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Solutions to the Ricci Flow on 4-Dimensional Principal Bundles

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Abstract

In many solutions to Ricci flow the blowdown limit features bounded curvature collapse and acquire a nilpotent symmetry group [Cheeger et al.(1992)Cheeger, Fukaya, and Gromov, Huang and Wang(2022), Lott(2010), Lott(2007)]. Investigation of Ricci flow on a manifold with such symmetry group, in particular a 4 dimensional principal bundle with 3 dimensional symmetry group, found type III immortal solutions for any invariant initial metric [Gindi and Streets(2021)]. Furthermore using a monotone energy specifically adapted to the principal bundle structure they could obtain clear conditions on the possible blowdown solutions. In this work we specialised to the case of the non-abelian simply connected symmetry group, the 3 dimensional Heisenberg group, to determine solutions satisfying these conditions. On the corresponding vector bundle it was found that in a parallel frame the metric must be an exponential of a Lie algebra derivation. However, global considerations to do with the holonomy of the vector bundle meant that such solutions are not possible. Thus in the case of the 3 dimensional Heisenberg symmetry group the blowdown limits must all collapse the base to a point, with no solutions existing over S^1.

Introduction

To understand the topology of manifolds it can often be helpful to give them additional structure, with the topology constraining the possible structures. One such structure is a Riemannian metric g. In the case of 2 dimensions the Gauss Bonnet theorem gives a good illustration of this. Here the integrated Gaussian curvature is proportional to the Euler characteristic, a topological invariant. However, even in this case it is much easier to obtain topological information from the metric if it has some distinguishing features. Having constant Gaussian curvature everywhere being one of these.

This presents the challenge of obtaining such distinguished metrics (a question that has been posed by Hopf, Thom and Yau). One such method that has found much success is the Ricci flow. It will evolve the metric on a 2 dimensional manifold to one that is conformally equivalent to one of constant curvature. In 3 dimensions it has also proven fruitful in allowing the Poincaré conjecture, one of the Millenium problems regarding the topological data characterising a 3-sphere, to be proven by Perelman. He in fact proved Thurston's geometrization conjecture, a further generalisation of the Poincaré conjecture, which seeks to classify the topology of 3 dimensional surfaces using geometrical data, has also yielded to Ricci flow.

We are interested in what distinguished metrics Ricci flow can give on 4 dimensional manifolds, but restrict to ones with some amount of symmetry to be amenable for study. These manifolds arise when considering group actions on manifolds and are called principal bundles. In particular due to many global Ricci flows acquiring nilpotent symmetry groups in the limit we will focus on nilpotent group actions. Such Ricci flows were studied by Gindi and Streets in [Gindi and Streets(2021)] for such a principal bundle with S^1 base. Their research obtained conditions on the possible Ricci flow limits on this space assuming that they exist and do not collapse the base to a point.

Our research focused on finding some of these Ricci flow limits for the non-abelian case assuming a simply-connected group. We took an initially orthonormal parallel frame as coordinates on the associated adjoint bundle with one of the vector fields in the centre. One of the conditions was equivalent to the determinant of the metric being constant along the frame, while another meant the central vector field stayed orthogonal. Together this reduced the independent parameters describing the metric to only two. The other conditions gave a linear second order ODE for these allowing an explicit local solution.

However, to obtain a global solution required consideration of how the frame joins up on S^1. The holonomy on the adjoint bundle along with the form of the local solution implied a global frame. This meant the metric at the initial point and after one wind must be the same but this was not possible without contradicting another condition. So our conclusion was that for the simply connected, non-abelian case that the limits must all collapse the base to a point.

Statement of Authorship

The workload was divided as follows:

- Jaco van Tonder produced the mathematical results, reported and interpreted the results, and wrote this report.
- Ramiro Lafuente helped with understanding the paper this work was based on, supervised the work, assisted with calculations and understanding the theory, and proofread this report.
- Timothy Buttsworth had some helpful comments, discussions on the work and proofread this report.

Remark on Prerequisites

In this research report we will assume a familiarity with Riemannian geometry, differential forms, vector bundles and fibre bundles. We understand that many working in these fields are not familiar with the concepts of principal bundles and potentially Ricci flow so these are reviewed in the appendix.

1 Summary of [Gindi and Streets(2021)]

1.1 Notation of [Gindi and Streets(2021)]

There is some slight difference in the notation used in the paper to the literature. The equivalence classes of $P \times \mathfrak{g}$ or points of $\mathfrak{G} := P \times_{\mathrm{Ad}} \mathfrak{g}$ are denoted by $\{p, x\}$ instead of $[p, x]$. The trace with respect to different metrics is denoted with dots and is defined in the following way. Let G be a fibre metric of $\mathfrak{G}$, which has base $\mathcal{M}$, and D be a connection on $\mathfrak{G}$, then

$$D_*G(\cdot, \cdot) := \mathrm{Tr}_G^{(1)} DG(*, \cdot_1, \cdot_1) = G^{ij} D_\alpha G_{ij}\, dx^\alpha \in \Gamma(T^*\mathcal{M})$$

with a dot denoting a trace over that component with respect to the metric on it and an asterisk denoting no trace. Sometimes the trace is explicitly left in if there is potential for confusion.

1.2 The Setup for [Gindi and Streets(2021)]

Before giving the main results of [**Gindi and Streets(2021)**] we need to first explain the setup. The manifold on which the flow occurs is a 4 dimensional principal $\mathcal{G}$-bundle $\pi : P \to \mathcal{M}$ where $\mathcal{M}$ is one dimensional and compact (so diffeomorphic to S^1 if connected) and $\mathcal{G}$ is a 3 dimensional nilpotent Lie group. $\mathfrak{g}$, the Lie algebra of $\mathcal{G}$, is the Heisenberg algebra $\mathfrak{h}_3(\mathbb{R})$ if $\mathcal{G}$ is non-abelian and $\mathbb{R}^3$ if $\mathcal{G}$ is abelian. We have a right invariant metric $\bar{g}$ on P which will be the initial value for Ricci flow on P.

1.2.1 Correspondence between $\mathcal{E}$ and TP

Introduction Due to the metric $\bar{g}$ on P being (right) invariant we can view it equivalently as a fibre metric on $\mathcal{E} = \mathfrak{G} \oplus TM$ where $\mathfrak{G}$ is the adjoint bundle and TM the tangent bundle to the base. To do so we use the connection $\bar{A}$ associated with $\bar{g}$ ($\ker \bar{A} = \mathcal{H} = \mathcal{V}^{\perp}$).

Informally we locally take a lift $\tilde{\gamma}$ of a (injective) curve on a neighbourhood U of the 1-dimensional base M and look at the restriction of $\bar{g}$ to it. Since $\bar{g}$ is invariant we can obtain it at any point of $\pi^{-1}(U)$ simply from the restriction to $\tilde{\gamma}$. Also due to the decomposition of the tangent spaces at a point $p \in \tilde{\gamma}$ into the vertical space $\mathcal{V}_p \cong \mathfrak{g}$ and horizontal space $\mathcal{H}_p = \mathcal{V}^{\perp} \cong T_{\pi(p)}M$ we see the restriction of $\bar{g}$ to each of these as a fibre metric g on TM and fibre metric G on $\mathfrak{G}$.

Formally we observe that there is a bundle isomorphism $\delta : TP \to \pi^*\mathcal{E}$, which at each point $p \in P$ restricts to an isomorphism $\delta_p : T_pP \to (\pi^*\mathcal{E})_p$ $(= \mathcal{E}_{\pi(p)} = \mathfrak{G}_{\pi(p)} \oplus T_{\pi(p)}M)$ given by

$$\begin{aligned}\delta_p(Z_p) &= (p, \{p, \bar{A}_p(Z_p)\} + \pi_{*,p}(Z_p)) \\ &= (p, \{p, \bar{A}_p(vZ_p)\} + \pi_{*,p}(hZ_p))\end{aligned}$$

for all $Z_p \in T_pP$ and $p \in P$ (note $\{p, \bar{A}_p(Z_p)\} \in (\pi^*\mathfrak{G})_p \subset \mathfrak{G}_{\pi(p)}$ and $\pi_{*,p}(Z_p) \in (\pi^*TM)_p \subset T_{\pi(p)}M$). We obtain an invariant vector field $\tilde{e}$ on P from a section e of $\mathcal{E}$ by

$$\tilde{e} := \delta^{-1}(\pi^* e)$$

giving a bijection $\tilde{\delta} := \delta^{-1} \circ \pi^*$ with inverse $\tilde{\delta}^{-1} = \mathrm{proj}_{\mathcal{E}} \circ \delta$ between sections of $\mathcal{E}$ and invariant vector fields on P (recall that $\pi^* e(p) = e(\pi(p))$).

Correspondence for fibre elements In terms of the fibre elements we then see that $\eta_i \in \mathfrak{G}_{\pi(p)}$ corresponds to $\tilde{\eta_i} \in \mathcal{V}_p$ by

$$\begin{aligned}\eta_i &= \tilde{\delta}^{-1}(\tilde{\eta_i}) \\ &= \{p, \bar{A}_p(\tilde{\eta_i})\}\end{aligned}$$

and $v \in TM_{\pi(p)}$ corresponds to $\tilde{v} \in \mathcal{H}_p$ by

$$\begin{aligned}v &= \tilde{\delta}^{-1}(\tilde{v}) \\ &= \pi_{*,p}(\tilde{v}).\end{aligned}$$

Explicit Correspondence for Tensors Let $\bar{\omega} \in \Omega^k_{\text{Ad}}(P, \mathfrak{g}) \subset \Gamma(\Lambda^k T^* P \otimes \mathfrak{g})$ be the connection form on $\pi : P \to M$ then the corresponding $\mathfrak{G}$-valued form $\omega \in \Omega^k(\mathcal{M}, \mathfrak{G}) = \Gamma(\Lambda^k T^* \mathcal{M} \otimes \mathfrak{G})$ is

$$\omega(\pi_{*,p} Z^{(1)}_p, \dots, \pi_{*,p} Z^{(k)}_p) := \{p, \bar{\omega}(Z^{(1)}_p, \dots, Z^{(k)}_p)\}.$$

This applies to the curvature $\bar{F}$ (which is tensorial) but not to the connection $\bar{A}$ (only pseudo-tensorial). For the metric $\bar{g}$ we obtain the fibrewise metric $g_{\mathcal{E}}$ defined as

$$\begin{aligned} g_{\mathcal{E}}(\tilde{\delta}^{-1} Z^{(1)}_p, \tilde{\delta}^{-1} Z^{(2)}_p) &:= \bar{g}(Z^{(1)}_p, Z^{(2)}_p) \\ &= \bar{g}(hZ^{(1)}_p, hZ^{(2)}_p) + \bar{g}(vZ^{(1)}_p, vZ^{(2)}_p) \\ &= g(\tilde{\delta}^{-1} hZ^{(1)}_p, \tilde{\delta}^{-1} hZ^{(2)}_p) + G(\tilde{\delta}^{-1} vZ^{(1)}_p, \tilde{\delta}^{-1} vZ^{(2)}_p) \end{aligned}$$

where

$$g_{\pi(p)}(v^{(1)}, v^{(2)}) = \bar{g}_p(\delta_p^{-1} \pi^* v^{(1)}, \delta_p^{-1} \pi^* v^{(2)}), \qquad G_{\pi(p)}(\eta^{(1)}, \eta^{(2)}) = \bar{g}_p(\delta_p^{-1} \pi^* \eta^{(1)}, \delta_p^{-1} \pi^* \eta^{(2)})$$

for $v^{(i)} \in T_{\pi(p)}\mathcal{M}$, $\eta^{(i)} \in \mathfrak{G}_{\pi(p)}$ (well defined as δ and $\bar{g}$ are $\mathcal{G}$ invariant, i.e. $R^*_g \delta = \delta$ and $R^*_g \bar{g} = \bar{g}$ for all $g \in G$, and as δ_p restricts to isomorphisms between $\mathcal{V}_p$ ($\mathcal{H}_p$) and $\mathfrak{G}_{\pi(p)}$ ($T_{\pi(p)}\mathcal{M}$)). This gives the decomposition $g_{\mathcal{E}} = g \oplus G$.

Equivalently

$$\begin{aligned} \bar{g}(\widetilde{Z^{(1)}_p}, \widetilde{Z^{(2)}_p}) &= g_{\mathcal{E}}(Z^{(1)}_p, Z^{(2)}_p) \\ &= g_{\mathcal{E}}(\{p, \bar{A}_p(\widetilde{Z^{(1)}_p})\} + \pi_{*,p}(\widetilde{Z^{(1)}_p}), \{p, \bar{A}_p(\widetilde{Z^{(2)}_p})\} + \pi_{*,p}(\widetilde{Z^{(2)}_p})) \\ &= g(\pi_{*,p}(\widetilde{Z^{(1)}_p}), \pi_{*,p}(\widetilde{Z^{(2)}_p})) + G(\{p, \bar{A}_p(\widetilde{Z^{(1)}_p})\}, \{p, \bar{A}_p(\widetilde{Z^{(2)}_p})\}). \end{aligned}$$

Remark. It is important to note that the fundamental vector fields are **not** *invariant* vertical vector fields on P since, in general, for all $g \in \mathcal{G}$

$$(R_g)_* \underline{A} = \underline{(\text{Ad}_{g^{-1}})A} \neq \underline{A}.$$

1.2.2 The Connection D on $\mathfrak{G}$

There is a natural connection on $\mathfrak{G}$ which corresponds to the Lie bracket on P. To define it we must first define the bracket $[,] : \Gamma(\mathcal{E}) \times \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ on $\mathcal{E}$ which evaluated on $e_1, e_2 \in \Gamma(\mathcal{E})$ is the unique section $[e_1, e_2]$ such that

$$\delta([\tilde{e}_1, \tilde{e}_2]_{Lie}) = \pi^*[e_1, e_2],$$

where $[,]_{Lie}$ is the Lie bracket on TP. This is well defined as the LHS is an invariant vector field. Note that this depends on the metric through δ. If we define what is called the **anchor** $\tau : \mathcal{E} = \mathfrak{G} \oplus TM \to TM$ to be the natural projection map

$$\eta + v \mapsto v$$

then Gindi and Streets show that $[,]$ on $\mathcal{E}$ satisfies

1. $\tau([e_1, e_2]) = [\tau(e_1), \tau(e_2)]_{Lie}$,
2. $[ge_1, fe_2] = fg[e_1, e_2] + g[\tau(e_1)](f)e_2 - f[\tau(e_2)](g)e_1$ (so $[v, f\eta] = f[v, \eta] + [\tau(v)](f)\eta$ where $v \in \mathfrak{X}(\mathcal{M})$ and $\eta \in \Gamma(\mathfrak{G})$),
3. $[*^1, [*^2, *^3]] + \text{cyclic}(1,2,3) = 0$,

for $e_1, e_2 \in \Gamma(\mathcal{E})$ and $f \in C^\infty(\mathcal{M})$. So we obtain a well defined **connection D on $\mathfrak{G}$** defined as

$$D_v\eta := [v, \eta]$$

for $v \in \mathfrak{X}(\mathcal{M}) \subset \Gamma(\mathcal{E})$ and $\eta \in \Gamma(\mathfrak{G}) \subset \Gamma(\mathcal{E})$.

D and $[,]$ satisfy the following useful properties, as was shown in [Gindi and Streets(2021)]. Let $s : U \to P$ be a local section of P, $x, y \in \mathfrak{g}$ and $v, w \in \Gamma(T\mathcal{M})$. Then

1. $[,]|_{\mathfrak{G}_m}$ is a Lie bracket on $\mathfrak{G}_m$ for $m \in \mathcal{M}$.
2. $[\{s, x\}, \{s, y\}] = -\{s, [x, y]\}$.
3. $D_v\{s, x\} = \{s, [(s^*\bar{A})v, x]\}$.
4. $[v, w] = [v, w]_{Lie}$ (note here that $[v, w]_{Lie} \in \Gamma(T\mathcal{M})$ and $[v, w] \in \Gamma(\mathfrak{G} \oplus T\mathcal{M})$).

1.2.3 $DG(\cdot, \cdot) = \mathbf{tr}_G DG$

Another important identity used in deriving the main results of their paper is

$$DG(\cdot, \cdot) = d(\ln h) \tag{1}$$

for some function h on M. Here d is an exterior derivative with $d^2 = 0$ defined in their paper.

1.2.4 The adjoint centre bundle $Z(\mathfrak{G})$

As a set **the adjoint centre bundle** is $Z(\mathfrak{G}) = \bigcup_{m\in M} Z(\mathfrak{G}|_m)$ where $Z(\mathfrak{G}|_m)$ is the centre of the Lie algebra $(\mathfrak{G}|_m, [,])$. It can be given differential structure making it a vector bundle over M. As shown in [Gindi and Streets(2021)], D also restricts to a connection on $Z(\mathfrak{G})$. It has the following useful property shown in [Gindi and Streets(2021)]:

1. If $x \in Z(\mathfrak{g})$ and s is a local section of P then $\{s, x\}$ is a local section of $Z(\mathfrak{G})$ and $D\{s, x\} = 0$.

1.3 The Main Results

Restated and with only the most relevant parts for our research, Gindi and Streets prove the following two main theorems in their paper:

Theorem 1. *Let $\bar{g}$ be a (right) invariant Riemannian metric on the nilpotent, four-dimensional $\mathcal{G}$-principal bundle $\pi : P \to M$, where M is diffeomorphic to S^1. Then the solution to the Ricci flow with initial metric $\bar{g}$ exists on $[0, \infty)$, and there is a universal constant C so that the solution satisfies the (type III) estimates*

$$\sup_{M\times[0,\infty)} t|\overline{\mathrm{Rm}}| \leq C, \quad \mathrm{diam}(g(t)) \leq \mathrm{diam}(g(0))t^{\frac{1}{2}}$$

where g_t is the one-parameter family of metrics on M induced by the family of invariant metrics on P.

The second theorem is the one most relevant to our research (we omit the results only applying to the abelian case, as we do not treat this case) and gives conditions on the blowdown limit of the flow.

Theorem 2. *Let $(P, \bar{g}(t))$ denote a solution to the Ricci flow from Theorem 1. Suppose there exists a blowdown limit $(P_\infty, \bar{g}_\infty(t))$ of $(P, \bar{g}(t))$. Then for $(P_\infty, \bar{g}_\infty(t))$ the following holds:*

5) $DG(\cdot, \cdot) = 0$.

6) $(D.DG).\,(\eta, \eta') = D_{\cdot_1}G\,(\eta, \cdot_2)\,D_{\cdot_1}G\,(\eta', \cdot_2)$, *for all* $\eta, \eta' \in \mathfrak{G}$.

7) $D|[,]|^2 = 0$.

8) $|DG|^2 = \frac{2}{t}$ *and* $g = tg|_{t=1}$.

In the case when $\mathcal{G}$ is nonabelian, we also have:

9) $DG|_{Z(\mathfrak{G})} = 0$.

10) $(D.DG).\,(\eta, \eta') = \frac{G(\eta,\eta')}{t}$, *for all* $\eta, \eta' \in \mathfrak{G}_0$.

Corollary. *Let $\bar{g}_\infty(t)$ be the limiting generalized Ricci flow solution given in Theorem 2. Then for nonabelian $\mathcal{G}$:*

a) $|[,]|^2 = \frac{1}{at+C}$, *where* $a = \frac{3}{2}$ *and* $C \geq 0$.

b) $G\,(\eta, \eta') = \left(\frac{at+C}{a+C}\right)^{\frac{1}{3}} G\Big|_{t=1}(\eta, \eta')$, *for all* $\eta, \eta' \in \mathfrak{G}_0 := Z(\mathfrak{G})^\perp$.

c) $G\,(\eta, \eta') = \left(\frac{at+C}{a+C}\right)^{-\frac{1}{3}} G\Big|_{t=1}(\eta, \eta')$, *for all* $\eta, \eta' \in Z(\mathfrak{G})$.

1.4 Methods for Theorem 1

1.4.1 Ricci flow on $\mathcal{E}$

Gindi and Streets first derive the Ricci flow equations

$$\begin{aligned}\frac{\partial G}{\partial t}(\eta_1, \eta_2) &= (D.DG).\,(\eta_1, \eta_2) - D_{\cdot_2}G\,(\eta_1, \cdot_1)\,D_{\cdot_2}G\,(\cdot_1, \eta_2) \\ &\quad + \mathrm{Tr}_G G\,([\cdot, \eta_1]\,, [\cdot, \eta_2]) - \frac{1}{2}\mathrm{Tr}_G G\,([\cdot_1, \cdot_2]\,, \eta_1)\,G\,([\cdot_1, \cdot_2]\,, \eta_2)\,, \end{aligned} \tag{2}$$

$$G\left(\frac{\partial A}{\partial t}v, \eta\right) = \mathrm{Tr}_G D_v G([\cdot, \eta], \cdot), \tag{3}$$

$$\frac{\partial g}{\partial t}(v_1, v_2) = \frac{1}{2}D_{v_1}G\,(\cdot_1, \cdot_2)\,D_{v_2}G\,(\cdot_1, \cdot_2) \tag{4}$$

for G and g where $\eta, \eta_1, \eta_2 \in \Gamma(\mathfrak{G})$ and $v, v_1, v_2 \in \Gamma(TM) = \mathfrak{X}(\mathcal{M})$ by using the correspondence between $\mathcal{E}$ and TP. Here $\partial A/\partial t$ is defined as

$$\left(\widetilde{\frac{\partial A}{\partial t}(w)}\right)(p) := \underline{\frac{\partial \bar{A}}{\partial t}(\tilde{w})}_p$$

for $w \in T_{\pi(p)}M$ and $\tilde{w} \in T_pP$.

To obtain the long time existence result from Theorem 1, Gindi and Streets note that from Theorem 5.23 of [Fernandez(2021)] it suffices to show that

$$\limsup_{t\to T} \sup_{M\times t} |\overline{\mathrm{Rm}}| = K < \infty.$$

In turn this will follow if one shows the type III estimate

$$\sup_{M\times[0,T)} t|\overline{\mathrm{Rm}}| \leq C$$

for a smooth existence interval $[0, T)$. Now using an expression for the curvature of an invariant metric from Theorem 2.12 of [Gindi and Streets(2020)] its norm can be estimated from the norms of DG, $[,]$ and D^2G. The estimates of these are derived from their evolution equations.

1.4.2 Evolution of $|[,]|^2$ and $|DG|^2$

Using the Ricci flow equations (2), (3) and (4) they calculate the evolution equations for $|[,]|^2$ and $|DG|^2$:

$$\frac{\partial}{\partial t}|[,]|^2 = \Delta|[,]|^2 - \frac{3}{2}|[,]|^4 - S_A, \tag{5}$$

$$\frac{\partial}{\partial t}|DG|^2 = \Delta|DG|^2 - \frac{1}{2}|DG|^4 - S_B \tag{6}$$

where

$$\begin{aligned} S_A = |T|^2 = &|DG\,(\cdot, *_1)\,G\,([*_2, *_3]\,, \cdot) - DG\,(*_2, \cdot)\,G\,([\cdot, *_3]\,, *_1) + DG\,(*_3, \cdot)\,G\,([\cdot, *_2]\,, *_1)\,|^2 \\ &\geq 0 \end{aligned}$$

and

$$\begin{aligned} S_B =&2\,|(D.DG).\,(*_1, *_2) - D._1 G\,(*_1, \cdot_2)\,D._1 G\,(*_2, \cdot_2)|^2 \\ &+ |[,]|^2\,(D_vG(\cdot,\cdot) - 2D_vG\,(\eta_3,\eta_3))^2 + 2|[,]|^2\left(D_vG\,(\cdot,\eta_3)\,D_vG\,(\cdot,\eta_3) - |DG|^2_{Z(\mathfrak{G})}\right) \\ &+ 2|DG([\cdot,*],\cdot)|^2 + 2\,|D_vG([\cdot,*],\cdot)|^2 \\ \geq& 0. \end{aligned}$$

They also derive an evolution equation for D^2G and from it, some other elementary estimates and the above two evolution equations obtain

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right)|D^2G|^2 &\leq -\left|D^3G\right|^2 + C_1\left|D^2G\right|^3 + C_2t^{-3}, \\ \left(\frac{\partial}{\partial t} - \Delta\right)|DG|^2 &\leq -\left|D^2G\right|^2 + Ct^{-2}. \end{aligned} \tag{7}$$

Remark. Here $D^2G(*_1, *_2) := [D.DG.](*_1, *_2)$ and $D^3G(*_1, *_2, *_3) := [D_{*_3}D.DG.](*_1, *_2)$.

1.4.3 Bounds using the Weak Maximum Principle

Now they apply the weak maximum principle for PDEs to obtain the bounds

$$|[,]|^2 \leq \frac{2}{3t}, \qquad |DG|^2 \leq \frac{2}{t}. \tag{8}$$

Recall that the principle states (see Thm 3.1.1 of [Topping(2006)]) that if $u(\boldsymbol{x},t)$ satisfies the PDE

$$\frac{\partial}{\partial t}u \leq \Delta u + f(u,t),$$

$y(t)$ the ODE

$$\frac{d}{dt}y = f(y,t), \quad y(0) = \alpha$$

where $f : \mathbb{R} \times [0,T] \to \mathbb{R}$ is a smooth function, and assuming that

$$u(\boldsymbol{x},0) \leq \alpha$$

for all $\boldsymbol{x}$ then

$$u(\boldsymbol{x},t) \leq y(t)$$

for all $t \in [0,T]$.

For $|[,]|^2$ they define $\Phi(x,t) := t|[,]|^2$ so that (5) becomes

$$\frac{\partial}{\partial t}\Phi \leq \Delta\Phi + |[,]|^2(1 - \frac{3}{2}\Phi)$$

due to

$$\frac{\partial}{\partial t}|[,]|^2 = \Delta|[,]|^2 - \frac{3}{2}|[,]|^4 - S_A \leq \Delta|[,]|^2 - \frac{3}{2}|[,]|^4.$$

We then see that $y(t) = 2/3$ satisfies

$$\frac{d}{dt}y = y(1 - \frac{3}{2}y), \quad y(0) = 2/3$$

and so, as $\sup_{\mathcal{M}\times\{0\}} \Phi = 0 \leq 2/3$, the maximum principle implies that for all smooth existence times t

$$\sup_{\mathcal{M}\times\{t\}} \Phi \leq \frac{2}{3},$$

so that we obtain the first bound. A very similar argument works for $|DG|^2$ to give the second bound.

Lastly the estimate of $|D^2G|^2$ is slightly more involved. They let

$$\Phi(x,t) = t^2|D^2G|^2 + tA|DG|^2$$

for some large constant $A > 0$ to be determined. Then using (7), the maximum principle from multivariable calculus and the arithmetic-geometric mean they show by contradiction that

$$\sup_{M\times[0,T]} \Phi \leq A^{\frac{3}{2}}$$

for sufficiently large A (independent of T) and any smooth existence time T.

1.4.4 Diameter estimate

The diameter estimate from Theorem 1, $\text{diam}(g(t)) \leq \text{diam}(g(0))t^{\frac{1}{2}}$, follows from noting that since M is 1 dimensional (4) is equivalent to

$$\frac{\partial g}{\partial t} = \frac{1}{2}|DG|^2 g_t$$

so we can express $g_t = e^{u_t} g_0$ where

$$\frac{\partial u}{\partial t} = \frac{1}{2}|DG|^2 \leq t^{-1}$$

by (8). So integrating we have

$$g_t \leq t g_0$$

giving the diameter estimate.

1.5 Methods for Theorem 2

1.5.1 The Monotone Energy

To prove the blowdown limit conditions Gindi and Streets define the monotone energy $\mathcal{I}$, a functional which is monotone along the Ricci flow. It is defined as

$$\mathcal{I}(\bar{g}(\tau), \tau) = \tau \int_M \left(|DG|^2 + \frac{2}{\tau}\right) \frac{dV_g}{\sqrt{\tau}}$$

and can be seen to be invariant under principle bundle isomorphisms and is scale invariant, i.e., $\mathcal{I}(s\bar{g}, s\tau) = \mathcal{I}(\bar{g}, \tau)$ for all $s > 0$. By showing that $\mathcal{I}$ is monotone along the flow, the existence of the limit as $t \to \infty$ of $\mathcal{I}(\bar{g}(t), t)$ follows and is easily seen to be equal to the blowdown limit for any t. Hence, the blowdown limits are fixed points of the functional.

To show monotonicity they prove that

$$\frac{d}{dt}\mathcal{I}(\bar{g}(t), t) = -t \int_{\mathcal{M}} S_B \frac{dV_{g(t)}}{\sqrt{t}} - \frac{t}{4} \int_{\mathcal{M}} \left(|DG|^2 - \frac{2}{t}\right)^2 \frac{dV_{g(t)}}{\sqrt{t}} \leq 0. \tag{9}$$

with the last inequality following from $S_B \geq 0$. This is done by using the flow equations for g and G, namely (2), (3) and (4), along with the evolution equation for $|DG|^2$, (6).

1.6 Fixed $\mathcal{I}$

The conditions following from $\mathcal{I}$ being fixed are then those of Theorem 2. For ease of reference we quote them again:

5) $DG(\cdot, \cdot) = 0$.

6) $(D.DG).(\eta, \eta') = D_{\cdot 1}G(\eta, \cdot_2)\, D_{\cdot 1}G(\eta', \cdot_2)$, for all $\eta, \eta' \in \mathfrak{G}$. #

7) $D|[,]|^2 = 0$.

8) $|DG|^2 = \frac{2}{t}$ and $g = tg|_{t=1}$.

In the case when $\mathcal{G}$ is nonabelian, we also have:

9) $DG|_{Z(\mathfrak{G})} = 0.$#

10) $(D.DG).\,(\eta,\eta') = \frac{G(\eta,\eta')}{t}$, for all $\eta,\eta' \in \mathfrak{G}_0$.

It is clear from (9) that for a fixed point, $S_B = 0$ and $|DG|^2 = 2/t$. The latter of gives 8), with the result for g following from a similar expression as in Section 1.4.4. Now, recalling that

$$\begin{aligned} S_B =& 2\,|(D.DG).\,(*_1,*_2) - D_{\cdot_1}G\,(*_1,\cdot_2)\,D_{\cdot_1}G\,(*_2,\cdot_2)|^2 \\ &+ |[,]|^2\,(D_vG(\cdot,\cdot) - 2D_vG\,(\eta_3,\eta_3))^2 + 2|[,]|^2\left(D_vG\,(\cdot,\eta_3)\,D_vG\,(\cdot,\eta_3) - |DG|^2_{Z(\mathfrak{G})}\right) \\ &+ 2|DG([\cdot,*],\cdot)|^2 + 2\,|D_vG([\cdot,*],\cdot)|^2 \end{aligned}$$

we directly obtain the following two identities

$$(D.DG).\,(*_1,*_2) = D_{\cdot_1}G\,(*_1,\cdot_2)\,D_{\cdot 1}G\,(*_2,\cdot_2) \tag{10}$$

$$D_vG(\cdot,\cdot) = 2D_vG\,(\eta_3,\eta_3) \tag{11}$$

for $v \in T_mM$ and $\eta_3 \in Z(\mathfrak{G})_m$. The first, (10), immediately gives 6) while using the identity

$$D_{\cdot 1}G\,(\eta,\cdot_2)\,D_{\cdot 1}G\,(\eta',\cdot_2) = \frac{1}{2}|DG|^2_{Z(\mathfrak{G})^\perp}G(\eta,\eta'),$$

which can easily be derived using an orthonormal basis for $Z(\mathfrak{G})_m$, gives 10). After tracing (10) one obtains an identity equivalent to

$$d^*(DG(\cdot,\cdot)) = 0.$$

Since $DG(\cdot,\cdot) = dh$ for some function h on M by (1), this implies $dh = DG(\cdot,\cdot) = 0$ which is 5). Substituting this result into the second identity above gives 9). Lastly, 7) follows from calculating that

$$D|[,]|^2 = -|[,]|^2\,(DG(\cdot,\cdot) - 2DG\,(\eta_3,\eta_3))$$

and using the second identity, (11).

1.7 The Corollary

For reference we quote it again:

a) $|[,]|^2 = \frac{1}{at+C}$, where $a = \frac{3}{2}$ and $C \geq 0$.

b) $G\,(\eta,\eta') = \left(\frac{at+C}{a+C}\right)^{\frac{1}{3}} G\Big|_{t=1}(\eta,\eta')$, for all $\eta,\eta' \in \mathfrak{G}_0 := Z(\mathfrak{G})^\perp$.

c) $G\,(\eta,\eta') = \left(\frac{at+C}{a+C}\right)^{-\frac{1}{3}} G\Big|_{t=1}(\eta,\eta')$, for all $\eta,\eta' \in Z(\mathfrak{G})$.

a) follows from integrating the evolution equation for $|[,]|^2$, (5). *(#why is $\Delta|[,]|^2 + S_A = 0$?#)* Whereas b) and c) both follow from the flow equation for G, (2), after applying 6) along with result a). This being most easily seen again by using an orthonormal basis.

2 Research

We henceforth let $M = S^1$ as we may without loss of generality assume M to be connected and take s as coordinates on S^1. Assuming that $\mathcal{G}$ is simply-connected and non-abelian we have $\mathcal{G} = H_3(\mathbb{R})$, the Heisenberg group. For simplicity of notation we denote the limit G_∞ (g_∞) just by G (g) as there is no possibility for confusion. Also we refer to the conditions in Thm 4.5 as i) and those in Cor 4.6 as i').

2.1 Time evolution

The metric g on S^1 at time $t = 1$ will have the form

$$g(1, s) := f(s)\, ds \otimes ds$$

which can be changed to

$$g(1, \theta) := d\theta \otimes d\theta$$

by a reparametrisation of s to arclength θ. It has time dependence

$$g(t, \theta) = tg(1, \theta) = t\, d\theta \otimes d\theta$$

by 8) while G has time dependence according to 3.b') and 3.c'), so

$$\begin{aligned} &b) \quad G(t,\theta)\,(\eta,\eta') = \alpha(t)G(1,\theta)\,(\eta,\eta') \quad \forall \eta,\eta' \in \mathfrak{G}_0 \\ &c) \quad G(t,\theta)\,(\eta,\eta') = \alpha(t)^{-1}G(1,\theta)\,(\eta,\eta') \quad \forall \eta,\eta' \in Z(\mathfrak{G}) \end{aligned} \tag{12}$$

where $\alpha(t) = ((at + C)/(a + C))^{1/3}$. Hence, it suffices to look at $t = 1$. We will henceforth sometimes omit the dependence on θ and t, with $t = 1$ assumed from now on, for clarity.

2.2 Setup

We take a parallel frame $\{\bar{\eta}_i\}$ with $\bar{\eta}_3 \in \Gamma(Z(\mathfrak{G}))$ as basis for the fibres $\mathfrak{G}_m$ for each $m \in S^1$ and θ as a coordinate on S^1. We obtain the frame from the parallel transport of an orthonormal basis at $m_0 \in S^1$ with coordinates $\theta = 0$. By 9) DG restricts to zero on $Z(\mathfrak{G})$ so that in this parallel frame

$$D_{\partial_\theta} G(1,\theta) = \begin{pmatrix} \partial_\theta G_{11}(1,\theta) & \partial_\theta G_{12}(1,\theta) & \partial_\theta G_{13}(1,\theta) \\ \partial_\theta G_{21}(1,\theta) & \partial_\theta G_{22}(1,\theta) & \partial_\theta G_{23}(1,\theta) \\ \partial_\theta G_{31}(1,\theta) & \partial_\theta G_{32}(1,\theta) & \partial_\theta G_{33}(1,\theta) \end{pmatrix} = \begin{pmatrix} \bar{a}(\theta) & \bar{b}(\theta) & 0 \\ \bar{b}(\theta) & \bar{c}(\theta) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{13}$$

From the orthonormality assumption these imply that

$$G_{13}(1,\theta) = G_{23}(1,\theta) = 0, \quad G_{33}(1,\theta) = 1.$$

The other conditions on the metric become in this parallel frame

$$\begin{aligned} &5) \quad G(1,\theta)^{ij}\partial_\theta G(1,\theta)_{ij} = 0, \\ &6) \quad \partial_\theta^2 G(1,\theta)_{ij} = G(1,\theta)^{kl}\partial_\theta G(1,\theta)_{ik}\partial_\theta G(1,\theta)_{jl}, \\ &8) \quad G(1,\theta)^{ij}G(1,\theta)^{kl}\partial_\theta G(1,\theta)_{ik}\partial_\theta G(1,\theta)_{jl} = 2. \end{aligned}$$

Remark. We remark that conditions 6) and 8) are the harmonic-Einstein equations as seen in *[Lott(2007)]* equations (4.5) and (4.6). As matrix equations these are simply

$$\begin{aligned}
&5)\quad \mathrm{Tr}\left(G^{-1}DG\right) = \mathrm{Tr}\left(G^{-1}\partial_\theta G\right) = 0,\\
&6)\quad D^2G = \partial_\theta^2 G = DGG^{-1}DG = (\partial_\theta G)G^{-1}(\partial_\theta G),\\
&8)\quad \mathrm{Tr}\left((DG)G^{-1}(DG)G^{-1}\right) = \mathrm{Tr}\left((\partial_\theta G)G^{-1}(\partial_\theta G)G^{-1}\right) = 2.
\end{aligned}$$

Now we observe that condition 5) is equivalent to

$$\partial_\theta \det(G) = 0$$

so the determinant of G in these coordinates is constant, so equal to 1 by the initial conditions. The inverse of G in these coordinates is

$$G^{-1} = \begin{pmatrix} G_{22} & -G_{12} & 0 \\ -G_{12} & G_{11} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so that the remaining conditions become (note that $\partial_\theta G^2 := (\partial_\theta G)^2$ and $\partial_\theta G\tilde{G} := (\partial_\theta G)\tilde{G}$)

$$6)\quad \begin{cases} \partial_\theta^2 G_{11} = G_{22}(\partial_\theta G_{11})^2 - 2G_{12}\partial_\theta G_{12}\partial_\theta G_{11} + G_{11}(\partial_\theta G_{12})^2 \\ \partial_\theta^2 G_{12} = \partial_\theta G_{12}\partial_\theta\left(G_{22}G_{11}\right) - G_{12}\left((\partial_\theta G_{12})^2 + \partial_\theta G_{11}\partial_\theta G_{22}\right) \\ \partial_\theta^2 G_{22} = G_{22}(\partial_\theta G_{12})^2 - 2G_{12}\partial_\theta G_{22}\partial_\theta G_{12} + G_{11}(\partial_\theta G_{22})^2 \end{cases}$$

$$\begin{aligned}
8)\quad 2 = {}& G_{11}^2(\partial_\theta G_{22})^2 + 2G_{12}\partial_\theta G_{22}\left(G_{12}\partial_\theta G_{11} - 2G_{11}\partial_\theta G_{12}\right)\\
&+ 2(\partial_\theta G_{12})^2\left(G_{12}^2 + G_{11}G_{22}\right) - 4G_{12}G_{22}\partial_\theta G_{11}\partial_\theta G_{12} + G_{22}^2(\partial_\theta G_{11})^2.
\end{aligned}$$

If we use the condition on the determinant to eliminate G_{22} then these simplify to the following three conditions

$$6)\quad \begin{cases} \partial_\theta^2 G_{11} = G_{11}\left(\frac{(G_{12}^2+1)(\partial_\theta G_{11})^2}{G_{11}^2} - \frac{2G_{12}\partial_\theta G_{12}\partial_\theta G_{11}}{G_{11}} + (\partial_\theta G_{12})^2\right) \\ \partial_\theta^2 G_{12} = G_{12}\left(\frac{(G_{12}^2+1)(\partial_\theta G_{11})^2}{G_{11}^2} - \frac{2G_{12}\partial_\theta G_{12}\partial_\theta G_{11}}{G_{11}} + (\partial_\theta G_{12})^2\right) \end{cases}$$

$$8)\quad 1 = \frac{\left(G_{12}^2+1\right)(\partial_\theta G_{11})^2}{G_{11}^2} - \frac{2G_{12}\partial_\theta G_{12}\partial_\theta G_{11}}{G_{11}} + (\partial_\theta G_{12})^2.$$

Condition 8) allows us to simplify condition 6) to give

$$6)\quad \begin{cases} \partial_\theta^2 G_{11} = G_{11} \\ \partial_\theta^2 G_{12} = G_{12} \end{cases}$$

$$8)\quad 1 = \frac{\left(G_{12}^2+1\right)(\partial_\theta G_{11})^2}{G_{11}^2} - \frac{2G_{12}\partial_\theta G_{12}\partial_\theta G_{11}}{G_{11}} + (\partial_\theta G_{12})^2.$$

2.3 Solutions

2.3.1 Local solution

The above condition 6) then gives the general solution

$$G(1,\theta)=\begin{pmatrix} A_{11}e^{\theta}+B_{11}e^{-\theta} & A_{12}e^{\theta}+B_{12}e^{-\theta} & 0 \\ A_{12}e^{\theta}+B_{12}e^{-\theta} & \frac{1+\left(A_{12}e^{\theta}+B_{12}e^{-\theta}\right)^2}{A_{11}e^{\theta}+B_{11}e^{-\theta}} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which we can substitute in condition 8) to give

$$A_{11}^2B_{12}^2+A_{12}^2B_{11}^2-A_{11}B_{11}\left(1+2A_{12}B_{12}\right)=0. \tag{14}$$

Also from the orthonormality of this frame at $\theta=0$ we require $B_{11}\to 1-A_{11},\ B_{12}\to -A_{12}$. (14) then becomes

$$A_{12}^2+\left(A_{11}-1\right)A_{11}=0.$$

Hence we have

$$\begin{aligned} G(1,\theta)&=\begin{pmatrix} A_{11}e^{\theta}+(1-A_{11})e^{-\theta} & A_{12}(e^{\theta}-e^{-\theta}) & 0 \\ A_{12}(e^{\theta}-e^{-\theta}) & \frac{1+A_{12}^2\left(e^{\theta}-e^{-\theta}\right)^2}{A_{11}e^{\theta}+B_{11}e^{-\theta}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &=\begin{pmatrix} \cosh(\theta)+(2A_{11}-1)\sinh(\theta) & 2A_{12}\sinh(\theta) & 0 \\ 2A_{12}\sinh(\theta) & \cosh(\theta)+(1-2A_{11})\sinh(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}=\exp\left(\theta D\right) \end{aligned} \tag{15}$$

for

$$D=\begin{pmatrix} 2A_{11}-1 & 2A_{12} & 0 \\ 2A_{12} & 1-2A_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with $(2A_{11}-1)^2+(2A_{12})^2=1$.

2.3.2 Global solution and Holonomy

By Proposition 2.4.3) for $x\in\mathfrak{g}$ and $s:U\to P$ a local section of P,

$$\begin{aligned} D_v\{s,x\}&=\{s,[(s^*\bar{A})v,x]\} \\ &=\{s,[\bar{A}ds(v),x]\} \\ &=\{s,[\bar{A}ds(v),x]\} \\ &=\{s,0\}=0 \end{aligned}$$

if s is horizontal. So these are the parallel sections of $\mathfrak{G}$ by the uniqueness condition for parallel transport. Locally then around m_0 ($\theta=0$ in coordinates) the parallel frame $\{\bar{\eta}_i\}$ with $\bar{\eta}_3\in\Gamma(Z(\mathfrak{G}))$ will then be

$$\{s_i,\tilde{H}_i\}$$

for some local horizontal sections $s_i : U \to P$ of P where $m_0 \in U$ and $H_i \in \mathfrak{g}$. At m_0 there exists $k_i \in \mathcal{G}$ such that $s_i(m_0) \cdot k_i = s(m_0)$ for a local horizontal section $s : U \to P$ so that

$$\bar{\eta}_i(m_0) = \{s_i(m_0), \tilde{H}_i\} = \{s(m_0), H_i\}$$

where $H_i := \mathrm{Ad}_{k_i} \tilde{H}_i$. As these are horizontal sections of a 1 dimensional base they are horizontal lifts and hence are unique once the initial point is fixed. So we see that $s_i \cdot k_i = s$. The parallel frames are locally then

$$\{s, H_i\}.$$

Similarly after parallel transport back to m_0 ($\theta = 2d$ in coordinates, with d being the diameter of S_1) the parallel frame will be

$$\{\tilde{s}, H_i\}$$

for some local horizontal section $\tilde{s} : V \to P$ of P where $m_0 \in V$. Noting that

$$\tilde{s}(0) = s(0) \cdot h$$

for $h \in \mathcal{G}$ we see that

$$\{\tilde{s}(m_0), H_i\} = \{s(m_0), \mathrm{Ad}_h H_i\}.$$

It then follows that

$$\bar{\eta}_i(2d) = \mathrm{Ad}_h \bar{\eta}_i(0).$$

Let $H := \mathrm{Ad}_h$ and $e_i := \bar{\eta}(0)$, then we have

$$\begin{aligned} G_{ij}(2d) &= G(2d) \left(\bar{\eta}_i(2d), \bar{\eta}_j(2d)\right) \\ &= G(2d) \left(He_i, He_j\right). \end{aligned}$$

Defining the coefficients of the endomorphism H by

$$He_i = H_i^l e_l$$

and using (15) we have

$$\begin{aligned} G_{ij}(2d) &= G(2d) \left(H_i^l e_l, H_j^k e_k\right) \\ &= H_i^l G(2d) \left(e_l, e_k\right) H_j^k \\ &= H_i^l G(2d)_{lk} H_j^k \\ &= H_i^l \exp(2dD)_l^s G(0)_{sk} H_j^k. \end{aligned}$$

Now by our initial condition assumption $G(0)_{sk} = \delta_{sk}$ so that

$$G_{ij}(2d) = H_i^l \exp(2dD)_l^k H_j^k.$$

Again using (15) and the initial condition assumption on the RHS we obtain

$$\begin{aligned}\exp(2dD)_i^j &= H_i^l \exp(2dD)_l^k H_j^k \\ &= H_i^l \exp(2dD)_l^k (H^T)_k^j.\end{aligned}$$

It then follows that

$$\exp(2dD) = H\exp(2dD)H^T. \tag{16}$$

We can now use the properties of the Heisenberg Lie algebra $\mathfrak{g}$. The most general expression for ad_x with $x \in \mathfrak{g}$ being in the basis $\{e_i\}$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y & z & 0 \end{pmatrix}, \quad y,\, z \in \mathbb{R}$$

and $H = \mathrm{Ad}_h = \exp(\mathrm{ad}_x)$ for some $x \in \mathfrak{g}$ we see that

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ y & z & 1 \end{pmatrix}$$

for some $y,\, z \in \mathbb{R}$. From the form of D we also have that

$$\exp(2dD) = \begin{pmatrix} E & 0 \\ 0 & 1 \end{pmatrix}$$

for some 2×2 matrix E. (16) then reads

$$\begin{aligned}\begin{pmatrix} E & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ y & z & 1 \end{pmatrix}\begin{pmatrix} E & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ y & z & 1 \end{pmatrix}\begin{pmatrix} E & E\begin{pmatrix} y \\ z \end{pmatrix} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} E & E\begin{pmatrix} y \\ z \end{pmatrix} \\ \begin{pmatrix} y \\ z \end{pmatrix}^T E & \begin{pmatrix} y \\ z \end{pmatrix}^T E\begin{pmatrix} y \\ z \end{pmatrix} \end{pmatrix}.\end{aligned}$$

$\exp(2dD)$ being invertible, E is invertible so that the above gives

$$E\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow y = z = 0$$

and hence that $H = \mathrm{Id}$. This further implies that $\bar{\eta}_i(2d) = \bar{\eta}_i(0)$, so that $G(2d) = G(0)$ which is only possible if $\exp(2dD) = \mathrm{Id}$, in other words if $D = 0$.

However, $D = 0$ means that G is constant and so $|DG|^2 = 0$, a contradiction to condition 8).

2.4 Discussion and Conclusion

Our result then shows that there are no solutions satisfying all the conditions of Theorem 2 meaning that all blowdown limits in fact collapse the base S^1 to a point. This is rather surprising and seems to be mainly due to topology of the base, S^1. This can be investigated in future work by looking at a 4 dimensional principal bundle with the same symmetry group but over the universal cover of S^1, $\mathbb{R}$. Another thing to look at is whether the same is true in the abelian case where there is less restrictions. Lastly, a much more ambitious future direction is to learn about the Generalised Ricci flow and see if there would be any solutions. This is the flow the paper by Gindi and Streets was mainly concerned with.

Appendix (Mathematical Background)

3 Ricci flow

Ricci flow is the evolution of a metric on a manifold according to a geometric PDE. More precisely there is a smooth one-parameter family of Riemannian metrics $\{g_t\}_{t\in I}$ on $\mathcal{M}$ which evolve according to the **Ricci flow PDE**

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}(g). \tag{$\star$}$$

So if $\{g_t\}_{t\in I}$ is a solution to this PDE then it is called a **Ricci flow**.

3.1 Examples and special solutions

The first and most distinguished examples of the Ricci flow are its fixed points or rather a slight generalisation of these.

3.1.1 Einstein manifolds

The simplest such generalisations are those which evolve by linear scaling of the metric with time. These have as initial metric $g(0) = g_0$ an Einstein metric so called as they satisfy the vacuum Einstein equation from General Relativity

$$\mathrm{Ric}(g_0) = \lambda g_0,$$

with λ some real constant. A Ricci flow with this initial metric is then found from the ansatz $g(t) = c(t)g_0$ where $c(0) = 1$ so that

$$\frac{\partial g}{\partial t}(t) = c'(t)g_0 = \frac{c'(t)}{\lambda}\,\mathrm{Ric}(g_0),$$

while

$$-2\mathrm{Ric}(g(t)) = -2\mathrm{Ric}(c(t)g_0) = -2\mathrm{Ric}(g_0)$$

as the Ricci curvature is invariant under scalings of the metric. Hence we require

$$\frac{c'(t)}{\lambda} = -2, \quad \lambda(0) = 1$$

so that $c(t) = 1 - 2\lambda t$ and the solution is $g(t) = (1 - 2\lambda t)g_0$.

The different signs of λ come from three geometrically different manifolds. Positive λ come from the **round unit sphere** (S^n, g_0), for which $\text{Ric}(g_0) = (n-1)g_0$. The Ricci flow on it is

$$g(t) = (1 - 2(n-1)t)g_0$$

meaning that the sphere shrinks to a point in time $T = \frac{1}{2(n-1)}$. Negative λ come from the **hyperbolic space with hyperbolic metric** $(\mathbb{H}^n, g_0)$, for which $\text{Ric}(g_0) = -(n-1)g_0$ so that

$$g(t) = (1 + 2(n-1)t)g_0$$

and the manifold expands for all time. Lastly, $\lambda = 0$ comes from **Euclidean space** with its flat metric $(\mathbb{R}^n, g_0)$, for which $\text{Ric}(g_0) = 0$ so that

$$g(t) = g_0$$

and the Euclidean space is steady.

3.1.2 Ricci solitons

Introduction A further generalisation of the Einstein manifolds are Ricci solitons which can evolve by linear scaling and diffeomorphisms. The motivation for this generalisation is that one can consider all Riemannian manifolds $\{(\mathcal{M}, g_\alpha)\}_{\alpha \in \Lambda}$ up to isometry, so

$$g_\alpha \sim \psi^* g_\alpha$$

for some diffeomorphism $\psi : \mathcal{M} \to \mathcal{M}$, and up to scaling

$$g_\alpha \sim c g_\alpha.$$

So the fixed points under the action of the Ricci flow will correspond to the Ricci solitons.

To define these solutions we start with a family of vector fields $\{X(t)\}_{t \in I} \subset \mathfrak{X}(\mathcal{M})$ which generate a family of diffeomorphisms ψ_t of $\mathcal{M}$. They generate the diffeomorphisms in the following manner. At a point $p \in \mathcal{M}$

$$X(\psi_{t_0}(p), t_0) f = \left. \frac{\partial f \circ \psi_t(p)}{\partial t} \right|_{t = t_0},$$

or equivalently (here $(X^t)_p$ denotes $X(p,t)$)

$$\partial_t \psi_{t_0}(p) = X(\psi_{t_0}(p), t_0) \iff \gamma_p'(t_0) = (X^{t_0})_{\gamma_p(t_0)}$$

where $\gamma_p(t) := \psi_t(p)$. Recalling that the integral curve φ_p^Y starting at p of vector field Y is defined by

$$(\varphi_p^Y)'(t_0) = Y_{\varphi_p^Y(t_0)}$$

we see that $\gamma_p(t) = \varphi_p^{X(t)}(t)$ so that

$$(X^{t_0})_{\varphi^{X(t_0)}(t_0)} = (\varphi_p^{X(t_0)})'(t_0).$$

Hence

$$\psi_t(p) = \varphi_p^{X(t)}(t),$$

so ψ_{t_0} is the diffeomorphism generated by flowing along $X(t_0)$ for time t_0.

Solution to Ricci flow Now to obtain a Ricci flow from this we let $\sigma : I \to \mathbb{R}$ be a smooth function of t and define a new family of metrics

$$\hat{g}(t) := \sigma(t)\psi_t^*(g(t)).$$

So then we have (Prop. 1.2.1 of [Topping(2006)])

$$\frac{\partial \hat{g}}{\partial t}(t_0) = \sigma'(t_0)\psi_{t_0}^*(g(t_0)) + \sigma(t_0)\psi_{t_0}^*\left(\frac{\partial g}{\partial t}(t_0)\right) + \sigma(t)\psi_t^*(\mathcal{L}_{X(t_0)}g(t_0)). \tag{17}$$

Meaning that if we have a metric g_0, vector field Y and $\lambda \in \mathbb{R}$ (with no time dependence) such that

$$-2\text{Ric}(g_0) = \mathcal{L}_Y g_0 - 2\lambda g_0 \iff \mathcal{L}_Y g_0 + 2\text{Ric}(g_0) = 2\lambda g_0, \tag{18}$$

then setting $g(t) := g_0$ and $\sigma(t) := 1 - 2\lambda t$, $\hat{g}(t)$ is a Ricci flow with $\hat{g}(0) = g_0$ (easily verified using (17)). The family of vector fields generating the diffeomorphisms ψ_t in this case is $X(t) := \sigma(t)^{-1}Y$.

Ricci soliton definition Motivated by the above solution we then call a Riemannian manifold $(\mathcal{M}, g_0)$ satisfying

$$\mathcal{L}_Y g_0 + 2\text{Ric}(g_0) = 2\lambda g_0,$$

for some smooth vector field Y and constant $\lambda \in \mathbb{R}$ a **Ricci soliton**. Since, as seen above, it is the initial condition to the Ricci flow solution $\hat{g}(t)$ (which may also be referred to as a Ricci solition), where

$$\hat{g}(t) = \sigma(t)\psi_t^*(g_0),$$

$\sigma(t) = 1 - 2\lambda t$ and ψ_t is generated by vector fields $X(t) = \sigma(t)^{-1}Y$. It is called **steady**, **expanding** or **shrinking** depending on whether $\lambda = 0$, $\lambda < 0$ or $\lambda > 0$. With this reflecting what was seen in the simpler case of the Einstein manifolds.

3.2 Blowdown limits

My research is concerned with the long-time behavior of Ricci flow solutions. Since the Ricci flow can cause the metric to expand exponentially or collapse to a point one needs to reparametrise it to capture the relevant geometric structures in the limit. To do so, one observes that given a Ricci flow solution $g(\cdot)$ and a parameter $s > 0$, there is another Ricci flow solution $g_s(\cdot)$ given by $g_s(t) = s^{-1}g(st)$. The time interval $[a, b]$ for g_s then becomes the time interval $[sa, sb]$ for g. So understanding the behavior of $g(t)$ for large t amounts to understanding the behavior of $g_s(\cdot)$ as $s \to \infty$. One calls the limit $g_\infty(\cdot)$ the **blowdown limit** if it exists.

4 Principal Bundles

Another essential topic for my research and the setting of the flow is (smooth) Principal Bundles. As motivated in the introduction, these arise when we have a smooth manifold P with, by convention, free right group action by a Lie group $\mathcal{G}$. Denoting the action by group element g on a point $p \in P$ by $R_g p = p \cdot g$, we mean by a free action that $R_g p = p$ for *some* $p \in \mathcal{M}$ if and only if $g = e$. The group action induces an equivalence relation on the points of P giving the quotient space $\mathcal{M} = P \backslash G$ and quotient map $\pi : P \to \mathcal{M}$. Since the action is free, for $m \in \mathcal{M}$ we have an isomorphism between $\pi^{-1}(m)$ and $\mathcal{G}$ once we choose a representative p with $\pi(p) = m$. This means that locally P 'looks like' a product $U \times \mathcal{G}$ for some open set U of $\mathcal{M}$. The proper definition just formalises this picture and requires that everything is smooth.

4.1 Definition

Let $\mathcal{M}$ be a smooth manifold and $\mathcal{G}$ a Lie group. A **smooth principal fibre bundle over $\mathcal{M}$ with group $\mathcal{G}$** (also called a **principal $\mathcal{G}$ bundle**) is a smooth fibre bundle $\pi : P \to \mathcal{M}$ with fibre $\mathcal{G}$ and a smooth action of $\mathcal{G}$ on P such that

1. $\mathcal{G}$ acts freely on P on the right,
2. $\mathcal{M}$ is the quotient space of P under the equivalence relation induced by $\mathcal{G}$, so $\mathcal{M} = P \backslash \mathcal{G}$, and π is the quotient map.
3. P is locally trivial: every point $m \in \mathcal{M}$ has a neighbourhood U such that there is a diffeomorphism (**fibre preserving local trivialisation)** $\psi_U : \pi^{-1}(U) \to U \times \mathcal{G}$ such that

   $$\psi_U(p) = (\pi(p), \varphi_U(p))$$

   where $\varphi_U : \pi^{-1}(U) \to \mathcal{G}$ is $\mathcal{G}$-equivariant so satisfies

   $$\varphi_U(p \cdot g) = \varphi_U(p) \cdot g$$

   for all $p \in \pi^{-1}(U)$ and $g \in \mathcal{G}$ (the $\mathcal{G}$ action on $U \times \mathcal{G}$ is that of a trivial bundle defined below).

$\mathcal{G}$ is called the **structure group** or **symmetry group**. A **local trivialisation** (which exists by property 3.) is an open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ of $\mathcal{M}$ together with a collection of fibre preserving local trivialisations $\{\psi_\alpha\}$ where $\psi_\alpha := \psi_{U_\alpha}$.

4.2 Foundational example

All principal bundles are locally diffeomorphic to a **trivial bundle**. This is the product principal $\mathcal{G}$-bundle $\mathcal{M} \times \mathcal{G}$ with free action on the right by $\mathcal{G}$ defined as

$$R_h(m, g) = (m, gh).$$

So understanding the trivial bundle allows the local structure of principal bundles to be understood. In particular we see that the tangent space will decompose as a direct sum $T_{(m,g)}(\mathcal{M} \times \mathcal{G}) \cong T_m\mathcal{M} \oplus T_g\mathcal{G}$. Since the tangent spaces on a Lie group are all isomorphic through the left action to that at the identity, the Lie algebra $\mathfrak{g}$, we see that $T_{(m,g)}(\mathcal{M} \times \mathcal{G}) \cong T_m\mathcal{M} \oplus \mathfrak{g}$. However, this decomposition is dependent on the local trivialisation. To remedy this we need a global way to define these two subspaces.

4.3 Principal bundle isomorphisms

Two smooth principal $\mathcal{G}$ bundles $\pi : P \to \mathcal{M}$ and $\tilde{\pi} : Q \to \mathcal{N}$ are isomorphic if there is a diffeomorphism $F : P \to Q$ which preserves fibres, descends to a diffeomorphism $f : \mathcal{M} \to \mathcal{N}$ (i.e. $\tilde{\pi} \circ F = f \circ \pi$) and commutes with the group action

$$f(p) \cdot g = f(p \cdot g)$$

for $g \in \mathcal{G}$ and $p \in P$. In our case where both bundles have a metric, respectively $\bar{g}_P$ and $\bar{g}_Q$, we also require that F is an isometry between P and Q.

4.4 Connections on Principal bundles

The way to do this is by introducing a connection. Firstly, let $\pi : P \to \mathcal{M}$ be a principal G-bundle. Then at $p \in P$ we call the subspace of vectors tangent to the fibre at p, $\mathcal{V}_p := \ker(\pi_{*,p})$, the **vertical subspace at** p. A **connection** Γ on P is an assignment of **horizontal subspace** $\mathcal{H}_p \subset T_pP$ (denoted Q_p in KN) at each $p \in P$ such that

1. $T_pP = \mathcal{V}_p \oplus \mathcal{H}_p$.
2. $\mathcal{H}_{p \cdot g} = (R_g)_*\mathcal{H}_p$ (called **right invariance**).
3. $\mathcal{H}_p$ depends smoothly on p, i.e. $\mathcal{H}$ is a distribution (subbundle of TP).

A vector $X_p \in T_pP$ is then called **vertical** (respectively **horizontal**) if $X_p \in \mathcal{V}_p$ (respectively $X_p \in \mathcal{H}_p$). Any vector can be written as

$$X_p = Y_p + Z_p,$$

for $Y_p \in \mathcal{V}_p$ and $Z_p \in \mathcal{H}_p$, respectively the **vertical** and **horizontal components** of X_p. These will be denoted respectively vX_p and hX_p (the corresponding maps being denoted respectively v and h). Condition (3.) implies that for $X \in \mathfrak{X}(P)$, both vX and hX are also smooth vector fields on P.

4.4.1 Connection associated with an invariant metric

Generally there is not enough data for the horizontal subspaces to be canonically chosen. However if there is a metric $\bar{g}$ on P (right invariant to satisfy condition (2.)) then one can canonically define the horizontal subspace, namely as the orthogonal complement with respect to $\bar{g}$ of $\mathcal{V}_p$ for each $p \in \mathcal{M}$. So we have

$$\mathcal{H}_p = \mathcal{V}_p^{\perp} = \{v \in T_pP \mid \bar{g}(v, w) = 0 \; \forall w \in \mathcal{V}_p\}.$$

It can be shown that this choice is smooth, so satisfies condition (3.). This connection is the one **associated with the metric** $\bar{g}$.

4.5 Fundamental vector field

Before introducing another way of looking at the connection we need to look at fundamental vector fields - important vertical vector fields on principal bundles which come from the Lie algebra $\mathfrak{g}$ of the structure group $\mathcal{G}$. These vector fields are associated to the Lie algebra elements, where given a Lie algebra element A we have an associated fundamental vector field $\underline{A}$ which is induced by the one-parameter group of diffeomorphisms $\varphi^A(t) := \exp tA$ by

$$\underline{A}(p) := \frac{d}{dt}\bigg|_{t=0} p \cdot \exp tA = \frac{d}{dt}\bigg|_{t=0} R_{\varphi^A(t)}(p).$$

If we let $\psi_p^A(t) = p \cdot \exp tA$ then we see that $\underline{A}(p) = (\psi_p^A)'(0)$, the initial tangent vector to the curve. Also note that $\psi_p^A(t)$ is the integral curve of $\underline{A}$ through p since

$$(\psi_p^A)'(t) = \frac{d}{ds}\bigg|_{s=0} \psi_p^A(t+s) = \frac{d}{ds}\bigg|_{s=0} p \cdot \exp{(t+s)A} = \frac{d}{ds}\bigg|_{s=0} (p \cdot \exp tA) \cdot \exp sA = \underline{A}_{\psi_p^A(t)}. \tag{19}$$

Denoting this map from $\mathfrak{g}$ to $\mathfrak{X}(P)$ by σ we remark that it is a Lie algebra homomorphism. In other words

$$\underline{[A,B]} = [\underline{A}, \underline{B}] \tag{20}$$

Also, since the action preserves fibres, the fundamental vector fields are tangent to the fibres, i.e.

$$\underline{A}_p \in T_p P_{\pi(p)}$$

for every $p \in P$. Defining the **point map** $\sigma_p : \mathcal{G} \to P, \sigma_p(g) = p \cdot g$ induced by a point $p \in P$ which satisfies

$$(\sigma_p)_{*,e} A_e = \frac{d}{dt}\bigg|_{t=0} \sigma_p(\exp tA) = \frac{d}{dt}\bigg|_{t=0} p \cdot \exp tA = \underline{A}_p = \sigma(A)_p, \tag{21}$$

we obtain a vector space isomorphism $(\sigma_p)_{e,*} : T_e\mathcal{G} \cong \mathfrak{g} \to T_p(P_{\pi(p)})$ since the dimension of each fibre is equal to that of $\mathfrak{g}$ and for non-zero $A \in \mathfrak{g}$, $\underline{A} \in \mathfrak{X}(\mathcal{M})$ vanishes nowhere. This follows since if it were to vanish at some point $p \in \mathcal{M}$ then its integral curve passing through p is the constant curve so $p \cdot \exp tA = p$ for all $t \in \mathbb{R}$ and hence $\exp tA = e$ for all $t \in \mathbb{R}$ and $A \equiv 0$, a contradiction. We make the remark that the connection form ω satisfies $\omega(\underline{A}) = A$.

4.6 Connection form

An equivalent and more effective way of looking at a connection Γ in P is as a $\mathfrak{g}$-valued 1-form ω on P called the **connection form**. To do so recall that we showed that the map $\sigma : \mathfrak{g} \to \mathfrak{X}(P), A \mapsto \underline{A}$ is a Lie algebra homomorphism and the map $\sigma_p : G \to \mathcal{M}, g \mapsto p \cdot g$ which satisfies

$$(\sigma_p)_{*,e} A_e = \underline{A}_p = \sigma(A)_p$$

is a linear isomorphism of $\mathfrak{g}$ and $T_p(P_{\pi(p)})$ (which is by definition $\mathcal{V}_p$). For $X_p \in T_pP$ we define $\omega_p(X_p)$ to be the unique $A \in \mathfrak{g}$ such that $\underline{A}_p = (vX)_p$. More concisely

$$\underline{\omega(X_p)}_p = (vX)_p \Leftrightarrow \omega_p = (\sigma_p)^{-1}_{*,e} \circ v.$$

From this it is clear that

$$\omega_p(X_p) = 0 \Leftrightarrow vX_p = 0.$$

From the above discussion it is then clear that the connection form ω satisfies the following useful property, $\omega(\underline{A}) = A$.

4.7 Curvature form and structure equation

Another key notion coming from the connection form on principal bundles is that of curvature. It can be understood as the generalisation of the curvature tensor in Riemannian geometry. Before we define it we need to understand the different types of distinguished forms on a principal bundle.

4.7.1 (Pseudo) tensorial and horizontal forms

Let ρ be a representation of G on finite dimensional vector space V (so $\rho : G \to \mathrm{GL}(V)$, $g \cdot v := \rho(g)v = w \in V$ is a group homomorphism). A **pseudotensorial r-form on P of ρ type** is a V-valued r-form φ on P such that

$$R_g^*\varphi = g^{-1} \cdot \varphi$$

for all $g \in G$. At a point $p \in P$ we have

$$(R_g^*\varphi)_p(v_1, \cdots, v_r) = g^{-1} \cdot \varphi_p(v_1, \dots, v_r)$$

for $v_1, \dots, v_r \in T_pP$. A V-valued r-form φ on P is **horizontal** if at each point $p \in P$,

$$\varphi_p(v_1, \dots, v_r) = 0$$

whenever one of $v_1, \dots, v_r \in T_pP$ is vertical. A horizontal pseudotensorial form is called a **tensorial form**. The set of all smooth tensorial V-valued forms of type ρ is denoted $\Omega^k_\rho(P, V)$.

4.7.2 New (pseudo) tensorial types from old

Proposition. *We can obtain other (pseudo) tensorial forms form (pseudo) tensorial forms in the following ways. If φ is a pseudotensorial r-form on P of ρ type then*

1. φh defined as
$$(\varphi h)_p(v_1(p), \dots, v_r(p)) := \varphi_p(hv_1(p), \dots, hv_r(p))$$
is a tensorial r-form of type ρ.

2. $d\varphi$ is a pseudotensorial $(r+1)$-form of ρ type.

3. $D\varphi := (d\varphi)h$ is a tensorial $(r+1)$-form of ρ type.

4.7.3 Curvature form

The tensorial form $D\varphi := (d\varphi)h$ is called the **covariant derivative** of φ. If ρ is the adjoint representation of G (so $\rho(g) = \mathrm{Ad}_g$) then a (pseudo) tensorial form is called of **adjoint type** or Ad type. An important property of the connection form ω is that it satisfies

$$(R_g)^*\omega = \mathrm{Ad}_{g^{-1}}\omega$$

so it is a pseudotensorial 1-form of Ad type. Using the covariant derivative we obtain $D\omega$, a tensorial 2-form of type Ad called the **curvature form** of ω and denoted Ω.

4.8 Associated bundle

Another important concept coming along with a principal bundle is an associated bundle to a principal bundle.

4.8.1 Set definition

Intuitively an associated bundle is a fibre bundle over the same base as the pricipal bundle P and is constructed from it by using the group action on P along with an action on the fibre. More precisely, if we have a manifold F with left $\mathcal{G}$ action then we can construct a fibre bundle E over $\mathcal{M}$ with fibre F from P. E is called the **associated bundle to P with fibre F**. To do so we let $\mathcal{G}$ act on $P \times F$ by

$$(p, f) \cdot g = (p \cdot g, g^{-1} \cdot f).$$

We denote the quotient space $E := P \times_{\mathcal{G}} F := (P \times F)/\mathcal{G}$ and the quotient map $\pi_{P\times F} : P \times F \to E$ with equivalence classes

$$[p, f] := \pi_{P\times F}(p, f).$$

Defining a map $\pi_{\mathcal{M}} : P \times F \to \mathcal{M}$ as

$$\pi_{\mathcal{M}}(p, f) = \pi(p)$$

induces a well-defined **bundle map** $\pi_E : E \to \mathcal{M}$ by

$$\pi_E \circ \pi_{P\times F} = \pi_{\mathcal{M}}, \quad \pi_E([p, f]) = \pi(p).$$

The set

$$E_{m=\pi(p)} := \pi_E^{-1}(m) = \{[p, g \cdot f] \mid g \in \mathcal{G}, f \in F\}$$

is the **fibre of E over** $m = \pi(p) \in \mathcal{M}$.

4.8.2 Differentiable structure

If we take a local trivialisation of $\{U, \psi_U\}$ of P with $m = \pi(p) \in U$ we see that the action of $\mathcal{G}$ on $\pi^{-1}(U) \times F$ is locally

$$(\psi_U \times \mathrm{Id}_F)((p, f) \cdot g) = (\psi_U \times \mathrm{Id}_F)\left(p \cdot g, g^{-1} \cdot f\right) = (\pi(p), hg, g^{-1} \cdot f)$$

where $h = \varphi_U(p)$. So a local trivialisation on P, $\psi_U : \pi^{-1}(U) \to U \times \mathcal{G}$ induces a **local trivialisation on** E, $\chi_U : \pi_E^{-1}(U) \to U \times F$ by

$$\chi_U \circ \pi_{P\times F} = \pi_{1,3} \circ (\psi_U \times \mathrm{Id}_F)$$

where $\pi_{1,3} : U \times \mathcal{G} \times F \to U \times F$ is the natural projection onto $U \times F$. So we see that

$$\chi_U : [p, f] \mapsto (\pi(p), f)$$

This gives a differentiable structure $\{\pi_E^{-1}(U_\alpha), \chi_{U_\alpha}\}_{\alpha\in A}$ turning $\pi_E : E = P \times_{\mathcal{G}} F \to \mathcal{M}$ into a **fibre bundle** with fibre F and base $\mathcal{M}$.

4.8.3 The adjoint bundle

The associated bundle in our research is the adjoint bundle which is $\mathfrak{G} := P \times_{\mathrm{Ad}} \mathfrak{g}$, with the fibre being $\mathfrak{g}$ and $\mathcal{G}$ acting on it through the adjoint representation.

4.8.4 Pullback/ induced bundle (see Tu2 Chapter 20.4 and replace vector with fibre)

More precisely given the data on the principal bundle it is isomorphic as fibre bundles to the pullback bundle of the vector bundle $\mathcal{E} := \mathfrak{G} \oplus TM$.

Definition If $\pi : E \to \mathcal{M}$ is a fibre bundle with fibre F and $f : \mathcal{N} \to \mathcal{M}$ a smooth map then we can define the **pullback bundle** $f^*\pi : f^*E \to \mathcal{N}$. It has fibres $(f^*E)_{x'} = E_{f(x')}$ for $x' \in \mathcal{N}$ and so as a topological space

$$f^*E := \{(x', p) \in \mathcal{N} \times E \mid f(x') = \pi(p)\}$$

endowed with the subspace topology. The projection map projects onto the first factor

$$f^*\pi(x', p) = x'$$

while projection onto the second factor gives the map $h : f^*E \to E$ for which the corresponding diagram commutes. It locally allows the differentiable structure on E to be put on f^*E to turn it into a fibre bundle. By taking a local trivialisation (U, φ) of E one constructs a local trivialisation $(f^{-1}(U), \psi)$ of f^*E by

$$\psi(x', p) = (x', \mathrm{proj}_F \circ \varphi(p)).$$

It has the same fibres as E but with base space $\mathcal{N}$, hence the name pullback. Note that any section of E, $\sigma : \mathcal{M} \to E$, can be pulled back to the **pullback section** $f^*\sigma : \mathcal{N} \to E$ defined by

$$f^*\sigma(x') = (x', \sigma \circ f(x')).$$

4.9 Lifts on Principal Bundles

One way of obtaining a local correspondence between the invariant data on the principal bundle $\pi : P \to \mathcal{M}$ and associated bundle $\pi_E : E = P \times_{\mathcal{G}} F \to \mathcal{M}$ is through lifts. Given a piecewise smooth curve $\gamma : [a, b] \to \mathcal{M}$

parametrised by t, $\tilde{\gamma} : [a,b] \to P$ is a **lift** of γ if

$$\gamma(t) = \pi \circ \tilde{\gamma}(t).$$

It is **horizontal lift** if in addition

$$\tilde{\gamma}'(t) \in \mathcal{H}_{\tilde{\gamma}(t)}$$

for all $t \in [a,b]$. One important property of horizontal lifts is that given a smooth curve $\gamma : [0,1] \to \mathcal{M}$ and $\tilde{\gamma}(0) \in \pi^{-1}(\gamma(0))$, there exists a unique horizontal lift $\tilde{\gamma}$ of γ which starts at $\tilde{\gamma}(0)$.

<u>**Lifts of vector fields**</u> Now we can also lift vector fields on the base $\mathcal{M}$ uniquely to horizontal vector fields of the total space P. To do so note that the projection $\pi : P \to \mathcal{M}$ induces a linear map $\pi_{*,p}|_{\mathcal{H}_p} : \mathcal{H}_p \to T_{\pi(p)}\mathcal{M}$ which we'll denote $\pi_{\mathcal{H},p}$. Also

$$\begin{aligned} \dim \mathcal{H}_p &= \dim T_pP - \dim \mathcal{V}_p \\ &= (\dim T_{\pi(p)}\mathcal{M} + \dim \mathfrak{g}) - \dim \mathfrak{g} = \dim T_{\pi(p)}\mathcal{M} \end{aligned}$$

and $\ker \pi_{*,p} =: \mathcal{V}_p$, so we see that $\pi_{\mathcal{H}}$ is a linear isomorphism. So we can define the **horizontal lift** $\tilde{X}$ of a vector field $X \in \mathfrak{X}(\mathcal{M})$ to be the vector field

$$\tilde{X}_p = \pi^{-1}_{\mathcal{H},p}(X_{\pi(p)}).$$

Equivalently it is the unique horizontal vector field $X \in \mathcal{H}$ such that

$$\pi_* \tilde{X} = X.$$

A key observation is that the horizontal lift of a curve corresponds to the horizontal lift of a vector field. To see this we let $\tilde{X}$ be the horizontal lift of $X \in \mathfrak{X}(\mathcal{M})$ and $\tilde{\gamma}(t)$ be the integral curve of $\tilde{X}$ starting at $p \in P$, then $\tilde{\gamma}$ is the lift of the integral curve of X starting at $x = \pi(p) \in \mathcal{M}$. Indeed

$$\pi \circ \tilde{\gamma}'(t) = \pi_{*,\tilde{\gamma}(t)} \tilde{X}_{\tilde{\gamma}(t)} = X_{\pi \circ \tilde{\gamma}(t)}.$$

Horizontal lifts satisfy the following two key properties: The horizontal lift $\tilde{X}$ of $X \in \mathfrak{X}(\mathcal{M})$ is right invariant and every horizontal smooth right invariant vector field $\tilde{X} \in \mathfrak{X}(P)$ is the lift of a vector field $X \in \mathfrak{X}(\mathcal{M})$.

4.10 Blowdown limits for Principal bundles

Let $\bar{g}(t)$ be a Ricci flow on $\mathcal{G}$-principal bundle $P \to M$ and define $\bar{g}_k(t) = s_k^{-1} \bar{g}(s_k t)$, where $\{s_k\}$ is a sequence of positive numbers such that $\lim s_k = \infty$. Then given a $\mathcal{G}$-principal bundle $P_\infty \to M_\infty$ with bundle isomorphisms $\psi_k : P_\infty \to P$ such that $\psi_k^*(\bar{g}_k)(t)$ uniformly converge to $\bar{g}_\infty(t)$ on $P_\infty \times [j^{-1}, j]$ for all $j > 0$ we call $(P_\infty, \bar{g}_\infty(t))$ a **principal bundle blowdown limit** for $(P, \bar{g}(t))$.

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