{8\cdot 48}\epsilon_{\gamma\delta\kappa\epsilon}\epsilon^{\lambda\epsilon\gamma\delta}W^{\kappa\lambda}\psi_\alpha^{\underline{i}} \tag{149}$$

$$= -\frac{5!}{24}\epsilon_{\gamma\delta\kappa\epsilon}\epsilon^{\lambda\epsilon\gamma\delta}W^{\kappa\lambda}\psi_\alpha^{\underline{i}} \tag{150}$$

$$= -\frac{5!}{24}\epsilon_{\kappa\delta\gamma\epsilon}\epsilon^{\delta\lambda\gamma\epsilon}W^{\kappa\lambda}\psi_\alpha^{\underline{i}} \tag{151}$$

$$= -\frac{5!}{4}\delta_\kappa^\lambda W^{\kappa\lambda}\psi_\alpha^{\underline{i}} \tag{152}$$

$$= -30W^{\kappa\kappa}\psi_\alpha^{\underline{i}} \tag{153}$$

Returning to (129),

$$\implies \nabla^{\alpha\beta}\psi_\beta^{\underline{i}} = -30W^{\kappa\kappa}\psi_\alpha^{\underline{i}} - \frac{36}{8}X^{\alpha k}q_k^{\underline{i}} \tag{154}$$

5 Conclusion

In this report, we have determined field content of the on-shell 6D hypermultiplet. We have also determined most of the equations of motion for the multiplet fields. It remains to compute the d'Alembertian of q ($\Box q$). Once this has been completed, compensating fields can be used and constraints applied to reduce the theory to a Poincare supergravity theory. This work can then be corroborated with the work of Jessica Hutomo, Gregory Gold, Saurish Khandelwal, William Kitchin and Gabriele Tartaglino Mazzucchelli, that should lead to a publication sometime this year.

6 References

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7 Appendix

7.1 Notation and conventions

Then notations and conventions for this paper are taken from Butter [4].

The Lorentzian metric is $\eta_{ab} = \text{diag}(-1, 1, 1, 1, 1, 1)$, the Levi-Civita tensor ε_{abcdef} obeys $\varepsilon_{012345} = -\varepsilon^{012345} = 1$, and the Levi-Civita tensor with world indices is given by $\varepsilon^{mnpqrs} := \varepsilon^{abcdef} e_a{}^m e_b{}^n e_c{}^p e_d{}^q e_e{}^r e_f{}^s$.

The Pauli-type 4×4 matrices $(\gamma^a)_{\alpha\beta}$ and $(\tilde{\gamma}^a)^{\alpha\beta}$ are antisymmetric and related by

$$(\tilde{\gamma}^a)^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta} (\gamma^a)_{\gamma\delta}\,, \quad (\gamma^a)^* = \tilde{\gamma}_a, \tag{155}$$

where $\varepsilon^{\alpha\beta\gamma\delta}$ is the canonical antisymmetric symbol of $\mathfrak{su}^*(4)$. They obey

$$\begin{aligned} (\gamma^a)_{\alpha\beta} (\tilde{\gamma}^b)^{\beta\gamma} + (\gamma^b)_{\alpha\beta} (\tilde{\gamma}^a)^{\beta\gamma} &= -2\eta^{ab}\delta_\alpha^\gamma, \\ (\tilde{\gamma}^a)^{\alpha\beta} (\gamma^b)_{\beta\gamma} + (\tilde{\gamma}^b)^{\alpha\beta} (\gamma^a)_{\beta\gamma} &= -2\eta^{ab}\delta_\gamma^\alpha, \end{aligned} \tag{156}$$

The Grassmann coordinates θ_i^α and the parameters η_α^i of S-supersymmetry are both symplectic Majorana-Weyl.

We define the antisymmetric products of two or three Pauli-type matrices as

$$\begin{aligned} \gamma_{ab} &:= \gamma_{[a}\tilde{\gamma}_{b]} := \frac{1}{2}\left(\gamma_a\tilde{\gamma}_b - \gamma_b\tilde{\gamma}_a\right), \quad & \tilde{\gamma}_{ab} &:= \tilde{\gamma}_{[a}\gamma_{b]} = -\left(\gamma_{ab}\right)^T \\ \gamma_{abc} &:= \gamma_{[a}\tilde{\gamma}_b\gamma_{c]}, & \tilde{\gamma}_{abc} &:= \tilde{\gamma}_{[a}\gamma_b\tilde{\gamma}_{c]} \end{aligned} \tag{157}$$

Note that γ_{ab} and $\tilde{\gamma}_{ab}$ are traceless, whereas γ_{abc} and $\tilde{\gamma}_{abc}$ are symmetric. Further antisymmetric products obey

$$\begin{aligned} \gamma_{abc} &= -\frac{1}{3!}\varepsilon_{abcdef}\gamma^{def}, & \tilde{\gamma}_{abc} &= \frac{1}{3!}\varepsilon_{abcdef}\tilde{\gamma}^{def}, \\ \gamma_{abcd} &= \frac{1}{2}\varepsilon_{abcdef}\gamma^{ef}, & \tilde{\gamma}_{abcd} &= -\frac{1}{2}\varepsilon_{abcdef}\tilde{\gamma}^{ef}, \\ \gamma_{abcde} &= \varepsilon_{abcdef}\gamma^{f}, & \tilde{\gamma}_{abcde} &= -\varepsilon_{abcdef}\tilde{\gamma}^{f}, \\ \gamma_{abcdef} &= -\varepsilon_{abcdef}, & \tilde{\gamma}_{abcdef} &= \varepsilon_{abcdef}. \end{aligned} \tag{158}$$

Making use of the completeness relations

$$\begin{aligned} \left(\gamma^a\right)_{\alpha\beta}\left(\tilde{\gamma}_a\right)^{\gamma\delta} &= 4\delta^{\gamma}_{[\alpha}\delta^{\delta}_{\beta]}, \\ \left(\gamma^{ab}\right)_\alpha{}^\beta\left(\gamma_{ab}\right)_\gamma{}^\delta &= -8\delta_\alpha{}^\delta\delta_\gamma{}^\beta + 2\delta_\alpha{}^\beta\delta_\gamma{}^\delta, \\ \left(\gamma^{abc}\right)_{\alpha\beta}\left(\tilde{\gamma}_{abc}\right)^{\gamma\delta} &= 48\delta_{(\alpha}{}^\gamma\delta_{\beta)}{}^\delta, \\ \left(\gamma^{abc}\right)_{\alpha\beta}\left(\tilde{\gamma}_{abc}\right)_{\gamma\delta} &= \left(\gamma^{abc}\right)^{\alpha\beta}\left(\tilde{\gamma}_{abc}\right)^{\gamma\delta} = 0, \end{aligned} \tag{159}$$

Vectors V^a and antisymmetric matrices $V_{\alpha\beta} = -V_{\beta\alpha}$ are related by

$$V_{\alpha\beta} := \left(\gamma^a\right)_{\alpha\beta}V_a \Longleftrightarrow V_a = \frac{1}{4}\left(\tilde{\gamma}_a\right)^{\alpha\beta}V_{\alpha\beta}. \tag{160}$$

Antisymmetric rank-two tensors F_{ab} are related to traceless matrices $F_\alpha{}^\beta$ via

$$F_\alpha{}^\beta := -\frac{1}{4}\left(\gamma^{ab}\right)_\alpha{}^\beta F_{ab}, \quad F_\alpha{}^\alpha = 0 \Longleftrightarrow \quad F_{ab} = \frac{1}{2}\left(\gamma_{ab}\right)_\beta{}^\alpha F_\alpha{}^\beta = -F_{ba}. \tag{161}$$

Self-dual and anti-self-dual rank-three antisymmetric tensors $T^{(\pm)}_{abc}$,

$$\frac{1}{3!}\varepsilon^{abcdef}T^{(\pm)}_{def} = \pm T^{(\pm)abc}, \tag{162}$$

are related to symmetric matrices $T_{\alpha\beta}$ and $T^{\alpha\beta}$ via

$$\begin{aligned} T_{\alpha\beta} &:= \frac{1}{3!}\left(\gamma^{abc}\right)_{\alpha\beta}T_{abc} = T_{\beta\alpha} \Longleftrightarrow T^{(+)}_{abc} = \frac{1}{8}\left(\tilde{\gamma}_{abc}\right)^{\alpha\beta}T_{\alpha\beta} \\ T^{\alpha\beta} &:= \frac{1}{3!}\left(\tilde{\gamma}^{abc}\right)^{\alpha\beta}T_{abc} = T^{\beta\alpha} \Longleftrightarrow T^{(-)}_{abc} = \frac{1}{8}\left(\gamma_{abc}\right)_{\alpha\beta}T^{\alpha\beta} \end{aligned} \tag{163}$$

7.2 Conformal superspace identities

These 6D identities are taken from Butter [5].

The Lorentz generators act on the superspace covariant derivatives $\nabla_A = \left(\nabla_a, \nabla_\alpha^i\right)$ as

$$\begin{aligned} [M_{ab}, M_{cd}] &= 2\eta_{c[a}M_{b]d} - 2\eta_{d[a}M_{b]c} \\ [M_{ab}, \nabla_c] &= 2\eta_{c[a}\nabla_{b]} \\ \left[M_\alpha{}^\beta, \nabla_\gamma^k\right] &= -\delta_\gamma^\beta \nabla_\alpha^k + \frac{1}{4}\delta_\alpha^\beta \nabla_\gamma^k \end{aligned} \tag{164}$$

where $M_\alpha{}^\beta = -\frac{1}{4}\left(\gamma^{ab}\right)_\alpha{}^\beta M_{ab}$. The $\mathrm{SU}(2)_R$ and dilatation generators obey

$$\begin{aligned} &\left[J^{ij}, J^{kl}\right] = \varepsilon^{k(i}J^{j)l} + \varepsilon^{l(i}J^{j)k}, \quad \left[J^{ij}, \nabla_\alpha^k\right] = \varepsilon^{k(i}\nabla_\alpha^{j)}, \\ &[\mathbb{D}, \nabla_a] = \nabla_a, \quad \left[\mathbb{D}, \nabla_\alpha^i\right] = \frac{1}{2}\nabla_\alpha^i. \end{aligned} \tag{165}$$

The Lorentz and $\mathrm{SU}(2)_R$ generators act on the special conformal generators $K^A = (K^a, S_i^\alpha)$ aS

$$[M_{ab}, K^c] = 2\delta_{[a}^c K_{b]}, \quad \left[M_\alpha^\beta, S_k^\gamma\right] = \delta_\alpha^\gamma S_k^\beta - \frac{1}{4}\delta_\alpha^\beta S_k^\gamma, \quad \left[J^{ij}, S_k^\gamma\right] = \delta_k^{(i}S^{\gamma j)} \tag{166}$$

while the dilatation generator acts on K^A as

$$[\mathbb{D}, K^a] = -K^a, \quad [\mathbb{D}, S_i^\alpha] = -\frac{1}{2}S_i^\alpha \tag{167}$$

Among themselves, the generators K_A obey the only nontrivial anti-commutation relation

$$\left\{S_i^\alpha, S_j^\beta\right\} = -2\mathrm{i}\varepsilon_{ij}\left(\tilde{\gamma}_c\right)^{\alpha\beta} K^c \tag{168}$$

The algebra of K^A with ∇_A is given by

$$\begin{aligned} [K_a, \nabla_b] =& 2\eta_{ab}\mathbb{D} + 2M_{ab}, \\ \left[K^a, \nabla_\alpha^i\right] =& -\mathrm{i}\left(\gamma^a\right)_{\alpha\beta} S^{\beta i}, \\ \left\{S_i^\alpha, \nabla_\beta^j\right\} =& 2\delta_\beta^\alpha \delta_i^j \mathbb{D} - 4\delta_i^j M_\beta{}^\alpha + 8\delta_\beta^\alpha J_i{}^j, \\ [S_i^\alpha, \nabla_b] =& -\mathrm{i}\left(\tilde{\gamma}_b\right)^{\alpha\beta}\nabla_{\beta i} + \frac{1}{10}W_{bcd}\left(\tilde{\gamma}^{cd}\right)^\alpha{}_\gamma S_i^\gamma - \frac{1}{4}X_i^\alpha K_b \\ &+ \left[\frac{1}{4}\left(\tilde{\gamma}_{bc}\right)^\alpha{}_\beta X_i^\beta + \frac{1}{2}\left(\gamma_{bc}\right)_\beta^\gamma X_{\gamma i}{}^{\beta\alpha}\right] K^c. \end{aligned} \tag{169}$$

The anticommutator of two spinor derivatives, $\left\{\nabla_\alpha^i, \nabla_\beta^j\right\}$, has the following non-zero torsion and curvatures

$$\begin{aligned} T_{\alpha\beta}^{ijc} &= 2\mathrm{i}\varepsilon^{ij}\left(\gamma^c\right)_{\alpha\beta}, \\ R(M)_{\alpha\beta}^{ijcd} &= 4\mathrm{i}\varepsilon^{ij}\left(\gamma_a\right)_{\alpha\beta} W^{acd}, \\ R(S)_{\alpha\beta\gamma}^{ijk} &= -\frac{3}{2}\varepsilon^{ij}\varepsilon_{\alpha\beta\gamma\delta}X^{\delta k}, \\ R(K)_{\alpha\beta c}^{ij} &= \mathrm{i}\varepsilon^{ij}\left(\gamma^a\right)_{\alpha\beta}\left(\frac{1}{4}\eta_{ac}Y - \nabla^b W_{abc} + W_a{}^{ef}W_{cef}\right). \end{aligned} \tag{170}$$

The non-zero torsion and curvatures in the commutator $\left[\nabla_a, \nabla^j_\beta\right]$ are:

$$
\begin{aligned}
T^{j\gamma}_{a\beta k} =& -\frac{1}{2}\left(\gamma_a\right)_{\beta\delta} W^{\delta\gamma}\delta^j_k, \\
R(\mathbb{D})^j_{a\beta} =& -\frac{\mathrm{i}}{2}\left(\gamma_a\right)_{\beta\gamma} X^{\gamma j}, \\
R(M)^{jcd}_{a\beta} =& \mathrm{i}\delta^{[c}_a\left(\gamma^{d]}\right)_{\beta\gamma} X^{\gamma j} - \mathrm{i}\left(\gamma^{cd}_a\right)_{\gamma\delta} X^{j\gamma\delta}_\beta + 2\mathrm{i}\left(\gamma_a\right)_{\beta\gamma}\left(\gamma^{cd}\right)_\delta^{\ \rho} X^{j\gamma\delta}_\rho, \\
R(J)^{jkl}_{a\beta} =& 2\mathrm{i}\left(\gamma_a\right)_{\beta\gamma} X^{\gamma\left(k^{l)j}_\varepsilon\right.}, \\
R(S)^{jk}_{a\beta\gamma} =& -\frac{\mathrm{i}}{4}\left(\gamma_a\right)_{\beta\delta} Y^{\delta jk}_\gamma + \frac{3\mathrm{i}}{20}\left(\gamma_a\right)_{\gamma\delta} Y^{\delta jk}_\beta - \frac{\mathrm{i}}{8}\left(\gamma_a\right)_{\beta\delta}\nabla_{\gamma\rho}W^{\delta\rho jk}_\varepsilon \\
&+ \frac{\mathrm{i}}{40}\left(\gamma_a\right)_{\gamma\delta}\nabla_{\beta\rho}W^{\delta\rho}\varepsilon^{jk} - \frac{\mathrm{i}}{8}\left(\gamma_a\right)_{\delta\epsilon}\varepsilon_{\beta\rho\tau\gamma}W^{\delta\rho}W^{\epsilon\tau}\varepsilon^{jk}, \\
R(K)^j_{a\beta^j} =& \frac{\mathrm{i}}{4}\left(\gamma_c\right)_{\beta\gamma}\nabla_a X^{\gamma j} - \frac{\mathrm{i}}{4}\left(\gamma_{acd}\right)_{\gamma\delta}\nabla^d X^{j\gamma\delta}_\beta + \frac{\mathrm{i}}{3}\left(\gamma_a\right)_{\beta\delta}\left(\gamma_{cd}\right)_\rho^{\ \gamma}\nabla^d X^{j\delta\rho}_\gamma \\
&- \frac{\mathrm{i}}{8}\left(\gamma_a\right)_{\beta\gamma}\left(\gamma_c\right)_{\delta\rho} W^{\gamma\delta}X^{\rho j} + \frac{5\mathrm{i}}{12}\left(\gamma_a\right)_{\beta\rho}\left(\gamma_c\right)_{\gamma\epsilon} W^{\gamma\delta}X^{j\rho\epsilon}_\delta \\
&+ \frac{\mathrm{i}}{4}\left(\gamma_a\right)_{\gamma\rho}\left(\gamma_c\right)_{\beta\epsilon} W^{\gamma\delta}X^{j\rho\epsilon}_\delta - \frac{\mathrm{i}}{2}\left(\gamma_a\right)_{\gamma\rho}\left(\gamma_c\right)_{\delta\epsilon} W^{\gamma\delta}X^{j\rho\epsilon}_\beta.
\end{aligned}
\tag{171}
$$

The commutator of two vector derivatives, $[\nabla_a, \nabla_b]$, has the following non-vanishing torsion and curvatures:

$$
\begin{aligned}
T^\gamma_{ab_k} &= \left(\gamma_{ab}\right)\beta^\alpha X^{\beta\gamma}_{\alpha k} \\
R(M)^{cd}_{ab} &= Y^{cd}_{ab} = \frac{1}{4}\left(\gamma_{ab}\right)^\alpha_\gamma\left(\gamma^{cd}\right)^\beta_\delta Y_{\alpha\beta}\gamma\delta \\
R(J)^{kl}_{ab} &= \frac{1}{2}\left(\gamma_{ab}\right)\delta^\gamma Y^{\delta kl}_\gamma = Y^{kl}_{ab}
\end{aligned}
\tag{172}
$$

$$
\begin{aligned}
R(S)^k_{ab\gamma} =& -\frac{\mathrm{i}}{3}\left(\gamma_{ab}\right)^\alpha_\delta\nabla_{\gamma\beta}X^{k\beta\delta}_\alpha - \frac{\mathrm{i}}{6}\left(\gamma_{abc}\right)_{\alpha\beta}\nabla^c X^{k\alpha\beta}_\gamma - \frac{i}{6}\varepsilon_{\gamma\beta\epsilon\rho}\left(\gamma_{ab}\right)^\rho_\delta W^{\alpha\beta}X^{k\delta\epsilon}_\alpha, \\
R(K)_{abc} =& \frac{1}{4}\nabla^d Y_{abcd} + \frac{\mathrm{i}}{3}X^{k\beta\gamma}_\alpha X^{\alpha\delta}_{\beta k}\left(\gamma_{abc}\right)_{\gamma\delta} + \mathrm{i}\left(\gamma_{ab}\right)^\alpha_\epsilon\left(\gamma_c\right)_{\gamma\delta} X^{k\beta\gamma}_\alpha X_{\beta k}\delta\epsilon \\
&+ \frac{\mathrm{i}}{4}X^{\alpha k}X_{\beta k}\gamma\delta\left(\gamma_{ab}\right)^\beta_\gamma\left(\gamma_c\right)_{\alpha\delta}.
\end{aligned}
\tag{173}
$$

7.3 Gamma matrix identities

These gamma matrix relations are taken from Tartaglino-Mazzucchelli [6].

Some useful identities are

$$
\begin{aligned}
\gamma^{ab}\gamma_{cd} &= \frac{1}{2}\epsilon^{ab}_{cdef}\gamma^{ef} + 4\delta^{[a}_{[c}\gamma^{b]}_{d]} - 2\delta^{[a}_{[c}\delta^{b]}_{d]} \\
\left(\gamma^{ab}\right)_\alpha^{\ \gamma}\left(\gamma_{cd}\right)_\gamma^{\ \beta} &= \frac{1}{2}\varepsilon^{ab}{}_{cdef}\left(\gamma^{ef}\right)_\alpha^{\ \beta} + 4\delta^{[a}_{[c}\left(\gamma^{b]}_{d]}\right)_\alpha^{\ \beta} - 2\delta^{[a}_{[c}\delta^{b]}_{d]}\delta^\beta_\alpha
\end{aligned}
\tag{174}
$$

Also,

$$
\begin{aligned}
\left(\gamma^{ab}\right)^\gamma_\alpha\left(\gamma_{bd}\right)^\beta_\gamma &= 4\delta^{[a}_{[b}\left(\gamma^{b]}_{d]}\right)_\alpha^{\ \beta} - 2\delta^{[a}_b\delta^{b]}_d\delta^\beta_\alpha \\
\left(\gamma^{ab}\right)_\alpha^{\ \gamma}\left(\gamma_{bd}\right)_\gamma^{\ \beta} &= \delta^a_b\left(\gamma^b_d\right)_\alpha^{\ \beta} - \delta^b_b\left(\gamma^a{}_d\right)_\alpha^{\ \beta} + \delta^b_d\left(\gamma\gamma_b{}^a\right)_\alpha^{\ \beta} - \delta^a_b\delta^b_d\delta^\beta_\alpha + \delta^b_b\delta^a_d\delta^\beta_\alpha \\
\left(\gamma^{ac}\right)_\alpha^{\ \gamma}\left(\gamma_{cb}\right)_\gamma^{\ \beta} &= -4\left(\gamma^a{}_b\right)_\alpha^{\ \beta} + 5\delta^a_b\delta^\beta_\alpha
\end{aligned}
\tag{175}
$$

next

$$\begin{aligned} \left(\tilde{\gamma}^{ab}\right)^{\alpha}{}_{\gamma}\left(\tilde{\gamma}_{cd}\right)^{\gamma}{}_{\beta} = \left(\gamma_{cd}\right)^{\gamma}_{\beta}\left(\gamma^{ab}\right)^{\alpha}_{\gamma} = \frac{1}{2}\varepsilon^{ab}{}_{cdef}\left(\gamma^{ef}\right)_{\beta}{}^{\alpha} + 4\delta^{[a}_{[c}\left(\gamma_{d]}{}^{b]}\right)_{\beta}{}^{\alpha} - 2\delta^{[a}_{[c}\delta^{b]}_{d]}\delta^{\alpha}_{\beta} \\ \left(\tilde{\gamma}^{ab}\right)^{\alpha}{}_{\gamma}\left(\tilde{\gamma}_{cd}\right)^{\gamma}{}_{\beta} = -\frac{1}{2}\varepsilon^{ab}{}_{cdef}\left(\tilde{\gamma}^{ef}\right)^{\alpha}{}_{\beta} + 4\delta^{[a}_{[c}\left(\tilde{\gamma}^{b]}_{d]}\right)^{\alpha}{}_{\beta} - 2\delta^{[a}_{[c}\delta^{b]}_{d]}\delta^{\alpha}_{\beta} \\ \tilde{\gamma}^{ab}\tilde{\gamma}_{cd} = -\frac{1}{2}\varepsilon^{ab}{}_{cdef}\tilde{\gamma}^{ef} + 4\delta^{[a}_{[c}\tilde{\gamma}^{b]}_{d]} - 2\delta^{[a}_{[c}\delta^{b]}_{d]} \end{aligned} \tag{176}$$

There is a completeness relation

$$\frac{1}{2}\left(\gamma^{m}\right)_{\alpha\beta}\left(\gamma_{m}\right)_{\gamma\delta} = \varepsilon_{\alpha\beta\gamma\delta} \tag{177}$$

Contraction with $\varepsilon^{\gamma'\delta'\gamma\delta}$ implies the completeness relation

$$\frac{1}{2}\left(\gamma^{m}\right)_{\alpha\beta}\left(\tilde{\gamma}_{m}\right)^{\gamma\delta} = \delta^{\gamma}_{\alpha}\delta^{\delta}_{\beta} - \delta^{\gamma}_{\beta}\delta^{\delta}_{\alpha} \tag{178}$$

and that

$$\frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}\left(\gamma_{m}\right)_{\gamma\delta} = \left(\tilde{\gamma}_{m}\right)^{\alpha\beta} \Rightarrow \left(\gamma_{m}\right)_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}\left(\tilde{\gamma}_{m}\right)^{\gamma\delta} \tag{179}$$

This gives

$$\frac{1}{4}\left(\tilde{\gamma}^{mn}\right)^{\alpha}{}_{\beta}\left(\gamma_{mn}\right)_{\gamma}{}^{\delta} = -\frac{1}{2}\delta^{\alpha}_{\beta}\delta^{\delta}_{\gamma} + 2\delta^{\delta}_{\beta}\delta^{\alpha}_{\gamma} \tag{180}$$

and

$$\left(\tilde{\gamma}^{abc}\right)^{\alpha\beta}\left(\gamma_{abc}\right)_{\gamma\delta} = 24\left(\delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta} + \delta^{\alpha}_{\delta}\delta^{\beta}_{\gamma}\right) \tag{181}$$

and also

$$\left(\gamma^{abc}\right)_{\alpha\beta}\left(\gamma_{abc}\right)_{\gamma\delta} = 0 \text{ and } \left(\tilde{\gamma}^{abc}\right)^{\alpha\beta}\left(\tilde{\gamma}_{abc}\right)^{\gamma\delta} = 0 \tag{182}$$

Note that in general

$$\begin{aligned} \varepsilon^{abcdef}\varepsilon_{a'b'c'd'e'f'} &= -6!\,\delta^{[a}_{[a'}\,\delta^{b}_{b'}\delta^{c}_{c'}\delta^{d}_{d'}\delta^{e}_{e'},\delta^{f]}_{f']} = -6!\,\delta^{a}_{[a'}\,\delta^{b}_{b'}\delta^{c}_{c'}\delta^{d}_{d'}\delta^{e}_{e'}\delta^{f}_{f']} = -6!\,\delta^{[a}_{a'}\delta^{b}_{b'}\delta^{c}_{c'}\delta^{d}_{d'}\delta^{e}_{e'}\delta^{f]}_{f'} \\ \varepsilon^{abcdem}\varepsilon_{a'b'c'd'e'm} &= -5!\,\delta^{[a}_{[a'}\,\delta^{b}_{b'}\delta^{c}_{c'}\delta^{d}_{d'}\delta^{e]}_{e']} = -5!\,\delta^{a}_{[a'}\,\delta^{b}_{b'}\delta^{c}_{c'}\delta^{d}_{d'}\delta^{e}_{e']} = -5!\,\delta^{[a}_{a'}\delta^{b}_{b'}\delta^{c}_{c'}\delta^{d}_{d'}\delta^{e]}_{e'} \\ \varepsilon^{abcdmn}\varepsilon_{a'b'c'd'mn} &= -2(4!)\delta^{[a}_{[a'}\,\delta^{b}_{b'}\delta^{c}_{c'}\delta^{d]}_{d']} = -2(4!)\delta^{a}_{[a'}\,\delta^{b}_{b'}\delta^{c}_{c'}\delta^{d}_{d']} = -2(4!)\delta^{a}_{a'}\delta^{b}_{b'}\delta^{c}_{c'}\delta^{d]}_{d'} \\ \varepsilon^{abcmnp}\varepsilon_{a'b'c'mnp} &= -(3!)^{2}\delta^{[a}_{[a'}\,\delta^{b}_{b'}\delta^{c]}_{c']} = -(3!)^{2}\delta^{a}_{[a'}\,\delta^{b}_{b'}\delta^{c}_{c']} = -(3!)^{2}\delta^{[a}_{a'}\delta^{b}_{b'}\delta^{c]}_{c'} \\ \varepsilon^{abmnpq}\varepsilon_{a'b'mnpq} &= -2(4!)\delta^{a}_{[a'},\delta^{b]}_{b']} = -2(4!)\delta^{a}_{[a'}\,\delta^{b}_{b']} = -2(4!)\delta^{a}_{a'}\delta^{b]}_{b'} \\ \varepsilon^{mnpqrt}\varepsilon_{mnpqrt} &= -6! \end{aligned} \tag{183}$$

Other useful relations are

$$
\begin{gathered}
\left(\gamma_a\right)_{\alpha\beta}\left(\gamma^{ab}\right)_{\gamma}{}^{\delta}=2\varepsilon_{\alpha\beta\gamma\epsilon}\left(\tilde{\gamma}^b\right)^{\epsilon\delta}+\left(\gamma^b\right)_{\alpha\beta}\delta_\gamma^\delta \\
\left(\gamma_a\right)_{\alpha\beta}\left(\gamma^{abc}\right)_{\gamma\delta}=-2\varepsilon_{\alpha\beta\gamma\epsilon}\left(\gamma^{bc}\right)_{\delta}{}^{\epsilon}+2\left(\gamma^{[b}\right)_{\alpha\beta}\left(\gamma^{c]}\right)_{\gamma\delta} \\
\left(\gamma_{abc}\right)_{\alpha\beta}\left(\gamma^{bc}\right)_{\gamma}{}^{\delta}=-8\left(\gamma_a\right)_{\gamma(\alpha}\delta_{\beta)}^{\delta} \\
\left(\tilde{\gamma}_a\right)^{\alpha\beta}\left(\gamma^{ab}\right)_{\gamma}{}^{\delta}=\frac{1}{2}\varepsilon^{\alpha\beta\alpha'\beta'}\left(2\varepsilon_{\alpha'\beta'\gamma\epsilon}\left(\tilde{\gamma}^b\right)^{\epsilon\delta}+\left(\gamma^b\right)_{\alpha'\beta'}\delta_\gamma^\delta\right)=4\delta_\gamma^{[\alpha}\left(\tilde{\gamma}^b\right)^{\beta]\delta}+\left(\tilde{\gamma}^b\right)^{\alpha\beta}\delta_\gamma^\delta \\
\left(\tilde{\gamma}_a\right)^{\alpha\beta}\left(\gamma^{abc}\right)_{\gamma\delta}=\frac{1}{2}\varepsilon^{\alpha\beta\alpha'\beta'}\left(-2\varepsilon_{\alpha'\beta'\gamma\epsilon}\left(\gamma^{bc}\right)_{\delta}^{\epsilon}+2\left(\gamma^{[b}\right)_{\alpha'\beta'}\left(\gamma^{c]}\right)_{\gamma\delta}\right)=4\delta_\gamma^{[\alpha}\left(\tilde{\gamma}^{bc}\right)^{\beta]}\delta+2\left(\tilde{\gamma}^{[b}\right)^{\alpha\beta}\left(\gamma^{c]}\right)_{\gamma\delta}
\end{gathered}
\tag{184}
$$

and also

$$
\begin{gathered}
\frac{1}{2}\left(\gamma_a\right)_{\alpha\beta}\left(\tilde{\gamma}^a\right)^{\gamma\rho}\left(\gamma^{bc}\right)_{\rho}{}^{\delta}=\delta_\alpha^\gamma\delta_\beta^\rho\left(\gamma^{bc}\right)_{\rho}{}^{\delta}-\delta_\beta^\gamma\delta_\alpha^\rho\left(\gamma^{bc}\right)_{\rho}{}^{\delta}=\delta_\alpha^\gamma\left(\gamma^{bc}\right)_{\beta}{}^{\delta}-\delta_\beta^\gamma\left(\gamma^{bc}\right)_{\alpha}{}^{\delta} \\
\left(\gamma_a\right)_{\alpha\beta}\left(\tilde{\gamma}^{abc}\right)^{\gamma\delta}=4\delta_{[\alpha}^{(\gamma}\left(\gamma^{bc}\right)_{\beta]}{}^{\delta)}
\end{gathered}
\tag{185}
$$

and also

$$
\left(\gamma^{ab}\right)_{\alpha}{}^{\beta}\left(\gamma_{ab}\right)_{\alpha'}{}^{\beta'}=2\delta_\alpha^\beta\delta_{\alpha'}^{\beta'}-8\delta_\alpha^{\beta'}\delta_{\alpha'}^{\beta}
\tag{186}
$$