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How American Option Optimal Exercise Boundary is Affected by Transaction Costs

Jeremy Sterjovski

Supervised by Dr. Xiaoping Lu

University of Wollongong

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Abstract

Fischer Black and Myron Scholes developed an option pricing scheme colloquially referred to as the 'Black-Scholes model', relying on hedging to produce a well studied Partial Difference equation with associated boundary conditions. Although Black-Scholes model is flexible enough to price several types of options, the unrealistic assumptions that underpin this option pricing model mean it has limited use in non-perfect markets that exist in reality. This project will involve studying the effects relaxing the no transaction cost assumption has on this model. Analysis of the corresponding effect on American option price and optimal exercise boundary from holder's perspective following the inclusion of transaction costs will be conducted.

1 Introduction

Options are a financial derivative, entitling the holder a right but not requirement to exercise its rights (whether it be buy or sell an asset). The two simplest forms of options are vanilla puts and calls, with these additionally being divided based on when the holder can exercise these rights. American options entitle the holder to exercise these rights anytime between settlement and a pre-determined exercise date. With this right to exercise early there exists a set of share values in which early exercise becomes economical for holder of option, with its boundary known as the optimal exercise boundary (Wilmott et al. 1995). The main motivator of this report will be to study in detail Leland's approach on relaxing the no transaction cost assumption in Black-Scholes model (Black & Scholes 1973) with proportional transaction costs (Leland 1985). Using Leland's model and Finite Difference Methods (FDM) (Seydell 2015) implemented in Financial Instruments MATLAB Toolbox (MathWorks 2021), it is considered how varying transaction cost rates and re-hedging interval affects price holder is willing to pay for option and their optimal exercise boundary. Finally, consideration is placed on pricing these American options from writer's perspective. (Lu et al. 2022)

1.1 Statement of Authorship

Lu developed this project's outline and key topics covered. Lu supervised this project. Sterjovski studied in detail the derivation of option pricing model with and without transaction costs with the support of Lu. Sterjovski used pre-built FDM code in Financial Instruments MATLAB Toolbox to price American options with transaction costs. Code was then created by Sterjovski to form the optimal exercise boundary from these numerical results. With the support of Lu these numerical results were then analysed from a financial perspective by Sterjovski. AMSI funded the project.

2 Option Pricing Fundamentals

This section aims to introduce the standard assumptions used by Fischer Black and Myron Scholes to price options. Additionally, the hedging strategy used to derive PDE for pricing options under these assumptions is studied in detail along with associated boundary conditions for American and European options. Finally, it is shown how Black-Scholes hedging method of pricing options is flexible enough to relax the no dividend paid assumption.

2.1 Black-Scholes Model

To price options under Black-Scholes model seven assumptions are taken for mathematical and simplifying reasons (Black & Scholes 1973):

Markets are perfectly liquid with no Transaction Costs for Re-Hedging- it is assumed that any portfolio re-hedging interactions with market do not incur any charges. Markets are additionally fully liquid with all market orders executed.

No Dividends Paid- underlying stock does not pay any dividends, this assumption can however easily be relaxed in proceeding sections.

Short-Selling Permitted- the short-selling of stocks are permitted and this short-selling is divisible into any real number.

No Arbitrage Opportunities- this assumes abnormal returns cannot be earned from a portfolio of assets. For example, if portfolio is fully predicatble with no risk it is assumed it can only earn the risk-free rate. This effectively states that no mis-pricing occurs within market.

Trading of Shares Occurs Continuously- trades of shares can occur at all points in time, with transactions immediately executed.

Shares follow a log-normal random walk- this model fits real time series data to a high degree of accuracy, especially for indices and equities (Wilmott et al. 1995). Share model is then given by:

$$\frac{dS}{S} = \sigma dX + \mu dt,\, dX = \phi\sqrt{dt} \tag{1}$$

Here ϕ is a random variable sampled from the standard normal distribution and σ is the share volatility, controlling the effect randomness has on the growth of a stock. Additionally μdt represents the expected return an investor can expect to earn from investing in the share based on historical returns over a given time-step. Conversely, σdX can be considered the asset response to external factors that cannot be anticipated.

Sigma and risk-free rate remain fixed over time- volatility and rate are assumed to not vary.

As options are financial derivatives their value is derived on the price movements of the underlying share. This share price is however stochastic, preventing us to know deterministically what share price will be at any time. To price an option the aim then becomes to remove the randomness of share price via hedging and obtain an equation that can be used to determine the value of option at any point in time over the life of an option.

From a holders perspective Black and Scholes accomplished this by developing a portfolio consisting of one long option and -Δ shares to create a porfolio with following value,

$$\Pi = V - \Delta S \tag{2}$$

Over a time step dt, the change in the value of portfolio is then given by,

$$d\Pi = dV - \Delta dS \tag{3}$$

where Δ remains fixed from t to $t + dt$, as holder of portfolio will only re-hedge shares held after dt time interval. Applying Ito's lemma to dV obtain,

$$dV = \sigma S \frac{\partial V}{\partial S} dX + (\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t})dt \tag{4}$$

Substituting Eq. (4) into Eq. (3) and collecting like terms leads to the expression;

$$d\Pi = \sigma S(\frac{\partial V}{\partial S} - \Delta)dX + (\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu \Delta S)dt \tag{5}$$

The random component in Eq. (5) can then be completely eliminated by selecting Δ such that it is equal to the value of $\frac{\partial V}{\partial S}$ at the start of the time step dt, i.e. at time t. Note, in above formulation that S and V are both driven by the random variable component dX. Thus, Black and Scholes exploited this fact to create a third variable (Black & Scholes 1973)(the portfolio of option and share, Π) to remove randomness. Substituting this selection for Δ into Eq. (5) the portfolio return simplifies to,

$$d\Pi = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt \tag{6}$$

The effect of selecting Δ in this manner and removing randomness means for a given time-step the return from portfolio Π is deterministic and wholly predictable. This effectively means that there is no risk contained within this portfolio, then as a result of the no-arbitrage assumption this portfolio's return has to be equivalent to the return one would earn from investing the value of the portfolio in a risk-free security. Taking r to be the risk-free rate, over time-step dt return earnt from Eq. (6) will be equivalent to $r\Pi dt$. This then results in;

$$r\Pi dt = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt \tag{7}$$

Rearranging and substituting value of portfolio $\Pi = V - \frac{\partial V}{\partial S} S$ gives,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0,\ 0 < S < \infty \tag{8}$$

This is referred to as the Black-Scholes PDE. Additionally, here derivation was taken from the holder's perspective however same procedure could be followed for the writer with a portfolio holding opposite positions, i.e. $\Pi = \Delta S - V$, leading to the same result. It should be noted above Black-Scholes PDE holds not only for puts and calls but for any portfolio of options, highlighting the flexibility of this model.

2.1.1 Black-Scholes Hedging Strategy

The hedging strategy utilised here to develop the Black-Scholes model is referred to dynamic Delta-Hedging (Wilmott et al. 1995). In dynamic Delta-Hedging, the Π portfolio is only instantenously risk-free. This Delta-hedged postion hence needs to be continuously monitored and appropriately adjusted for all points in time. The idea behind this form of hedging is to continuously rebalance the portfolio of option and stock so overall the change of the portfolio with respect to the share is always 0 at any point in time, i.e. $\frac{\partial \Pi}{\partial S} = 0$. Thus, when Delta-Hedging between an option and share, the position taken is called 'Delta-neutral' as sensitivity of hedged portfolio to share is 0. This hedging method also means that writer and holder both have their position fully covered in the case holder's right is exercised. For example, if holder exercises option that requires writer to purchase a share, holder will have exactly 1 share to provide to writer and correspondingly writer will have an opposite short position in that stock. This is expected as delta-hedging is a risk-free trading strategy until the expiration of the option. However, when there are transaction costs for rehedging such Dynamic-Hedging is not possible as continuous trading would fully erode the value of portfolio with no return earnt.

2.2 American & European Options

To price options using the Black-Scholes PDE boundary conditions are required. As European options can only be exercised on one pre-determined exercise date, they have the simplest associated boundary conditions. That is, conditions are the terminal value of option is equal to the associated value that can be earnt from the rights an option entitles holder to and present value of the option at t if the share price reaches $S = 0$. As a result, European calls have boundary conditions:

$$V(S,T) = max(S - E, 0),\ V(0,t) = 0,\ V(S,t) \sim S \text{ as S } \rightarrow \infty \tag{9}$$

European puts have boundary conditions:

$$V(S,T) = max(E - S, 0),\ V(0,t) = Ee^{-r(T-t)},\ V(S,t) \rightarrow 0 \text{ as S } \rightarrow \infty \tag{10}$$

From these boundary conditions European calls and puts both have well-defined analytical solutions for all points in time and share prices. As the solution for European options are known they are not included in this report but rather noted they exist. In the more complex case of American options, previous boundary conditions hold however the right to exercise option early produces free-boundary conditions. Consider the case of an American put option with an exercise price of E. For such an option there will be an optimal exercise price, $S_f(t)$. In the case the price of share at a given point in time t lies in the region $0 \leq S < S_f(t)$ the holder will view early exercise of the option as being economical. The value of the option at this point will then be equal to the value earnt from exercising the option, V(S, t) = E - S. This region where early use of rights is optimal is then referred to as the exercise region, with the option then closed. Conversely, when the value of share is in the region $S_f(t) < S < \infty$, the holder of the option will view it as economical to not exercise and wait for share price to fall further. This is referred to as the continuation region in which case the value of the option then satisfies the Black-Scholes PDE at that point in time:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,\ S_f(t) < S < \infty \tag{11}$$

For an American put at optimal exercise boundary $S_f(t)$, the free-boundary conditions then arise. First condition is known as the smooth pasting condition and follows from arbitrage arguments that gradient of option is continuous at this optimal exercise boundary. The second condition can be thought of as determining the value of the option at this optimal price with the first condition determining the location (Wilmott et al. 1995) where this price occurs:

$$\frac{\partial V}{\partial S}(t, S_f(t)) = -1,\ V(t, S_f(t)) = E - S_f(t) \tag{12}$$

For an American call, exercise region is $S_f(t) < S < \infty$ and continutation region is $0 \leq S < S_f(t)$ with the associated free-boundary conditions being:

$$\frac{\partial V}{\partial S}(t, S_f(t)) = 1\ ,\ V(t, S_f(t)) = S_f(t) - E \tag{13}$$

Figure 1 shows an optimal exercise boundary for an American call option. For prices in the blue region the

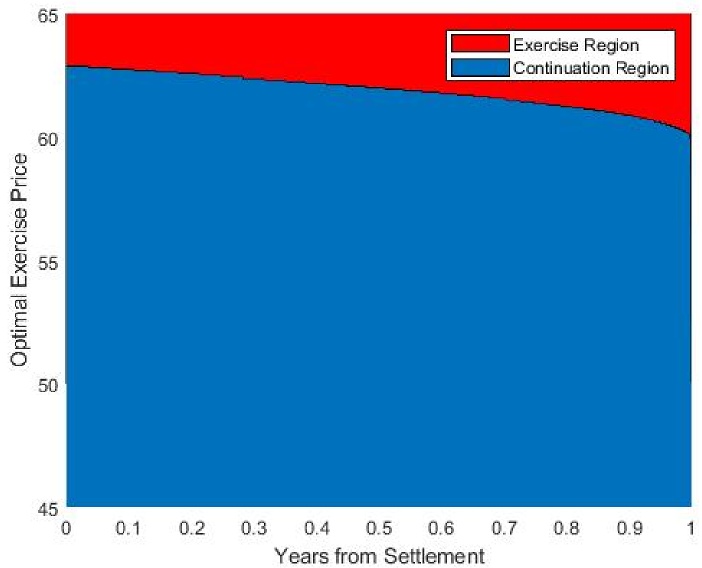

Figure 1: Example of Optimal Exercise Boundary

holder of option will view early exercise as not being economical and will choose to hold the option and wait for share price to increase further. Conversely, in red zone holder will view early exercise as being economical and will close out the option by exercising its rights early. Due to this unknown optimal boundary $S_f(t)$, from the ability to exercise option early, the PDE problem becomes a non-linear moving boundary problem. This is a difficult mathematical problem with no analytical solution available apart from special cases, unlike European options. As a result to price American options many numerical techniques exist to approximate their value, with Finite Difference Method being widely used and easy to implement, being used in this report.

2.2.1 Black-Scholes Model with Dividends

Certain assumptions within Black-Scholes model can be relaxed with these changes being captured in the process of deriving the Black-Scholes PDE (Wilmott et al. 1995). 2 examples of these include:

- Risk-free rate and volatility can be made deterministic functions of time rather than being a fixed constant.
- Assumption of shares paying no dividends can be relaxed.

Dividend assumption will be relaxed in this report, with model used to price American call options. For simplicity, it is assumed that dividends are paid continuously and proportional to the value of shares held by an investor at that point in time. For example if dividend rate is D at a given point in time, share value is S and an investor holds v shares, the return from dividends earnt by investor over a given time-step dt is given by $DvSdt$. As investor now has an additional source of return the lognormal random walk of a share is adjusted to account for this:

$$\frac{dS}{S} = \sigma dX + (\mu - D)dt,\ dX = \phi\sqrt{dt} \tag{14}$$

The same notations exist in Eq. (14) as prior however the deterministic return of the share is adjusted down by the amount of dividends earnt over a given time interval. This need to adjust down the return that can be earnt from the share again is a result of the no-aribtrage assumption previously stated. To price options with dividends again a portfolio is created with a long position in option and Δ shorted in share, earning following return over dt:

$$d\Pi = dV - \Delta dS - DS\Delta dt \tag{15}$$

As the holder of option shorts Δ shares in portfolio, dividends have to be paid out to party providing shorted stocks, thus representing a drain on the portfolio return. As the additional dividend return in portfolio is of coefficient dt they do not affect selection of delta-hedging strategy as only dX has to removed to hedge portfolio Π. Thus, following the same dynamic delta-hedging as under Black-Scholes, the risk-free rate is earnt by portfolio leading to the modified Black-Scholes PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S\frac{\partial V}{\partial S} - rV = 0,\ 0 < S < \infty \tag{16}$$

The same boundary conditions for European option and free-boundary conditions for American options hold under this Black-Scholes model with dividends, with European options again having well-defined analytical solutions. Additionally, for American call options at expiry optimal exercise boundary will always converge to following value independent of σ when $r > D_0 > 0$:

$$S_f(T) = \frac{Er}{D_0} \tag{17}$$

This is the price in which the return from the asset that can be earnt, D_0S, exeeds the risk-free return that can be earnt from investing exercise price, Er, making it economical for holder to exercise the option at this point (Wilmott et al. 1995). Additionally, it is observed as the dividend rate goes to 0 optimal exercise boundary becomes undefined in which case early exercise of the option is never optimal, resulting in American call then reverting to a European call. Thus, dividend model is introduced to facilitate pricing of American call options in Section 4.

3 Option Pricing with Transaction Costs

With the framework for pricing options developed this section aims to consider how relaxing transaction cost assumption affects previous Black-Scholes derivation (Leland 1985). This is accomplished by formulating the American option PDE with transaction costs proportional to transaction size following LeLand (Li & Abdullah 2012) approach from the holder's and writer's perspective.

3.1 Black-Scholes with Transaction Costs

The previous no transaction cost assumption is now relaxed to better reflect the market imprefections that exist within real-world transactions. The costs associated with re-hedging may include a bid-offer spread on the underlying share, i.e. the difference between the lowest price seller is willing to accept for a share and the highest price a buyer is willing to pay for a share. Such types of transaction costs that are independent of the time scale of re-hedging period would lead to an infinite amount of costs incurred for keeping portfolio in Section 2.1 hedged at all points in time until expiry of option. These significant costs that are accrued in the prescence of rehedging costs leads to a breakdown of Black-Scholes model derivation, requiring an alternate method to produce a tractable model.

With the addition of transaction costs it is expected that the option writer and holder may not agree on the price of an option unlike previous models derived. This arises as both parties want to recover their costs from re-hedging, resulting in different prices between party's. Leland proposed the following modifications to assumptions in Black-Scholes PDE to price options with continuous rehedging changed to discrete (Wilmott et al. 1995) :

- The portfolio is now rehedged every δt points in time rather than $dt \to 0$. This rehedging interval is then selected by investor so rehedging occurs for discrete points in time over life over option an example may be investor choosing to rehedge once every second or third day.
- Log-normal random walk for share is adjusted such that return is earnt over discrete points in time:

$$\frac{\delta S}{S} = \sigma \delta X + \mu \delta t,\ \delta X = \phi \sqrt{\delta t} \tag{18}$$

- Transaction costs from buying or selling a share are proportional to the value of the transaction in market. Consider if v shares are bought or sold in market, transaction costs incurred by investor are then $kS|v|$. k is kept constant, varying for the investor holding the portfolio.
- Hedged portfolio return is now only *expected* to earn risk-free rate rather than actually earning risk-free return. As hedging is only done at discrete points in time under this approach, an investor may not

actually fully remove risk from portfolio for a given point in time but overall expects risk-free rate to be earnt.

With the continuous hedging adjusted to discrete to account for transaction costs, consideration will now be placed on how such options can be priced. Options will be priced from the perspective of a holder and then writer.

3.1.1 Holder Price

To price an option with transaction costs from the holder's perspective a portfolio is constructed as in Eq. (2). The difference however is that return of portfolio becomes $\delta\Pi$ over a discrete time-step δt and additionally includes the drain on the portfolio from re-hedging costs:

$$\delta\Pi = \delta V - \Delta\delta S - kS|v| \tag{19}$$

Where v is the amount of share bought or sold over the period t to $t + \delta t$. Absolute sign is necessary to ensure that transaction costs are always a drain on the portfolio. Following same procedure as in Section 3.1, Ito's lemma is again applied to relate δV to its random component ϕ. However, as hedging is now discrete, a slight modification is made to Eq. (4) with ϕ^2 now added to $\frac{\partial^2 V}{\partial S^2}$:

$$\delta V = \sigma S\frac{\partial V}{\partial S}\phi\sqrt{\delta t} + (\mu S\frac{\partial V}{\partial S} + \phi^2\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t})\delta t \tag{20}$$

Note in Eq. (20) as δt does not tend to 0 we cannot replace the random variable ϕ^2 with its expected value 1 as in Eq. (4) (Wilmott et al. 1995). Once again substituting Eq. (20) into Eq. (19) and grouping like terms obtain:

$$\delta\Pi = \sigma S(\frac{\partial V}{\partial S} - \Delta)\phi\sqrt{\delta t} + (\mu S\frac{\partial V}{\partial S} + \phi^2\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu\Delta S)\delta t - kS|v| \tag{21}$$

Again select Delta-Hedging strategy to remove risk from this portfolio. As a result, the amount of shares sold from t to $t + \delta t$ is the difference between the shares held at end and start of this interval, that is:

$$v = \frac{\partial V}{\partial S}(S + \delta S, t + \delta t) - \frac{\partial V}{\partial S}(S, t) \tag{22}$$

Can then apply multivariate taylor theorem around (S, t) to the first term in Eq. (22) to obtain:

$$\frac{\partial V}{\partial S}(S + \delta S, t + \delta t) = \frac{\partial V}{\partial S}(S, t) + \delta S\frac{\partial^2 V}{\partial S^2}(S, t) + \delta t\frac{\partial^2 V}{\partial S\partial t}(S, t) + ... \tag{23}$$

$$\Rightarrow v = \delta S\frac{\partial^2 V}{\partial S^2}(S, t) + \delta t\frac{\partial^2 V}{\partial S\partial t}(S, t) + ... \tag{24}$$

Recalling that $\delta S = \sigma S\sqrt{\delta t} + \mu S\delta t$; for δt close to 0 terms of order $\sqrt{\delta t}$ will be dominant. Thus, dropping terms that are not of this leading order obtain the approximation:

$$v \approx \frac{\partial^2 V}{\partial S^2}\sigma S\phi\sqrt{\delta t} \tag{25}$$

As prior approximation still contains the random variable ϕ, an investor cannot be sure of the total amount of shares transacted over a given time-step and thus transaction costs incurred. As a result, we find the *expected* total amount of transaction cost that are accrued over our re-hedging period:

$$E[kS|v|] = E[kS|\frac{\partial^2 V}{\partial S^2}\sigma\phi\sqrt{\delta t}S|]$$

As σ, S and $\sqrt{\delta t}$ will be positive by defintion, absolute values are removed from these values. Additionally, from linearity of the expectation operator these constants are subsituted out yielding:

$$\Rightarrow kS^2\sigma E[|\phi|]|\frac{\partial^2 V}{\partial S^2}|$$

Finding value of $E[|\phi|]$:

$$E[|\phi|] = \int_{-\infty}^{\infty} \frac{|\phi|e^{\frac{\phi^2}{2}}}{\sqrt{2\pi}}d\phi = -\int_{-\infty}^{0} \frac{\phi e^{\frac{\phi^2}{2}}}{\sqrt{2\pi}}d\phi + \int_{0}^{\infty} \frac{\phi e^{\frac{\phi^2}{2}}}{\sqrt{2\pi}}d\phi$$

Using integral subsitution it is then trivial to see $\int_{-\infty}^{0} \frac{\phi e^{\frac{\phi^2}{2}}}{\sqrt{2\pi}}d\phi = -\frac{1}{\sqrt{2\pi}}$ and $\int_{0}^{\infty} \frac{\phi e^{\frac{\phi^2}{2}}}{\sqrt{2\pi}}d\phi = \frac{1}{\sqrt{2\pi}}$. Thus, total expected amount of transaction costs over a given re-hedging period is then:

$$E[kS|v|] = \sqrt{\frac{2}{\pi}}k\sigma S^2|\frac{\partial^2 V}{\partial S^2}|\sqrt{\delta t} \tag{26}$$

It should be noted in Eq. (26) the absolute value remains around the $\frac{\partial^2 V}{\partial S^2}$. This value can be thought of as the rate of change the delta with respect to S as $\Delta = \frac{\partial V}{\partial S}$. This absolute value is due to the fact for a portfolio of options this value may be negative at a given point in time, however for a single vanilla put or call (which is the consideration of this report) this value will always be positive, with absolute values able to be dropped. Now with selection of Δ as $\frac{\partial V}{\partial S}$ Eq. (21) becomes:

$$\delta\Pi = (\phi^2\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t})\delta t - kS|v| \tag{27}$$

It can be seen however that a component of randomness still exists Eq. (27) via ϕ^2. Thus, again take the expectation for what the portfolio should earn over a given time step, using Eq. (26) and fact $E[\phi^2] = 1$:

$$E[\delta\Pi] = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - \sqrt{\frac{2}{\pi\delta t}}k\sigma S^2\frac{\partial^2 V}{\partial S^2})\delta t \tag{28}$$

Due to rehedging then expect this return to be equal to risk-free return that can be earnt, $E[\delta\Pi] = r(V - \frac{\partial V}{\partial S})\delta t$. Substituting this and re-arranging gives:

$$\frac{\partial V}{\partial t} + \frac{1}{2}S^2\frac{\partial^2 V}{\partial S^2}(\sigma^2 - 2k\sigma\sqrt{\frac{2}{\pi\delta t}}) + rS\frac{\partial V}{\partial S} - rV = 0,\ 0 < S < \infty \tag{29}$$

Thus, Eq. (29) is the PDE for holder of option when proportional transaction costs are included. Should be noted this the same Black-Scholes PDE in Eq. (8) with volatility adjusted such that $\sigma^2_{\text{Holder}} = \sigma^2 - 2k\sigma\sqrt{\frac{2}{\pi\delta t}}$.

3.1.2 Writer Price

Option with transaction costs can be priced from writer perspective by taking opposite position to holder, that is create a portfolio with short position in option and Δ long in shares, that is $\Pi = \Delta S - V$. Considering rate of change as done for the holder's perspective, obtain:

$$d\Pi = \Delta \delta S - \delta V - kS|v| \tag{30}$$

Applying Eq. (20) and grouping terms:

$$\delta\Pi = \sigma S(\Delta - \frac{\partial V}{\partial S})\phi\sqrt{\delta t} + (\mu \Delta S - \mu S\frac{\partial V}{\partial S} - \phi^2 \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \frac{\partial V}{\partial t})\delta t - kS|v| \tag{31}$$

Again selecting the delta-hedging method and using Eq. (26) for the expected transaction costs over a re-hedging period and taking overall expectation of Eq. (31) obtain:

$$E[\delta\Pi] = (-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial S^2} - \sqrt{\frac{2}{\pi \delta t}} k\sigma S^2 \frac{\partial^2 V}{\partial S^2})\delta t \tag{32}$$

Now subsituting the risk-free rate of the portfolio over time-step δt:

$$r(\frac{\partial V}{\partial S} - V)\delta t = (-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \sqrt{\frac{2}{\pi \delta t}} k\sigma S^2 \frac{\partial^2 V}{\partial S^2})\delta t$$

$$\Rightarrow \frac{\partial V}{\partial t} + \frac{1}{2}S^2 \frac{\partial^2 V}{\partial S^2}(\sigma^2 + 2k\sigma\sqrt{\frac{2}{\pi \delta t}}) + rS\frac{\partial V}{\partial S} - rV = 0,\ 0 < S < \infty \tag{33}$$

Again the Black-Scholes equation with modified variance is arrived at, with modified variance now being $\sigma^2_{\text{Writer}} = \sigma^2 + 2k\sigma\sqrt{\frac{2}{\pi \delta t}}$.

From the derivations from the prespective of holder and writer it has been shown the Black-Scholes PDE with modified variance has been reached under Leland's approach with proportional transaction costs. Thus, as buyer and seller have different measures of volatility for the same option this will lead to a pricing mismatch on the expected fair value of the option between these two parties. The financial implications for why this occurs and how varying costs and re-hedging period affects value of American call options will be explored in Section 4. It should also be noted that free-boundary conditions and boundary conditions remain the same as in Section 2.2. This arises from the delta-hedging method employed here, whenever option is exercised be it early in time under American or at expiry under European, writer and holder will have their positions fully covered. For example, if a call option is exercised writer will hold 1 stock in portfolio to provide to holder and holder will have 1 stock shorted with the position closed when the option is exercised. Thus, due to the nature of delta-hedging whenever an option is exercised writer and holder have no interaction with the market, thus, allowing the boundary and free-boundary conditions to remain the same.

4 Results & Discussions

Financial Instruments MATLAB Toolbox (MathWorks 2021) was used here to price American call options with dividends to consider how prices are affected by increases in transaction costs and decreases in rehedging interval. It is then considered what effect this correspondingly has on the option's optimal exercise boundary over time, considering the financial implications of these results. It should be noted that in this report analysis is restricted to call options with dividends however studies could easily be extended into American puts with similar results being produced. Options priced have the parameters; T = 1, E = \$50, $D_0 = 0.05$, $r = 0.06$, $\sigma = 0.1$ and $\delta t = 0.01$.

4.1 American Call Price

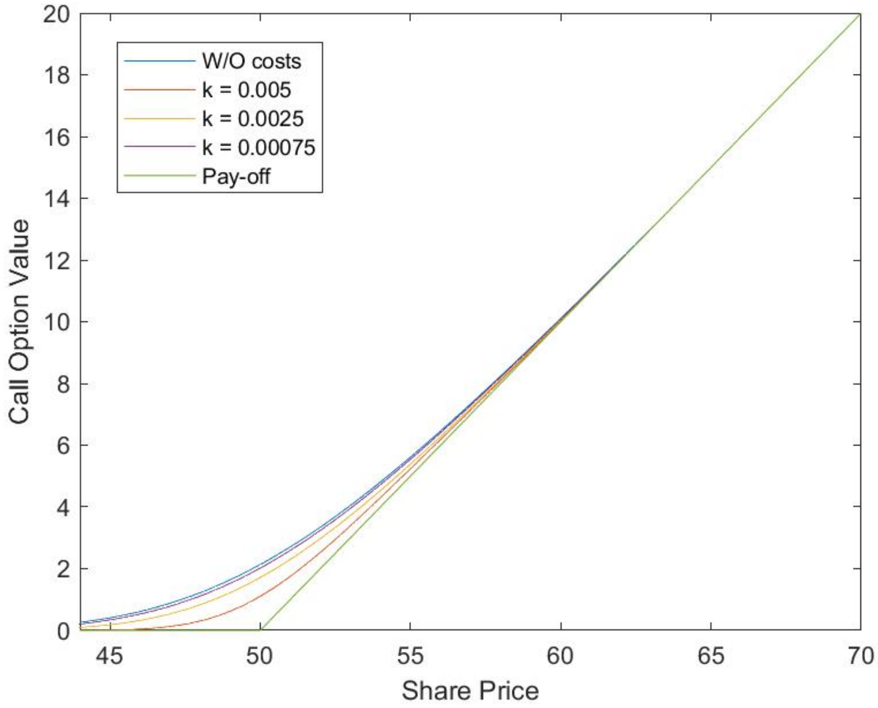

Figure 2: American Call Price with $\delta t = 0.01$

Figure 2 considers the effect of increasing the transaction cost rate on the price of an American option from the holder's perspective for a fixed re-hedging peiod, $\delta t = 0.01$. As the cost of re-hedging increases more of the portfolio wealth held by an investor will be eroded by these costs. The holder of option will then demand compensation for these costs incurred and the only way this can be facilitated is by decreasing the price they would be willing to pay for an option. Thus, it is observed as the rate of transaction costs increase the price a holder would be willing to pay for the option decreases.

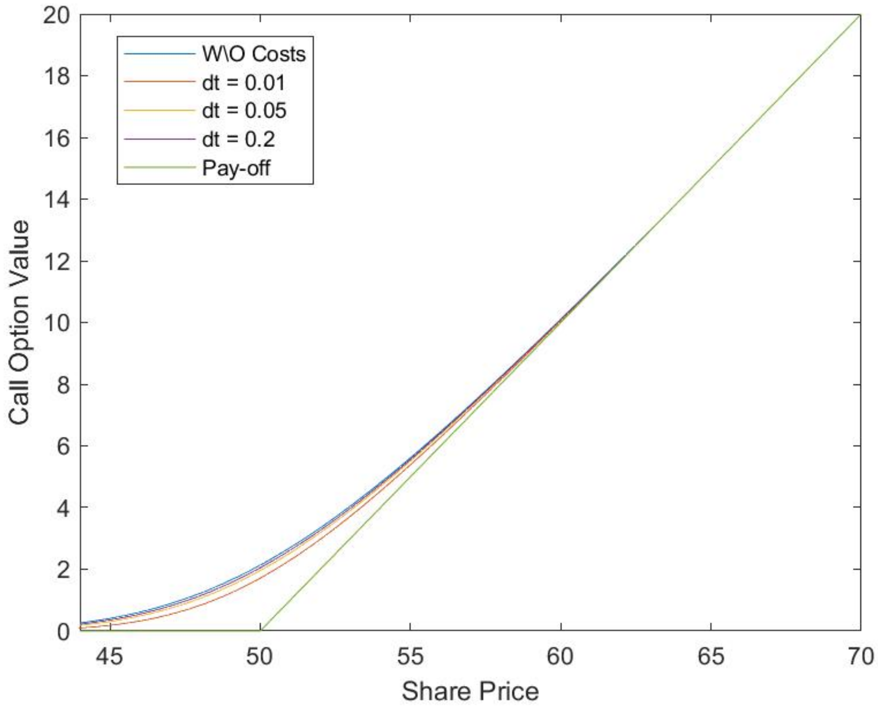

Figure 3: American Call Price with $k = 0.0025$

Figure 3 now shows the effect of decreasing the re-hedging interval for a fixed transaction cost size, $k = 0.0025$. As the interval of re-hedging decreases there are more re-hedging interactions that occur over the life of the option. This leads to more value in the portfolio being erroded by these costs, again leading to the holder of option to demand compensation for these costs. As a result, as the interval of re-hedging decreases the price investor would be willing to pay for such option decreases. It should be noted that the effect from these price decreases are felt the smaller the rehedging period is. For example, there is a greater drop between 0.05 to 0.01 re-hedges when compared with 0.2 to 0.05. This is a result that volatility in Eq. (20) has a square root and decrease in volatility for different re-hedging periods is not linear. Consideration must additionally be placed on the hedging interval an investor chooses. Leland's approach to pricing with transaction cost relies on enough adjustments in portfolio such that the investor can reasonably expect portfolio to be risk-free. Thus, $\delta t = 0.2$ would not be a reasonable hedging period as this would only lead to 5 re-hedging adjustments over life of the option. This value was chosen for illustration purposes but serves to underline some of the limitations that exist in this model and its inability to determine when enough re-hedges have been done to the portfolio.

It should be finally noted that Figure 2 - 3 display the financial implications of transaction costs and re-hedging interval from the holder's view but not writer. The writer of option will also expect compensation for these costs from portfolio which can be earnt by expecting higher price to be paid for options in market. This is where the pricing mismatch occurs between the holder and writer of option, due to time constraints however it was not possible to develop code to price options from writer's perspective.

4.2 American Call Optimal Exercise Boundary

In this section by finding the share prices that are tangential to the payoff function over the mesh time points and applying interpollation in MATLAB, optimal exercise boundaries for Figure 2 - 3 are produced.

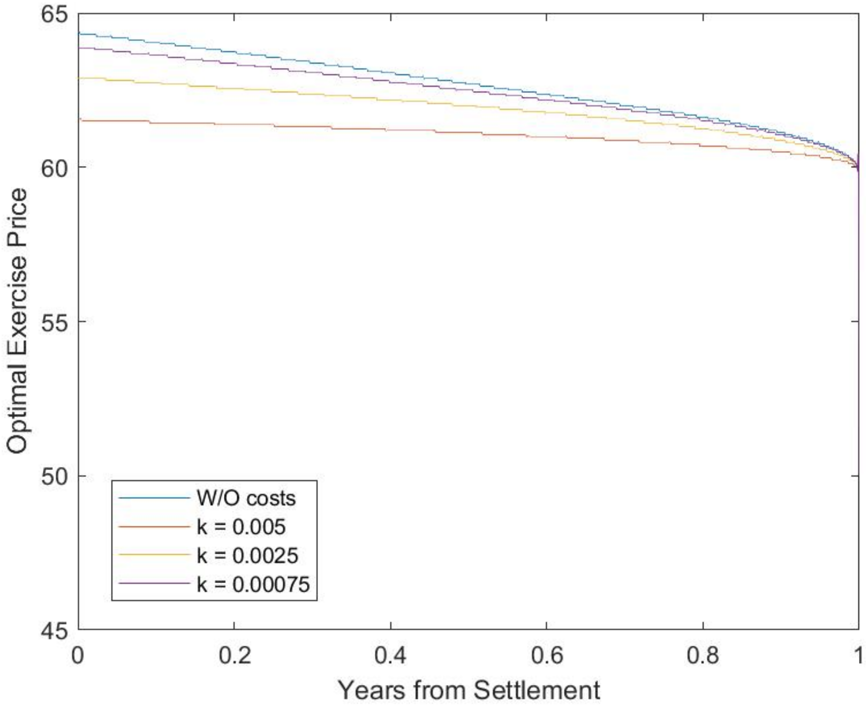

Figure 4: Optimal Exercise Boundary with $\delta t = 0.0025$

In Figure 4 again re-hedging period is fixed at $\delta t = 0.01$ and transaction costs are varied to observe the effect on options optimal exericse boundary. It is observed as the cost of rehedging increases optimal exercise price correspondingly decreases, becoming flatter and less variable over time. This decrease in price is a consequence of more of the holder's portfolio wealth being errored by market interactions as the cost of re-hedging increases. Thus, to limit the errosion of the portfolio's value, the holder of option will accept a lower exercise price in order to close out their position and end the costs incurred by re-hedging. Conversely, it is seen for lower (or no) transaction costs there is greater variability in the optimal exercise price over the life of the American call option. This results from the lower amount of costs incurred by holding and re-hedging portfolio, the holder of option then requires greater return in form of a higher exercise boundary to close out its position as it has more tolerance to wait for greater increases in share price. It should finally be noted that all optimal boundary curves converge to value Eq. (17) at maturity as expected, being \$ 60 for this option.

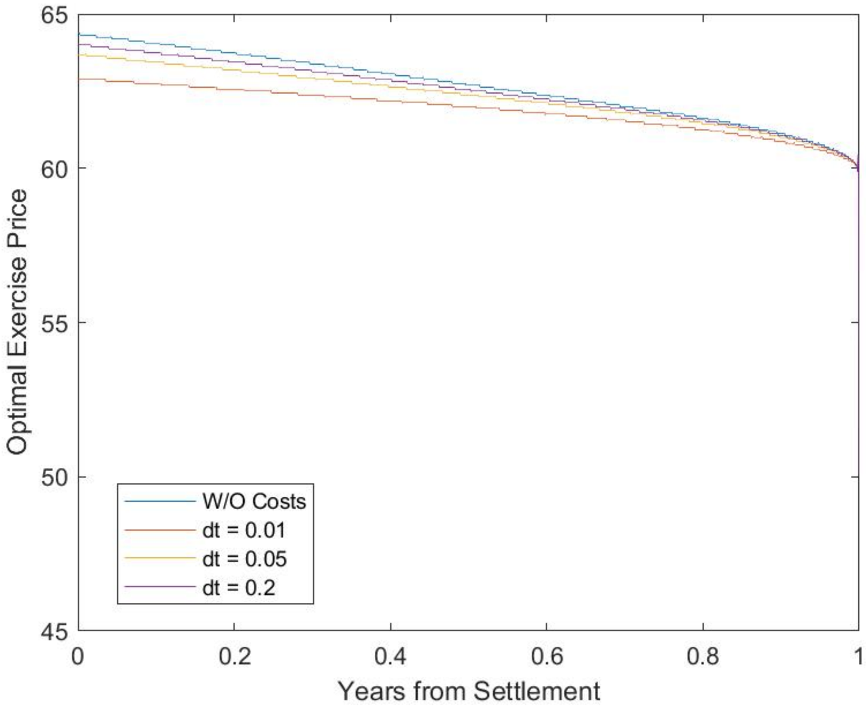

Figure 5: Optimal Exercise Boundary with $k = 0.0025$

In Figure 5 the transaction costs are fixed at $k = 0.0025$ and the re-hedging interval is varied. It is observed as the re-hedging interval decreases the optimal exercise boundary becomes less variable and flatter over-time. This is due to the fact that as re-hedging interval decreases there are more overall interactions with the market as the time between adjustments of portfolio decreases. As a result there will be more costs incurred as a result of re-hedging which effects the wealth of the portfolio held by the option holder. Thus, they aim to close out position earlier and hence accept a lower exercise boundary than if re-hedging interval was larger. It should be additionally noted that with more re-hedges of portfolio the more risk-free the overall portfolio becomes, as more adjustments of portfolio are made over the life of option. From holder's perspective selection of re-hedging interval is a balancing act between making sure portfolio is re-hedged enough such that it earns risk-free return but not re-hedged too often such that too much of its value is erroded by transaction costs.

It should be noted that Figure 4 - 5 are from the prespective of the holder not the writer. This is due to the fact that right to exercise an American option lies with the holder of that option not the writer. Thus, to price American option from the writer's perspective it is necessary to have the optimal exercise boundary for holder. This results from the writer's need to consider when holder will choose to exercise the option and factor that into the price they expect to receive for the option. As optimal exercise boundary decreases early exercise of the American option is more likely, requiring additional compensation for writer in the form of an increased expected price for the option. (Lu et al. 2022)

5 Conclusions & Future Work

This report considered the effects of varying transaction costs and re-hedging interval on the price and optimal exercise boundary of American options. It was observed as the transaction costs were increased or re-hedging interval decreased, option call holder demanded compensation for these costs in the form of lower option value. It was shown how this decrease in price option holders were willing to pay corresponded to a lower, less variable and flatter optimal exercise boundary for American calls. A strong motivator for future work would then be using these optimal exercise boundaries for holder to price these American call options from the writer's perspective. Further, it would be important to use market data to determine an appropriate value for transaction cost rate and consider what the most optimal re-hedging interval would be for holder to strike the balance between hedging held portfolio and costs accrued from transaction costs.

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