AMSI VACATIONRESEARCH SCHOLARSHIPS 2021–22

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Bach Flow

Adam Thompson

Supervised by Dr Ramiro Lafuente

University of Queensland

Vacation Research Scholarships are funded jointly by the Department of Education and the Australian Mathematical Sciences Institute.

Abstract

Geometric evolution equations provide a powerful method of studying the geometry of manifolds by evolving a metric in a particular "direction of improvement". A famous example is the use of the Ricci flow by Perelman to resolve the Poincaré conjecture. The Bach flow is a fourth order geometric flow defined on four-manifolds which arises as the gradient of the Weyl curvature energy functional on compact manifolds. In this paper we study the Bach flow on four-dimensional simply connected, indecomposable nilpotent Lie groups. Combining our results with previous results of Helliwell gives a complete description of the behaviour of the Bach flow on simply-connected nilmanifolds.

Statement of Authorship.

All research towards new results included in this report was carried out by the stated author. Any previously established results are cited appropriately and clearly.

Acknowledgements.

I would like to thank my supervisor Dr Ramiro Lafuente for his advice and encouragement he gave me throughout this project. In particular, his commitment to my development as a mathematician and his explanations of various geometric concepts have benefited not just this report, but my mathematical ability on the whole.

1 Introduction

A central question in Riemannian geometry is if a given manifold M admits some class of metrics which could be considered distinguished in some sense [3]. Of related interest is how we can improve an initial metric g_0 on M. Geometric evolution equations provide a method to deform a metric in a chosen 'direction of improvement' [14]. In this paper we study the Bach flow. The Bach flow is a fourth order geometric flow on 4-manifolds that was introduced by Bahuaud and Helliwell in [1].

In this project, we study the Bach flow on four-dimensional simply connected nilpotent Lie groups which have an indecomposable Lie algebra. The Bach flow has been studied previously on simply-connected Lie groups which are a product of a 3-dimensional unimodular Lie group with $\mathbb{R}$ by Helliwell [7]. In particular, Helliwell determines the behaviour of the Bach flow on simply-connected Lie groups whose Lie algebra is a four-dimensional, decomposable, nilpotent Lie algebra.

Our main result is the following:

Theorem. Let N^4 be a four-dimensional, simply connected, nilpotent Lie group whose Lie algebra is indecomposable. Then, the Bach flow, $g(t)$, beginning at an arbitrary left-invariant metric g on N^4 exists for all $t > 0$ and converges to the Euclidean metric on $\mathbb{R}^4$ in the pointed Cheeger-Gromov sense as $t \to \infty$. Moreover, if $s_t = s(g(t))$ denotes the scalar curvature of $g(t)$ then the normalised metrics

$$\tilde{g}(t) := |s_t| g(t)$$

converge in the pointed Cheeger-Gromov sense to a Bach soliton, g_∞.

We study the Bach flow by following the framework set out by Lauret to study the Ricci flow on a simply connected Nilpotent Lie group [11]. In particular, we apply Lauret's method of varying the bracket [12] rather than varying the metric.

The main challenge we face is the complexity of the Bach tensor due to it being fourth order. To overcome this, we use the symmetries of the system to reduce the number of variables so that we can give an explicit computation of the Bach tensor. We show that up to isometry, any simply connected nilpotent Lie group equipped with a left invariant metric can be described by a three real variables a, b, c (§2.2 and §2.3) and that the Bach tensor is then given by a fourth order polynomial in a, b, c (§3.3). By gauging the flow the solution can be described by a curve $t \mapsto (a(t), b(t), c(t)) \in \mathbb{R}^3$ (§3.2) so our study reduces to the study of ODEs in $\mathbb{R}^3$.

2 Background

There is a deep theory manifolds (c.f [15]), however, since our considerations can essentially be reduced to the study of vector spaces and of $\mathbb{R}^n$ we will largely avoid this to simplify the exposition. The point of view adopted here is taken from [11].

2.1 Nilpotent Lie groups as Euclidean Space

Consider the group $N := \{x \in \mathbb{R}^n : x_i > 0 \,\forall i = 1, ..., n\}$ under component wise multiplication. The Lie algebra of this group is $\mathbb{R}^n$ with the abelian Lie bracket, $\mu(x, y) = 0$ for all $x, y \in \mathbb{R}^n$. The exponential map $\exp : \mathbb{R}^n \to N$ is the diffeomorphism given by

$$\exp(x) = (e^{x_1}, ..., e^{x_n}).$$

We can identify N with $(\mathbb{R}^n, +)$ via exp since

$$\exp(x + y) = \exp(x) \cdot \exp(y) \qquad \forall x, y \in \mathbb{R}^n$$

For any $x \in \mathbb{R}^n$, the left translation map $L(x) : \mathbb{R}^n \to \mathbb{R}^n$, $L(x)y := x + y$, is a diffeomorphism. The differential $dL(x)_0 : T_0\mathbb{R}^n \to T_x\mathbb{R}^n$ is therefore an isomorphism of vector spaces. A metric g on $\mathbb{R}^n$ is left-invariant if the isomorphisms $dL(x)_0$ are isometries, that is, if

$$g(x)(dL(x)_0 v, dL(x)_0 w) = g(0)(v, w), \qquad \forall v, w \in T_0\mathbb{R}^m \simeq \mathbb{R}^n, \forall x \in \mathbb{R}^n.$$

A simply connected Nilpotent Lie group is a generalisation of the above situation. Any $\mu \in \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$ which is *nilpotent* and satisfies the *Jacobi identity* defines a Nilpotent Lie algebra. The bracket μ is nilpotent and satisfies the Jacobi identity if for all $x \in \mathbb{R}^n$ the map $\operatorname{ad}_\mu x : \mathbb{R}^n \to \mathbb{R}^n$ defined by $(\operatorname{ad}_\mu x)y = \mu(x, y)$ is a nilpotent derivation. That is, if for each $x \in \mathbb{R}^n$ we have $(\operatorname{ad}_\mu x)^n = 0$ and

$$\operatorname{ad}_\mu x(\mu(y, z)) = \mu(\operatorname{ad}_\mu x(y), z) + \mu(y, \operatorname{ad}_\mu x(z)), \qquad \forall y, z \in \mathbb{R}^n.$$

The set

$$\mathcal{N}_n = \{\mu \in \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n : \mu \text{ is nilpotent and satisfies the Jacobi Identity.}\} \tag{1}$$

parametrises n dimensional nilpotent Lie algebras (see Section 3 in [11]).

If $(\mathbb{R}^n, \mu)$ is a Lie algebra, there is a unique simply connected Lie group N_μ which has μ as its Lie algebra ([8], Section I.10). When μ is nilpotent, the exponential map $\exp_\mu : \mathbb{R}^n \to N_\mu$ is a diffeomorphism and the Campbell-Baker-Hausdorff formula implies

$$\exp_\mu(x)\exp_\mu(y) = \exp_\mu(x + y + p_\mu(x, y)), \qquad \forall x, y \in \mathbb{R}^n$$

where $p_\mu(x, y)$ is a polynomial in x, y. Therefore, in the same manner as we did above, we can identify N_μ with $\mathbb{R}^n$ under the operation

$$x \cdot_\mu y := x + y + p_\mu(x, y), \qquad \forall x, y \in \mathbb{R}^n.$$

The left translation maps, $L_\mu(x) : \mathbb{R}^n \to \mathbb{R}^n$ defined by $L_\mu(x)y = x \cdot_\mu y$, are again diffeomorphisms. A metric g on $(\mathbb{R}^n, \cdot_\mu)$ is left-invariant if

$$g(x)(dL_\mu(x)_0 v, dL_\mu(x)_0 w) = g(0)(v, w), \qquad \forall v, w \in T_0\mathbb{R}^m \simeq \mathbb{R}^n, \forall x \in \mathbb{R}^n.$$

Left-invariant metric are completely determined by their value at the identity. Therefore, there is a one-to-one correspondence between left-invariant metrics on $(\mathbb{R}^n, \cdot_\mu)$ and inner products on $T_0\mathbb{R}^n \simeq \mathbb{R}^n$.

2.2 Correspondence Between Bracket and Metrics

Recall that the subset $\mathcal{N}_n \subset \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$ defined by (1) parametrises n dimensional nilpotent Lie algebras. There is a natural 'change of basis' action of $\mathrm{GL}_n(\mathbb{R})$ on $\Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$ given by

$$h \cdot \mu = h\mu(h^{-1}\cdot, h^{-1}\cdot), \qquad \forall h \in \mathrm{GL}_n(\mathbb{R}).$$

Observe that two Lie brackets μ_1 and μ_2 define isomorphic Lie algebras if and only if there is a $h \in \mathrm{GL}_n(\mathbb{R})$ such that $h \cdot \mu_1 = \mu_2$.

Remark. If $h \in \mathrm{GL}_n(\mathbb{R})$ takes the basis $\{e_i\}$ to the basis $\{\tilde{e}_i\}$ then $h \cdot \mu = \tilde{\mu}$ if and only if $\mu(e_i, e_j) = \tilde{\mu}(\tilde{e}_i, \tilde{e}_j)$. This follows from

$$h \cdot \mu(\tilde{e}_i, \tilde{e}_j) = h\mu(h^{-1}\tilde{e}_i, h^{-1}\tilde{e}_j) = h\mu(e_i, e_j) = h\mu_{ij}^k e_k = \mu_{ij}^k \tilde{e}_k.$$

Let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{R}^n$. Then any other inner product on $\mathbb{R}^n$ can be written as

$$(\cdot, \cdot) = \langle h\cdot, h\cdot \rangle$$

for some $h \in \mathrm{GL}_n(\mathbb{R})$. If $(\cdot, \cdot)$ is an inner product on $\mathbb{R}^n$, we denote by $g_{\mu,(\cdot,\cdot)}$ the left-invariant metric on $(\mathbb{R}^n, \cdot_\mu)$ which agrees with $(\cdot, \cdot)$ on $T_0\mathbb{R}^n$. When $(\cdot, \cdot)$ is the standard inner product on $\mathbb{R}^n$, we will write g_μ instead of $g_{\mu,(\cdot,\cdot)}$. We then have

Proposition 2.1 ([19], see also Thm 4.1 in [11]). For $\mu, \lambda \in \mathcal{N}_n$ and an inner product $(\cdot,\cdot) = \langle h\cdot, h\cdot\rangle$ on $\mathbb{R}^n$, the metrics g_μ and $g_{\lambda,(\cdot,\cdot)}$ are isometric if and only if $\lambda = h\cdot\mu$. In particular, g_μ and g_λ are isometric if and only if $\lambda \in O(n)\cdot\mu$.

The previous proposition allows us to see the main equivalence which we exploit in this project: The family of left-invariant metrics $g_\mu(t) = g_{\mu,(\cdot,\cdot)_t}$ on the fixed Nilpotent Lie group $(\mathbb{R}^n, \cdot_\mu)$ is isometric at each point in time to the metric $g_{\mu(t)}$ on $(\mathbb{R}^n, \cdot_{\mu(t)})$ where $(\cdot,\cdot)_t = \langle h(t)\cdot, h(t)\cdot\rangle$ and $\mu(t) = h(t)\cdot\mu$. That is, rather than varying the inner product we may instead vary the bracket μ. This equivalence was first exploited by Lauret in [12] to study the Ricci flow on homogeneous manifolds.

Observe that there is a geometric rescaling action of $\mathbb{R}^* = \mathbb{R}^*\cdot I \subset \mathrm{GL}_n(\mathbb{R})$ on $\Lambda^2(\mathbb{R}^n)^*\otimes\mathbb{R}^n$(Section 3.2 in [13]). Under this, we have $c\cdot\mu = c^{-1}\mu$. By Proposition 2.1, the bracket $g_{c\cdot\mu}$ is isometric to the metric which agrees with $\langle cI\cdot, cI\cdot\rangle = c^2\langle\cdot,\cdot\rangle$ at the origin. But this is $c^2 g_\mu$. This says that rescaling the bracket by c^{-1} corresponds to rescaling the metric by a factor of c^2. We can make sense of this as follows: If we start with a fixed bracket μ, then shrinking μ homothetically towards the abelian bracket should flatten $(\mathbb{R}^n, g_\mu)$ (since we are approaching Euclidean space). But to make a metric flatter we need to expand it (think of a sphere, for example).

We also note that the inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^n$ induces the following inner products of $\Lambda^2(\mathbb{R}^n)^*\otimes\mathbb{R}^n$ and $\mathfrak{gl}_n$:

$$\langle\mu,\lambda\rangle = \sum_{i,j=1}^n \langle\mu(e_i,e_j),\lambda(e_i,e_j)\rangle\,,\ \forall\mu,\lambda\in\Lambda^2(\mathbb{R}^n)^*\otimes\mathbb{R}^n \quad \langle\alpha,\beta\rangle = \sum_{i=1}^n\langle\alpha e_i,\beta e_i\rangle = \operatorname{tr}\alpha^t\beta,\ \forall\alpha,\beta\in\mathfrak{gl}_n$$

where $\{e_i\}$ is any orthonormal basis.

2.3 Four-Dimensional Nilpotent Lie algebras

In dimension 4 we can give a concrete description of Nilpotent Lie algebras since they have been classified (Section 5.5 in [5]). By [5], any element $\mu\in\mathcal{N}_4$ is isomorphic to one of the three following Lie algebras:

1. $\mathbb{R}^4$: with the abelien Lie bracket $\mu(x,y) = 0$ for all $x, y\in\mathbb{R}^4$.

2. $\mathbb{R}\oplus\mathfrak{h}_3$: This is the product of the 3 dimensional Heisenberg algebra with $\mathbb{R}$. The non-trivial bracket relations are $\mu(e_1,e_2) = -\mu(e_2,e_1) = e_3$.

3. $\mathfrak{n}_4$: This is not a product Lie algebra. The non-trivial relations are

$$\mu(e_1,e_2) = -\mu(e_2,e_1) = e_3,\quad \mu(e_1,e_3) = -\mu(e_3,e_1) = e_4.$$

These three isomorphism classes correspond to three $\mathrm{GL}_4(\mathbb{R})$ orbits in $\mathcal{N}_4$. Note that 1. and 2. are decomposable Lie algebras.

Suppose now that $\mu\in\mathcal{N}_4$ is a four dimensional Nilpotent Lie algebra and $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{R}^4$. Proposition 2.2 below gives a convenient basis of $\mathbb{R}^4$ for us to work in.

Proposition 2.2. Let $\mu \in \mathcal{N}_4$ such that $\mu \simeq \mathfrak{n}_4$ and let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{R}^4$. Then, there is a orthonormal basis $\{e_i\}$ of $\mathbb{R}^4$ such that

$$\mu(e_1, e_2) = ae_3 + be_4, \qquad \mu(e_1, e_3) = ce_4 \tag{2}$$

for some $a, b, c \in \mathbb{R}$ with $a, c > 0$.

Proof. It follows from Theorems 3.1 and 3.2 in [18] that there is a basis such that (2) holds for some $b \in \mathbb{R}$ and $a, c \neq 0$. To see that we may assume $a, c > 0$, we observe that replacing e_1 with $-e_1$ changes the sign of both a and c and replacing e_4 with $-e_4$ changes the sign of only c. Clearly neither of these break orthonormality. □

Suppose now that $\{\tilde{e}_i\}$ is the standard basis of $\mathbb{R}^n$ and $\tilde{\mu} \in \mathcal{N}_4$ has the relations

$$\tilde{\mu}(\tilde{e}_1, \tilde{e}_2) = -\tilde{\mu}(\tilde{e}_2, \tilde{e}_1) = a\tilde{e}_3 + b\tilde{e}_4, \quad \tilde{\mu}(\tilde{e}_1, \tilde{e}_3) = -\tilde{\mu}(\tilde{e}_3, \tilde{e}_1) = c\tilde{e}_4.$$

Then if $h \in \mathrm{GL}_4(\mathbb{R})$ is the matrix which sends $e_i \to \tilde{e}_i$ we have $h \cdot \mu = \tilde{\mu}$ by Remark . Clearly it also holds that $h \in \mathrm{O}(n)$, so $(\mathbb{R}^n, g_\mu)$ and $(\mathbb{R}^n, g_{\tilde{\mu}})$ are isometric Riemannian manifolds by Proposition 2.1.

Let us define

$$\mathcal{O} = \{\mu \in \mathcal{N}_4 : \mu = \mu_{a,b,c} \text{ with respect to the standard basis of } \mathbb{R}^4\}. \tag{3}$$

It follows from the above discussion that if $(\mathbb{R}^n, g_\mu)$ is a simply connected Nilpotent Lie group with left-invariant metric g_μ, then up to isometry we may assume that $\mu \in \mathcal{O}$. Clearly we can identify $\mathcal{O}$ with the set

$$\{(a, b, c) \in \mathbb{R}^3 : 0 < a, c\}.$$

We will use this in §3 in order to reduce our study of the bracket flow (Definition 7). In particular, our problem reduces to the study of an ODE in an open subset of $\mathbb{R}^3$.

It will also be useful to have an explicit description of an arbitrary derivation $D \in \mathrm{Der}(\mu)$ of a bracket $\mu \in \mathcal{O}$. We can obtain this by differentiating the description of an automorphism of $\mathfrak{n}_4$ given in [18].

Lemma 2.1. Let $\mu = \mu_{a,b,c} \in \mathcal{O}$. Then, any derivation $D \in \mathrm{Der}(\mu)$ has the form

$$D = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ * & \beta & 0 & 0 \\ * & a\gamma + b\beta & \alpha + \beta & 0 \\ * & * & c\gamma & 2\alpha + \beta \end{pmatrix}.$$

For convenience, we summarise our progress in Proposition 2.3 below. Note that if g is a left-invariant metric on a simply connected nilpotent Lie group N^4, then $(N^4, g) = (\mathbb{R}^4, g_{\tilde{\mu},,(\cdot,\cdot)})$ where $(\cdot, \cdot) := g(e)$ is the inner product induced by restricting g to $\mathrm{Lie}(N)$ and $\tilde{\mu}$ is the Lie bracket of $\mathrm{Lie}(N)$.

Proposition 2.3. Let $(\mathbb{R}^n, g_{\tilde{\mu},(\cdot,\cdot)})$ be an arbitrary four dimensional, simply connected Nilpotent Lie group equipped with a left-invariant metric $g_{\tilde{\mu},(\cdot,\cdot)})$. Then, there exists $\mu = \mu_{a,b,c} \in \mathcal{O}$ such that $(\mathbb{R}^n, g_{\tilde{\mu},(\cdot,\cdot)})$ is isometric to $(\mathbb{R}^n, g_\mu)$. That is, up to isometry, we can describe any four dimensional simply connected Lie group with left-invariant metric by a bracket $\mu = \mu_{a,b,c} \in \mathcal{O}$.

Of course, there will be no guarantee that a solution to the bracket flow (7), to be introduced later on, will remain in $\mathcal{O}$. However, we will show in §3.3 that we can gauge our flow so that the solution does remain within $\mathcal{O}$.

3 The Homogeneous Bach Flow

3.1 Bach Flow

Let M^4 be a four dimensional Riemannian manifold. A one parameter family of metrics, $(g(t))_{t\in I}$, is said to evolve by *Bach Flow* if

$$\frac{\partial}{\partial t}g = \mathrm{Bac}(g) + \frac{\Delta s}{12}g, \quad \forall t \in I \qquad g(0) = g_0, \tag{4}$$

where $\mathrm{Bac}(g)$ is the Bach tensor given in local coordinates by

$$\mathrm{Bac}(g)_{ij} = \nabla^k\nabla^l W_{kijl} + \frac{1}{2}R^{kl}W_{ikjl},$$

and W is the Weyl tensor, $R^{kl} = g^{ka}g^{lb}R_{ab} = g^{ka}g^{lb}(\mathrm{Rc})_{ab}$ are the components of the Ricci tensor with both indices raised and $s = \mathrm{tr}_g \mathrm{Rc}$ is the scalar curvature.

The Bach tensor arises in the study of manifolds which are conformally Einstein (Section 6 in the survey article [9]). In particular, Theorem 6.6 in [9] says $\mathrm{Bac}(g) \equiv 0$ is a necessary condition for (M^4, g) to be conformal to an Einstein manifold (the converse holds if g is conformal to a metric with harmonic Weyl tensor, see Corollary 6.8 in [9]). The Bach tensor is trace and divergence free (proof of this can be found in [6]).

On a compact manifold the Bach tensor is -4 times the gradient of the Riemannian functional

$$g \longmapsto \int_M |W|_g^2 \, dv(g)$$

(4.76 in [3]), note that the definition given here is only agrees with their definition up to a factor of -4). As a result of this, if $(M^4, g(t))$ is a compact solution to (4), the L^2 norm of the Weyl tensor decreases along the flow:

$$\frac{d}{dt}\|W(g(t))\|_{L^2(M,g)} = -4\|B\|^2_{L^2(M,g)} \leq 0 \tag{5}$$

Short time existence for the flow (4) was shown by Bahuaud and Helliwell in [1]. Moreover, Bahuaud and Helliwell showed uniqueness for (4) on compact manifolds [2].

The Bach flow has been studied on Homogeneous manifolds whose universal cover has a product structure (that is, $(\widetilde{M}, g) = (N_1 \times N_2, g_1 \oplus g_2)$ where (N_i, g_i) are Lie groups of lower dimension) by Helliwell in [7]. Helliwell uses Milnor frames to diagonalise the Ricci and Bach tensor and then studies the resulting ODE's.

3.2 The Bach Flow on a Nilpotent Lie group

Suppose now that $(M^4, g) = (\mathbb{R}^4, g_\mu)$ is a Nilpotent Lie group with left-inavriant metric and that we have a $(\mathbb{R}^n, \cdot_\mu)$-invariant solution $\{g(t)\}_{t\in(a,b)}$ to the Bach flow with $g(0) = g_\mu$ (a solution $\{g(t)\}_{t\in(a,b)}$ is $(\mathbb{R}^n, \cdot_\mu)$-invariant if it is for all $t \in (a, b)$). Since g, $\mathrm{Bac}(g)$ and s are determined by their values at $0 \in \mathbb{R}^n$,$\Delta s = 0$ and

$\mathrm{Bac}(g(t))$ is determined by $\mathrm{Bac}(g(t))(0)$ for all $t \in (a,b)$. By evaluating at the identity, we find that $\mathrm{Bac}(g(t))(0)$ satisfies the ODE

$$\frac{d}{dt}\langle\cdot,\cdot\rangle_t = \mathrm{Bac}(\langle\cdot,\cdot\rangle_t) \qquad \langle\cdot,\cdot\rangle_0 = g(0). \tag{6}$$

Here, we have written $\mathrm{Bac}(\langle\cdot,\cdot\rangle_t) = \mathrm{Bac}(g(t))(0)$. Conversely, given a solution $(\cdot,\cdot)_t$ to the ODE (6) we obtain a $(\mathbb{R}^n, \cdot_\mu)$-invariant solution $\{g(t)\}$ to the Bach flow by defining $g(t) = g_{\mu,(\cdot,\cdot)}$ for all t. By the usual existence and uniqueness of ODE's, we are guaranteed a unique G-invariant solution [13]. The need for this reasoning is that uniqueness of the Bach flow is an open problem on general manifold.

Let us fix once and for all initial metric $g_0 = g_{\mu_0}$ which is invariant under a Nilpotent Lie group $(\mathbb{R}^4, \cdot_{\mu_0})$. Note that we have we have tacitly assumed that the initial inner product on $T_0\mathbb{R}^n$ is the standard inner product, but this is not an issue since up to isometry this is always the case (this follows from Proposition 2.1 in §2.2).

By the above discussion, there is a unique curve of left-invariant metrics of inner products $(\langle\cdot,\cdot\rangle_t$ on $\mathbb{R}^n$ satisfying (6) which corresponds to the unique $(\mathbb{R}^4, \cdot_{\mu_0})$-invariant solution of the Bach flow beginning at g_{μ_0}. It follows that for each t there is a $h(t) \in \mathrm{GL}_4(\mathbb{R})$ such that

$$\langle\cdot,\cdot\rangle_t = \langle h(t)\cdot, h(t)\cdot\rangle .$$

One can show that the one parameter family of matrices $\{h(t)\}$ can be chosen to be a smooth curve (see section 4.1 in [13] or Proposition 3.1 below). The corresponding curve of brackets is given by $\mu(t) = h(t) \cdot \mu_0$. The bracket flow, introduced by Lauret to study Ricci flow on homogeneous manifolds in [12], is motivated by considering what equation the curve $\mu(t) \in \mathrm{GL}_4(\mathbb{R}) \cdot \mu_0$ satisfies. For more examples of the bracket flow technique see [11, 4, 17].

Definition 3.1. Let $(\mathbb{R}^4, g_{\mu_0})$ as above. The bracket Bach flow is the ODE

$$\frac{d}{dt}\mu = \frac{1}{2}\pi(B_\mu)\mu, \qquad \mu(0) = \mu_0 \tag{7}$$

where B_μ is defined by $\mathrm{Bac}(g_{\mu(t)})(0) = g(0)(B_\mu\cdot,\cdot)$ is the Bach tensor determined by μ and π is the representation defined by

$$\pi(A)\mu := A\mu - \mu(A\cdot,\cdot) - \mu(\cdot, A\cdot), \qquad \forall A \in \mathfrak{gl}_n.$$

Remark. A few remarks are in order.

1. The representation π is the derivative of the $\mathrm{GL}_n(\mathbb{R})$ on $\Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$ action defined in §2.2. That is,

$$\pi(A)\mu = \left.\frac{d}{ds}\right|_s e^{sA} \cdot \mu, \qquad \forall A \in \mathfrak{gl}_n.$$

2. The solution $\mu(t)$ of (7) remains in the orbit $\mathrm{GL}_4(\mathbb{R}) \cdot \mu_0$. This is because

$$\frac{1}{2}\pi(B_\mu)\mu = \left.\frac{d}{ds}\right|_s e^{s\frac{1}{2}B_\mu} \cdot \mu \in T_\mu(\mathrm{GL}_4(\mathbb{R}) \cdot \mu) \subset \mathcal{N}_4.$$

(see Lemma 3.2 in [12]).

Beginning at our inital metric g_{μ_0} we now have two families of Riemannian manifolds

$$(\mathbb{R}^4, g(t)) \qquad (\mathbb{R}^4, g_{\mu(t)})$$

where $g(t)$ is the unique $(\mathbb{R}^n, \cdot_{\mu_0})$-invariant solution of the Bach flow and $\mu(t)$ is the solution of the bracket flow (7). The next proposition shows that these are equivalent in a precise way.

Proposition 3.1 ([11], Theorem 5.1)**.** Let $(\mathbb{R}^4, g(t))$, $(\mathbb{R}^4, g_{\mu(t)})$ be solutions of the homonegeous Bach flow and the bracket flow respectively. Then, there exists a family of isomorphisms $\varphi(t) : (\mathbb{R}^4, \cdot_{\mu_0}) \to (\mathbb{R}^4, \cdot_{\mu(t)})$ such that

$$g(t) = \varphi(t)^* g_{\mu(t)} \quad \forall t \in I.$$

Moreover, $d\varphi(t) = h(t)$ where $h(t)$ is the solution of any of the following ODEs

1. $h' = \frac{1}{2} h B(\langle \cdot, \cdot \rangle_t), \quad h(0) = I$
2. $h' = \frac{1}{2} B_{\mu(t)} h, \qquad h(0) = I$

Moreover, this satisfies

1. $\langle \cdot, \cdot \rangle_t = \langle h\cdot, h\cdot \rangle$
2. $\mu(t) = h \cdot \mu_0$.

For a proof of Proposition 3.1, one should consult Theorem 5.1 in [11] (note that the proof in [11] is for the Ricci flow however only symmetry of the Ricci tensor is used). In particular, Proposition 3.1 shows that the solutions of (4) and (7) have the same maximal interval of existence and the same curvature (see the Remark after Theorem 3.3 in [12]).

Recall from §2.2 that if $\lambda \in \mathrm{O}(n) \cdot \mu$ then the metrics g_μ and g_λ were isometric. This was due to the fact that if $h \in \mathrm{GL}_4(\mathbb{R})$ gives rise to the inner product $(\cdot, \cdot) = \langle \cdot, \cdot \rangle_0$ and $k \in \mathrm{O}(n)$ then $hk \in \mathrm{GL}_4(\mathbb{R})$ gives rise to the same inner product. It will be useful to exploit this $\mathrm{O}(n)$ equivariance when studying the bracket flow (7). Böhm and Lafuente describe this as a refinement of Uhlenbeck's trick of moving frames (see Sections 2 and 3 in [4]).

Proposition 3.2 ([4], Proposition 3.1)**.** Let $R : \mathrm{GL}_4(\mathbb{R}) \cdot \mu_0 \to \mathfrak{so}(4)$ be a smooth map and let $\mu(t), \bar{\mu}(t)$ denote respectively solutions to the bracket flow (7) and to the modified bracket flow

$$\frac{d\bar{\mu}}{dt} = \frac{1}{2} \pi (B_\mu - R_\mu) \bar{\mu}, \qquad \bar{\mu}(0) = \mu_0. \tag{8}$$

Then, there is a smooth curve $\{k(t)\} \subset \mathrm{O}(\mathfrak{g}, \langle \cdot, \cdot \rangle$ such that $\bar{\mu} = k \cdot \mu$.

In particular, the solutions $\mu, \bar{\mu}$ to the bracket flow (7) and the gauged bracket flow (8) have the same maximal interval of existence and the same curvature.

3.3 Behaviour of The Flow on Simply Connected Nilpotent Lie group

Proposition 3.3. The Bach tensor of $\mu = \mu_{a,b,c} \in \mathcal{O}$ is given by the following matrix

$$B_\mu = \begin{pmatrix} b_1 & 0 & 0 & 0 \\ 0 & b_2 & b_5 & 0 \\ 0 & b_5 & b_3 & b_6 \\ 0 & 0 & b_6 & b_4 \end{pmatrix}, \tag{9}$$

where

$$\begin{aligned}
b_1 &= \frac{1}{8}\left(4a^4 + 8a^2b^2 - a^2c^2 + 4b^4 + 8b^2c^2 + 4c^4\right), \\
b_2 &= \frac{1}{24}\left(12a^4 + 24a^2b^2 - a^2c^2 + 12b^4 + 8b^2c^2 - 4c^4\right), \\
b_3 &= \frac{-1}{24}\left(20a^4 - a^2c^2 + 24a^2b^2 + 4b^4 - 8b^2c^2 - 12c^4\right), \\
b_4 &= \frac{1}{24}\left(-4a^4 + 3a^2c^2 - 8a^2b^2 - 20\left(b^2 + c^2\right)^2\right), \\
b_5 &= \frac{2}{3}bc\left(a^2 + b^2 + c^2\right), \\
b_6 &= -\frac{2}{3}ab\left(a^2 + b^2 + c^2\right),
\end{aligned}$$

Since the Bach tensor of a simply connected nilmanifold is determined by the actions of $\langle \cdot, \cdot \rangle$ and ∇ on $T_0\mathbb{R}^n \simeq \mathbb{R}^n$, Proposition 3.3 follows from a computation. This was carried out using Mathematica (see Appendix).

Remark. From the equations in Proposition 3.3 we can see the following:

1. We can see explicitly the rescaling formula for the Bach tensor $B_{c\cdot\mu} = c^4 B_\mu$.

2. The expressions for b_5 and b_6 show that B_μ is diagonal if and only if $b = 0$.

3. It is interesting to note that
$$b_1 = \left|W_\mu\right|^2 \geq 0.$$
It is not clear to us why this is the case.

We have noted in §2.3 that the solution $\mu(t)$ to the bracket flow (7) beginning at $\mu_0 \in \mathcal{O}$ may not remain in $\mathcal{O}$. However, we have also seen in §2.3 that for any $\mu \in \mathcal{N}_4$, the orbit $\mathrm{O}(4) \cdot \mu$ intersects $\mathcal{O}$. That is to say, for each t we can find a $k(t) \in \mathrm{O}(4)$ such that $k(t) \cdot \mu(t) \in \mathcal{O}$. Since μ and $k \cdot \mu$ determine isometric Riemannian manifolds for $k \in \mathrm{O}(4)$, we may use $\mathrm{O}(4)$ to gauge our flow, readjusting at each point in time to ensure that the solution remains in $\mathcal{O}$. We formalise this in Proposition 3.4 below by appealing to Proposition 8 in §3.2.

Proposition 3.4. Let $\mu_0 = (a_0, b_0, c_0) \in \mathcal{O}$ and let $B_\mu(t)$ be the Bach tensor along the solution $\mu(t)$ of (7). Then, the solution $\bar{\mu}(t)$ of the gauged bracket flow (8) remains in $\mathcal{O}$ where the gauging, R_μ, is given by

$$R_\mu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & b_5 & 0 \\ 0 & -b_5 & 0 & b_6 \\ 0 & 0 & -b_6 & 0 \end{pmatrix}.$$

Proof. To show that the symmetries are preserved, it suffices to show that $\dot{\mu}_{ij}^k = 0$ whenever $i < j$ and $(i,j,k) \notin \{(1,2,3),(1,2,4),(1,3,4)\}$ since then μ_{ij}^k will solve the system $\dot{u} = 0, u(0) = 0$ and hence be $u \equiv 0$ by uniqueness. Here a dot denotes a derivative with respect to time (i.e $\dot{} := d/dt$).

The effect of gauging is that $B_\mu - R_\mu = L_\mu$ is lower triangular for all t. With respect to the basis, $\{e_i\}$, (8) is

$$\dot{\mu}_{ij}^k = \frac{d}{dt}\langle \mu(e_i,e_j), e_k\rangle = \langle L_\mu \mu(e_i,e_j) - \mu(L_\mu e_i, e_j) - \mu(e_i, L_\mu e_j), e_k\rangle \tag{10}$$
$$= \sum_{l=1}^{4}\left(\mu_{ij}^l L_l^k - L_i^l \mu_{lj}^k - L_j^l \mu_{il}^k\right) = \sum_{i<j\le l\le k}\left(\mu_{ij}^l L_l^k - L_i^l \mu_{lj}^k - L_j^l \mu_{il}^k\right).$$

(Note that we are only summing over l). The last equality follows since $B_l^k = 0$ for $l > k$ since it is lower triangular and $\mu_{il}^k = -\mu_{li}^k = 0$ for $l > k$ by our choice of structure constants. For $k = 1,2$ the right hand side is zero since each term will have a factor of μ_{ij}^1 or μ_{ij}^2, all of which are equal to 0. If $k = 3$, then the only triples (i,j,k) we need to check are $(i,3,3)$ for $i = 1,2$. But

$$\dot{\mu}_{i3}^3 = \sum_{i<3\le l\le 3}\left(\mu_{ij}^l L_l^k - L_i^l \mu_{lj}^k - L_j^l \mu_{il}^k\right) = \mu_{i3}^3 L_3^3 - L_i^3 \mu_{33}^3 - L_3^3 \mu_{i3}^3 = 0.$$

If $k = 4$ then we must check $(2,j,4)$ for $j = 3,4$. This is

$$\dot{\mu}_{2j}^4 = \sum_{2<j\le l\le 4}\left(\mu_{ij}^l L_l^k - L_i^l \mu_{lj}^k - L_j^l \mu_{il}^k\right) = \mu_{2j}^l L_l^4 - L_2^l \mu_{lj}^4 - L_j^l \mu_{2l}^4 = 0$$

since $\mu_{2j}^k = 0$ for $j > 2$ and $\mu j l^4 = 0$ for $j,l > 2$. $\square$

Therefore, the bracket flow (7) is equivalent to the following ODE for $\mu = (a,b,c) \in \mathcal{O}$:

$$\begin{cases} a' &= \frac{-a}{48}(44a^4 + 72a^2b^2 - 5a^2c^2 + 28b^4 + 24b^2c^4 - 4c^4) \\ b' &= \frac{-b}{48}(60a^4 + 104a^2b^2 + 57a^2c^2 + 44b^4 + 104b^2c^2 + 60c^4) \\ c' &= \frac{-c}{48}(-4a^4 + 24a^2b^2 - 5a^2c^2 + 28b^4 + 72b^2c^2 + 44c^4) \end{cases} \tag{11}$$

With these in hand we are in a position to study the long time behaviour of the flow.

Lemma 3.1. The following evolutions hold along the gauged bracket flow:

$$\frac{d}{dt}\log\frac{a}{c} = (c^2 - a^2)\|\mu\|^2, \tag{12}$$
$$\frac{d}{dt}\frac{b^2}{a^2} \le \frac{-2}{3}\frac{b^2}{a^2}\|\mu\|^4, \tag{13}$$
$$\frac{d}{dt}\|\mu\|^2 \le \frac{-1}{12}\|\mu\|^6. \tag{14}$$

Proof. 1. Since $a,c > 0$, $a/c > 0$ so we can that its logarithm. Taking derivatives gives

$$\frac{d}{dt}\log\frac{a}{c} = \frac{d}{dt}\log a - \frac{d}{dt}\log c = \frac{a'}{a} - \frac{c'}{c} = (c^2 - a^2)\|\mu\|^2.$$

2. Taking the derivative gives
$$\frac{d}{dt}\frac{b^2}{a^2} = 2\frac{b}{a}\frac{ab' - a'b}{b^2} = 2\frac{b^2}{a^2}\left(\frac{b'}{b} - \frac{a'}{a}\right).$$
The claim then follows since
$$\left(\frac{a'}{a} - \frac{b'}{b}\right) = \frac{-1}{24}(8\|\mu\|^4 + 3c^2(5a^2 + b^2 + c^2)) \leq \frac{-1}{3}\|\mu\|^4.$$
3. Similarly to 2,
$$48\frac{d}{dt}\|\mu\|^2 = -44\|\mu\|^6 + 3a^2c^2(47a^2 + 53b^2 + 47c^2).$$
Now, we observe that by the multinomial theorem
$$\|\mu\|^6 = (a^2 + b^2 + c^2)^3 \geq a^6 + c^6 + 3a^4c^2 + 3a^2c^4 + 6a^2b^2c^2.$$
Hence,
$$\begin{aligned}-40\|\mu\|^6 + 3a^2c^2(47a^2 + 53b^2 + 47c^2) &\leq -40a^6 + 21a^4c^2 + 21a^2c^4 - 40c^6\\ &= (a^2 + c^2)(-40a^4 + 61a^2c^2 - 40c^4)\\ &\leq (a^2 + c^2)(-40a^4 + 80a^2c^2 - 40c^4) = -40(a^2 + c^2)(a^2 - c^2)^2 \leq 0.\end{aligned}$$
Therefore,
$$\frac{d}{dt}\|\mu\|^2 \leq \frac{-1}{48}\cdot 4\|\mu\|^6 = \frac{-1}{12}\|\mu\|^6.$$
□

Corollary 3.1. The Bach flow on a four dimensional simply connected Nilpotent Lie group is immortal, that is, the maximal interval of existence contains $(0, \infty)$.

Proof. Since $\|\mu\|^2$ decreases along the flow, it remains within the closed ball of radius $\|\mu_0\|$ which is a compact set. □

4 Self Similar Solutions to the Bach Flow

4.1 Solitons of The Bach Flow

In this section we study *solitons* of the Bach flow. A Bach soliton is metric which cannot be 'improved' by deforming the metric in the direction of the Bach tensor (see the Introduct of [14]). Formally, a Bach soliton is a Riemannian manifold (M^4, g) such that

(15) $$\mathrm{Bac}(g) = \lambda g + \mathcal{L}_X g$$

for a constant $\lambda \in \mathbb{R}$, and a complete vector field, $X \in \mathfrak{X}(M)$ ($\mathcal{L}_X$ is the Lie derivative in the direction X). The soliton is called expanding, steady or shrinking if $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$ respectively. If the vector field arises as the gradient of a potential function $u \in C^\infty(M)$, then (15) becomes

(16) $$\mathrm{Bac}(g) = \lambda g + 2\,\mathrm{Hess}\, u.$$

In this case, we say the soliton is a gradient soliton. Homogeneous gradient Bach solitons were studied by Griffin in [6]. Griffin applies Theorem 3.6 from [16] to reduce the study to Riemannian products of manifolds of dimension less than 4.

Proposition 4.1 ([16], [6])**.** Let (M^4, g) be a homogeneous Riemannian manifold. If there is a non-constant function $u \in C^\infty(M)$ such that

$$\mathrm{Bac}(g) = \lambda g + 2\,\mathrm{Hess}\, u, \qquad \lambda \in \mathbb{R}$$

then M splits isometrically as a product $(M, g) = (M' \times \mathbb{R}^k, g' \oplus \overline{g})$ where $\overline{g}$ is the Euclidean metric and $k > 0$.

We deduce from Proposition 4.1 that any non-product solitons we find cannot be gradient.

Bach solitons are self similar solutions in the following sense. If (M^4, g_0) is a Bach soliton, then there is function $\lambda : (a, b) \to \mathbb{R}$ with $\lambda(0) = 1$ and a family of diffeomorphisms $\varphi(t) \in \mathrm{Diff}(M)$ with $\varphi_0 = \mathrm{Id}_M$ such that

$$g(t) := \lambda(t)\varphi(t)^* g_0 \tag{17}$$

is a solution to the Bach flow with $g(0) = g_0$ (see for instance Theorem 4.10 in [13]). Since $\varphi(t) : M \to M$ is trivially an isometry between $(M, \varphi(t)^* g_0)$ and (M, g_0), the family of metrics in (17) are isometric to the initial metric g_0 up to the scale factor $\lambda(t)$.

4.2 Bach Solitons on Nilpotent Lie groups

If $(M^4, g) = (\mathbb{R}^n, g_\mu)$ is a simply connected Nilpotent Lie group, then an analogous condition to the metric evolving self similarly is that the solution $\mu(t)$ to the bracket flow evolves only by scaling (see the discussion before Theorem 6 in [13]). If we consider a solution of this form, $\mu(t) = \lambda(t) \cdot \mu_0$ then taking a derivative we find

$$\pi(B_\mu) = \frac{d}{dt}\mu(t) = \frac{d}{dt}\lambda(t) \cdot \mu_0 = \pi(\lambda'(t)I)$$

(Recall that our scaling is given by $\lambda \cdot \mu = (\lambda I) \cdot \mu$ for $c \in \mathbb{R}^*$.) Therefore, we find

$$0 = \pi(B_\mu - \lambda I),$$

for some $\lambda \in \mathbb{R}$. Since $\pi(A) = 0$ if and only if $A \in \mathrm{Der}(\mu)$, this is equivalent to

$$B_\mu = \lambda I + D, \qquad \lambda \in \mathbb{R}, D \in \mathrm{Der}(\mu). \tag{18}$$

A simply connected Nilpotent Lie group $(\mathbb{R}^n, g_\mu)$ whose Bach endomorphism satisfies (18) is called an *algebraic Bach solitons.* Algebraic solitons were introduced by Lauret to study Ricci Nilsolitons (Ricci solitons on Nilpotent Lie groups) in [10]. An important observation made by Lauret is that an algebraic soliton is indeed a soliton in the sense of (15). To see this we observe that for simply connected Lie groups an automorphism is determined by its differential at the identity. If $D \in \mathrm{Der}(\mu)$, then $\exp D \in \mathrm{Aut}(\mu)$ where exp is the usual matrix exponential. Therefore, for each $t \in \mathbb{R}$ we integrate $\exp tD \in \mathrm{Aut}(\mu)$ to a Lie group automorphism.This one parameter family of automorphisms $\{\varphi_t\}$ generates a vector field X which will satisfy (15). This discussion is summarised below.

Proposition 4.2 (Theorem 6, [13])**.** For a simply connected Nilpotent Lie group $(\mathbb{R}^n, g_{\mu_0})$ the following are equivalent:

1. The solution to the bracket flow (7) beginning at μ_0 is given by

$$\mu(t) = \lambda(t) \cdot \mu_0, \quad \text{for some } \lambda(t) > 0,\ \lambda(0) = 1.$$

2. The operator B_μ associated to the Bach tensor satisfies (18).

Moreover, whenever either of these conditions hold, the Riemannian manifold $(\mathbb{R}^n, g_{\mu_0})$ is a Bach soliton.

Note that in general, not all solitons are algebraic solitons. Since we have an explicit description of what a derivation $D \in \operatorname{Der}(\mu)$ looks like for a bracket $\mu \in \mathcal{O}$, the notion of an algebraic soliton reduces our search for a soliton to simply solving a system of equations in terms of a, b, c, and the components of D (note that since B_μ is trace free, $\lambda = -\operatorname{tr} D/4$). In fact, we can reduce the difficulty of this system further.

Lemma 4.1. If $\mu = (a, b, c) \in \mathcal{O}$ is an algebraic soliton, then B_μ is diagonal.

Proof. Since $D = B_\mu - cI$, $D \in \operatorname{Der}(\mu)$ must be symmetric as the difference of two symmetric matrices. But by Lemma 2.1, D is lower triangular. Hence, D must be diagonal and so $B_\mu = cI + D$ must also be diagonal. □

Since B_μ is diagonal if and only if $b = 0$, this allows us to set $b = 0$ when searching for algebraic solitons. By Lemma 2.1, a diagonal derivation has eigenvalues $\alpha, \beta, \alpha + \beta, 2\alpha + \beta$ for some $\alpha, \beta \in \mathbb{R}$. Therefore, existence and uniqueness of algebraic solitons reduces to existence and uniqueness of solutions to a set of polynomials in a, b, α, β.

Theorem 1. The bracket $\mu \in \mathcal{O} \subset \mathcal{N}_4$ given by

$$\mu(e_1, e_2) = e_3 \quad \mu(e_1, e_3) = e_4$$

is a Bach soliton. Moreover, this soliton is a non-gradient expanding soliton and is the unique Bach soliton up to isometry and scaling within the orbit $\mathrm{GL}_4(\mathbb{R}) \cdot \mu \subset \mathcal{N}_4$.

Proof. By Proposition 4.2, for μ to be a soliton it suffices to show that (18) holds. By setting $a = c = 1$ and $b = 0$ in Proposition 3.3 it is not difficult to check that (18) is satisfied for $\alpha = -7/12$ and $\beta = -7/6$. This gives $\lambda = -21/16 < 0$ so the soliton is expanding. The soliton is not of gradient type due to Proposition 4.1.

To see uniqueness we assume that (18) holds for $a, b, c, \alpha, \beta \in \mathbb{R}$ with $(a, b, c) \in \mathcal{O}$ and $a \neq c$, $b = 0$. We then show that this leads to a contradiction. □

5 Normalised Bach Flow and Convergence

5.1 Normalised Bach Flow

Let $r : [0, T) \to \mathbb{R}$ be a smooth function, where $T \in (0, \infty]$ is the maximal existence time for (4). The

$r-$normalised Bach flow is the equation

$$\frac{\partial}{\partial t}g(t) = \mathrm{Bac}(g(t)) + r(t)g(t), \quad g(0) = g_0. \tag{19}$$

Remark. The trace free property of the Bach tensor implies that if $(M, g^r(t))$ is a compact solution of (19) then

$$\frac{d}{dt}\int_M |W|^2\, dv(g) = -4(\mathrm{Bac}(g(t)) + r(t)g(t), \mathrm{Bac}(g(t))) = -4\|\mathrm{Bac}(g(t))\|^2.$$

In the varying brackets perspective (19) becomes the following.

Definition 5.1. An r-normalised bracket Bach flow for a normalisation function $r : \mathbb{R} \to \mathbb{R}$ is a curve $\{\mu^r(t)\} \subset \mathcal{N}_4$ such that

$$\frac{d}{dt}\mu^r = \frac{1}{2}\pi(B_{\mu^r} - rI)\mu^r, \quad \mu(0) = \mu_0. \tag{20}$$

Remark. There is an equivalence between (19) and (20) which is analogous to the case of un-normalised flows given in Proposition 3.1 ([11], Section 7).

The usefulness of (19) and (20) is that the addition of the r term allows us to keep a keep a geometric quantity fixed along the flow. Moreover, the next proposition shows that solutions of the normalised flows only differ from the solutions of the original flows by a scaling and parametrisation of time.

Proposition 5.1. Let $\mu(t)$ and $\mu^r(t)$ be solutions of the bracket flow (7) and the $r-$normalised bracket flow (20) respectively. Then, there are functions $\tau : [0, \infty) \to [0, \infty)$, $\lambda : [0, \infty) \to \mathbb{R}$ such that

$$\mu^r(t) = \lambda(t)\mu(\tau(t)) \qquad \forall t \in [0, \infty). \tag{21}$$

The fucntions τ and λ are the solutions of the ODE's

$$\tau' = \lambda^4, \quad \tau(0) = 0 \qquad\qquad \lambda' = \frac{1}{2}r\lambda, \quad \lambda(0) = 1. \tag{22}$$

Proof. Let $\mu(t)$ be a solution to (7) and define

$$\mu^r(t) = \lambda(t)\mu(\tau(t))$$

where τ, λ are the solutions of (22). Clearly $\mu^r(0) = \mu_0$. Differentiating gives

$$\begin{aligned}\frac{d}{dt}\mu^r(t) = \frac{d}{dt}(\lambda(t)\mu(\tau(t))) = \lambda'\mu(\tau(t)) + \lambda\tau'\frac{d}{dt}\Big|_{\tau(t)}\mu &= \frac{1}{2}r\lambda\mu(\tau(t)) + \frac{1}{2}\lambda^5\pi(B_\mu)\mu \\ &= \frac{1}{2}r\mu^r + \frac{1}{2}\pi(B_{\mu^r})\mu^r = \frac{1}{2}\pi(B_{\mu^r} - rI)\mu^r\end{aligned}$$

where we have used that the Bach tensor scales by $B_{\lambda\cdot\mu} = \lambda^4 B_\mu$. □

Corollary 5.1. Let $u : \Lambda^2(\mathbb{R}^4)^* \otimes \mathbb{R}^4 \to \mathbb{R}$ be scale invariant (i.e. $u(c\mu) = u(\mu)$ for any $c \in \mathbb{R}^*$) and smooth away from 0. Then u increases (resp. decreases) along a solution of the bracket flow of and only if it u increases (resp. decreases) along a solution of the normalised bracket flow.

Proof. Let μ, μ^r be solutions to the bracket flow and r-normalised bracket flow respectively, $u(t) = u(\mu(t)), u^r(t) = u(\mu^r(t))$ the restriction of u to these solutions. Then, $\mu^r(t) = c(t)\mu(\tau(t))$, so that

$$u^r(t) = u(\overline{\mu}(t)) = u(c(t)\mu(\tau(t))) = u(\tau(t)).$$

Differentiating both sides gives

$$\frac{d}{dt}u^r = \tau' \frac{d}{dt}u.$$

Since $\tau' = c^4 > 0$, the claim follows. □

5.2 Scalar Curvature Normalisation

One useful normalisation function is

$$r = \frac{-\langle \pi(B_\mu)\mu, \mu\rangle}{\|\mu\|^2}.$$

With this normalisation, we have

$$\frac{d}{dt}\|\mu\|^2 = 2\left\langle \frac{1}{2}\pi(B_\mu - rI)\mu, \mu \right\rangle = \langle \pi(B_\mu)\mu, \mu\rangle - \frac{\langle \pi(B_\mu)\mu, \mu\rangle \|\mu\|^2}{\|\mu\|^2} = 0.$$

Therefore, the norm of μ remains constant (and consequently so does the scalar curvature since $s_\mu = -\|\mu\|^2/4$ on Nilpotent Lie groups, c.f. 7.39 in [3]).

Since the quantities a/c and b^2/a^2 are scale invariant, we can use Lemma 3.1 and Corollary 5.1 to determine the behaviour of the normalised flow.

Theorem 2. Let $\mu_0 \in \mathcal{O}$ with $\|\mu_0\| = 1$. Then, the normalised Bracket flow

$$\frac{d}{dt}\mu = \frac{1}{2}\pi(B_\mu - \alpha_\mu I)\mu, \qquad \mu(0) = \mu_0$$

where $\alpha_\mu = -\langle \pi(B_\mu)\mu, \mu\rangle$ converges to an expanding soliton μ_∞ as $t \to \infty$.

Proof. Let $\mu_0 = (a_0, b_0, c_0) \in \mathcal{O}$ with $a_0^2 + b_0^2 + c_0^2 = 1$. The claim amounts to showing that $a, c \to \sqrt{2}/2$ and $b \to 0$ as $t \to \infty$.

We first show that $b \to 0$. By Lemma 3.1 and Corollary 5.1, the quantity b^2/a^2 is monotone decreasing. Since $b^2/a^2 \geq 0$ it must converge to some limit as $t \to \infty$. But then, $d/dt(b^2/a^2)$ must converge to zero as $t \to \infty$ which implies $b^2/a^2 \to 0$ as $t \to 0$ since $\tau' = c^4$ is bounded away from 0 for large t.

Next, we have that

$$\frac{d}{dt}\log\frac{a}{c} = \tau'(c^4 - a^4)\|\mu(\tau(t))\| = \lambda^3(c^4 - a^4).$$

Observe that if $a(t_0) = c(t_0)$ for some $t_0 \geq 0$, then $a \equiv c$ for all $t > t_0$ by uniqueness of ODE solutions. Assume $a(t) > c(t)$ for all t (resp. $a(t) < c(t)$). In this case, $\log\frac{a}{c}$ is bounded and monotone and hence must be convergent. But then $d/dt \log\frac{a}{c} \to 0$ as $t \to \infty$ so $c^4 - a^4 \to 0$. Since $a^2 + c^2 \to 1$ as $t \to \infty$, it must hold that

$$\lim_{t\to\infty} a = \lim_{t\to\infty} c = \frac{\sqrt{2}}{2}.$$

□

6 Conclusion

Our results for the Bach flow presented here should be compared with the corresponding results for the Ricci flow on simply connected nilmanifolds [11]. Similar to the Ricci flow case, the Bach flow on simply connected nimanifolds always exists for all positive time and the normalised flow converges to an expanding soliton.

In order to study the Bach flow in greater generality, one would need to have an approach which dealt with the complexitity of the Bach tensor. Since the Bach tensor is the gradient of a functional on a compact manifold, and the norm of the Weyl tensor is constant on homogeneous manifolds, it would be interesting to understand the relationship between B_μ and

$$\mu \mapsto |W_\mu|^2.$$

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Appendix: Mathematica Code

The following Mathematica code computes the Bach tensor of a simply connected, four-dimensional Lie group.

```
g := DiagonalMatrix[{1, 1, 1, 1}];(*The Metric*)
invg := Inverse[g];(*Inverse Metric*)

c :=
 ReplacePart[ConstantArray[0, {4, 4, 4}],(*Structure Constants of the Lie Algebra*)
 {{1, 2, 3} -> a, {1, 2, 4} -> b, {1, 3, 4} -> \gamma}]; (*\gamma is 'c' in the report*)
c = c - Transpose[c];

G[i_, j_, m_] := G[i, j, m] = (1/2) (c[[i, j, m]]
                   - Sum[g[[l, i]]*c[[j, k, l]]*invg[[m, k]], {k, 1, 4}, {l, 1, 4}]
               + Sum[g[[l, j]]*c[[k, i, l]]*invg[[m, k]], {k, 1, 4}, {l, 1, 4}]);
(*Defines the Christoffel Symols*)

Rm[i_,j_,k_,l_] := Rm[i,j,k,l] =
   Sum[g[[l, p]] (Sum[G[j, k, q]*G[i, q, p] - G[i, k, q]*G[j, q, p]
        - c[[i, j, q]]*G[q,k,p], {q, 1, 4}]), {p, 1, 4}];
(*(0,4) Curvature*)
```

```
R[i_,j_,k_] := (Sum[G[j,k,q]*G[i,q,p] - G[i,k,q]*G[j,q,p]
                - c[[i, j, q]]*G[q, k, p], {q, 1, 4}]);
(*(1,3) Curvature*)
Ric[i_, j_] :=  Ric[i, j] = Sum[R[p, i, j], {p, 1, 4}];
(*Ricci Curvature*)
S := Sum[invg[[i, j]]*Ric[i, j], {i, 1, 4}, {j, 1, 4}];
(*Scalar Curvature*)

P[i_,j_] := (1/2) (Ric[i,j] - (S/6)*g[[i,j]]);
(*Schouten Tensor*)

Pog[i_,j_,l_,m_] := Pog[i,j,l,m] =
   (P[i, m]*g[[j,l]] + P[j,l]*g[[i,m]] - P[i,l]*g[[j,m]] - P[j,m]*g[[i,l]]);
   (*Kulkarni-Nomizu Product of P and g*)


W[i_,j_,k_,l_] := W[i,j,k,l] = Rm[i,j,k,l] - Pog[i,j,k,l];
(*Weyl Curvature*)

DP[i_,j_,k_] := DP[i,j,k] =Sum[-G[k,i,p]*P[p,j] - G[k,j,p]*P[i,p], {p, 1, 4}];
DRic[i_,j_,k_]:= DRic[i,j,k] = Sum[-G[k, i, p]*Ric[p, ] - G[k, j, p]*Ric[i,p], {p, 1, 4}];

CovP[k_,l_] := CovP[k, l] = Sum[invg[[p, k]]*invg[[q, l]]*P[p, q], {p, 1, 4}, {q, 1, 4}];
 (*Schouten Tensor with an index rasied*)

Cotton[i_,j_,k_] := Cotton[i,j,k] = (DP[i, j, k] - DP[i, k, j]);
DC[i_,j_,k_,l_] :=DC[i,j,k,l] = Sum[-G[l,i,p]*Cotton[p,j,k]
                - G[l,j,p]*Cotton[i,p,k] - G[l,k,p]*Cotton[i, j, p], {p, 1, 4}];
(*Covariant derivative of the Cotton Tensor*)



b[i_,j_] := b[i, j] = Sum[invg[[l,q]]*DC[i,j,l,q], {l, 1, 4}, {q, 1, 4}] +
                                    Sum[CovP[k,l]*W[k,i,j,l], {k, 1, 4}, {l, 1, 4}]
  (*Bach Tensor*)
```