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Classification of Orbits in the Kepler-Heisenberg Problem

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Abstract

The Kepler-Heisenberg Problem is an extension of the well studied Kepler Problem to the non-Riemannian Heisenberg geometry. Dods and Shanbrom have previously shown that the $H = 0$ subsystem is completely integrable and exhibits self-similar orbits. The existence of self-similar orbits is novel for Hamiltonian systems. On the $H = 0$ subsystem, we are able to parametrise the orbits in terms of two integrals of motion, the angular momentum and dilational momentum. We have classified the orbits in terms of two invariant quantities, the rotation number $\mathcal{R}$ and the dilation number $\mathcal{D}$. The rotation and dilation numbers are found to codify the self-similar properties of the $H = 0$ orbits. They correspond proportionally to rotation by $\mathcal{R}$ and dilation by $e^{\mathcal{D}}$, respectively. We have verified that periodic orbits exist when $W = 0$ and are parametrised by rational rotation number $r \in (-1, 1) \setminus \{0\}$.

Statement of Authorship The project was supervised by Holger Dullin and the main work was completed by Jonathan Skelton. Jonathan found and evaluated the $u \mapsto \xi$ transform, polynomial $P(\xi)$, action, and rotation number integrals. Jonathan expressed the rotation number functions analytically in terms of elliptic functions. Holger sourced the handbook of elliptic functions in order to solve the integrals. Jonathan evaluated and plotted the rotation number functions in Mathematica, and computed the orbit trajectories in Matlab via the ODE78 numerical ordinary differential equation solver.

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Contents

1 Introduction

The Kepler problem is a well studied Hamiltonian system. The solution orbits describe the motion of bodies about a fixed central sun. The bodies have an intrinsic kinetic energy and exhibit a potential energy generated by the gravitational potential of the sun. For negative energy solutions, the curves form closed orbits, and exhibit three important properties known as Kepler's laws: 1) the trajectories are ellipsoidal, 2) conservation of orbital angular momentum, and 3) that scaling the orbital size and period of a solution by a dilational factor is also a valid solution [1]. The ambient geometry of the Kepler problem is Euclidean space, which gives rise to Kepler's laws as the metric is isotropic and allows for dilations.

Analogous variants of the classical Kepler problem can be formulated in non-Euclidean geometries. The first of which was Lobachevsky, who in 1835 posed the Kepler problem in three dimensional hyperbolic space (see Diacu [2] for more examples). Typically one constructs the kinetic energy from the Riemannian metric of the geometry. For the potential, the standard approach is to consider the fundamental solution to Laplace's equation defined on the geometry. In the context of Euclidean geometry, the fundamental solution to Laplace's equation in 3-dimensional space corresponds to the $1/r$ dependence of the potential in the classical Kepler problem.

An interesting question to pose is, what geometries offer analogous forms of Kepler's three laws? As noted by Montgomery and Shanbrom [1], the only Riemannian geometry with these properties is Euclidean space [3]. For example, in hyperbolic geometry, Kepler's third law fails as this space does not permit dilations. Consequently, to pose the Kepler problem in non-euclidean geometry such that analogous forms of Kepler's laws hold, it is necessary to leave the realm of Riemannian geometries and consider Sub-Riemannian geometries. One of the simplest sub-Riemannian geometry that admit dilations is the Heisenberg geometry [4]. This geometry is defined such that the vector fields X, Y satisfy the Heisenberg Lie Algebra $[X, Z] = [Y, Z] = 0$, where $Z := [X, Y]$ [4]. The choice of such commutator relations is motivated by quantum mechanics. When $X = \hat{x}$, $Y = \hat{p}$, and $Z = i\hbar$, the relations become the Heisenberg canonical commutation relations.

The Kepler problem was posed on the Heinsenberg group by Montgomery and Shanbrom [1]. Solving for the solution curves is known as the Kepler-Heisenberg Problem. A surprising property of the Kepler-Heisenberg problem is the existence of self-similar (fractal-like) orbits, which were relatively recently identified by Dods and Shanbrom [5]. They also noted that these solutions can be classified into three distinct types, future collision, past collision, and quasi-periodic or periodic. Currently, little is known about the analytic and geometric properties of the self-similar solutions. In this report we aim to classify the orbits of the zero-Hamiltonian subsystem by computing invariant quantities known as rotation numbers.

2 System

The Hamiltonian of the Kepler-Heisenberg Problem is given by [6],

$$H = \tfrac{1}{2}\Big((p_x - \tfrac{1}{2}yp_z)^2 + (p_y + \tfrac{1}{2}xp_z)^2\Big) - \frac{1}{8\pi\sqrt{(x^2+y^2)^2+16z^2}} \tag{1}$$

Where x, y, z are the coordinates and p_x, p_y, p_z are the conjugate momenta.

2.1 Change of Coordinates

The form of the Hamiltonian can be simplified by changing to new coordinates (s, θ, u). Explicitly, we use the canonical transformation derived by [5],

$$\begin{aligned} s &= \frac{1}{4}\log\left((x^2+y^2)^2+16z^2\right) \\ \theta &= \arg(x,\, y) \\ u &= \arg(x^2+y^2,\, 4z) \end{aligned}$$

with conjugate momenta,

$$\begin{aligned} p_s &= xp_x + yp_y + 2zp_z \\ p_\theta &= xp_y - yp_x \\ p_u &= \frac{1}{4}p_z(x^2+y^2) - 2z\frac{xp_x+yp_y}{x^2+y^2} \end{aligned}$$

Heuristically the point transformation corresponds to a form of spherical coordinates within the context of the Heisenberg geometry. Here e^s is the radial component, θ the classical azimuthal angle, and u loosely the angle between the x-y plane and the z axis. We call p_s the dilational momentum, and p_θ the angular momentum.

As the coordinate transformation is canonical, the form of Hamilton's equations is preserved in the new coordinate system. That is for coordinate q and momentum p, the time derivatives are, $\frac{dq}{dt} = \frac{\partial H}{\partial p}$ and $\frac{dp}{dt} = -\frac{\partial H}{\partial q}$. As the coordinate transform is time independent, it follows that the new Hamiltonian is simply the old Hamiltonian with new coordinates substituted.

In the new $(s, \theta, u, p_s, p_\theta, p_u)$ phase space, the Hamiltonian is $H = e^{-2s}(T+U)$, [6], where,

$$T = \tfrac{1}{2}\sec u\,(p_s\cos u + p_\theta \sin u)^2 + \tfrac{1}{2}\cos u\,(p_\theta + 2p_u)^2 \qquad \text{and} \qquad U = -\frac{1}{8\pi}$$

Notice that the Hamiltonian is independent of time and angle θ. Likewise, the Hamiltonian is 'independent' of coordinate s, up to multiplicative factor e^{-2s}. Applying Hamilton's equations thus gives the following proposition.

Proposition 1. *In the Kepler-Heisenberg system, the Hamiltonian $H = E$ and angular momentum $p_\theta = J$ are two integrals of motion. The dilational angular momentum satisfies $\frac{dp_s}{dt} = 2H$. Thus, on the $H = 0$ subsystem, $p_s = W$ is also an integral of motion.*

Consequently, on the $H = 0$ submanifold, the system is completely integrable. That is the number of degrees of freedom (three) is equal to the number of independent constants of motion. In the general case $H \neq 0$ it is not clear whether the system is integrable. As such, we initially analyse the general system, but then focus our discussion to the zero Hamiltonian subsystem.

However, to an extent we are able to 'scale' away the non-integral nature of dilational momentum p_s by introducing a parameterised time τ defined by the relation $\frac{d\tau}{dt} = e^{-2s}$. In τ-time the equations of motion are thus, $\frac{dq}{d\tau} = e^{2s}\frac{\partial H}{\partial p}$ and $\frac{dp}{d\tau} = -e^{2s}\frac{\partial H}{\partial q}$. For all coordinates except p_s, we can bring e^{2s} inside the partial derivative and form the new 'pseudo-Hamiltonian', $G := e^{2s}H$. The q's and p's satisfy Hamilton's equations in τ-time respect to Hamiltonian G, except p_s which satisfies $\frac{dp_s}{d\tau} = 2G$. To simplify notation we let $\dot{q}$ and $\dot{p}$ be the τ-time derivatives.

Proposition 2. *In τ-time, H and p_θ remain integrals of motion. The important equations of motion are,*

$$\begin{aligned} \dot{s} &= p_s \cos u + J \sin u \\ \dot{\theta} &= p_s \sin u + J \sec u + 2p_u \cos u \\ \dot{u} &= 2\cos u (J + 2p_u) \\ \dot{p}_s &= 2e^{2s}H = 2G \end{aligned}$$

Rewriting the Hamiltonian, we find the following general relationship between u and p_u.

Proposition 3. *The relationship between angle u and conjugate momentum p_u is,*

$$2(e^{2s}E - U)\cos u = (p_s \cos u + J \sin u)^2 + \cos^2 u\,(J + 2p_u)^2 \tag{2}$$

2.2 Integrable Subsystem and Rotation Numbers

For completely integrable systems, with n-degrees of freedom and autonomous of time, it is well known by the Liouville-Arnold theorem, that compact connected energy level sets lie on invariant n-tori (T^n). However, for the Kepler-Heisenberg problem, the dilational action is non-compact. Although, we can appeal to the Liouville–Arnold–Nekhoroshev Theorem (see [7]), which gives analogous results. In particular,

Theorem 1. *The Kepler-Heisenberg $H = 0$ subsystem solution curves lie on invariant $T^2 \times \mathbb{R}$ manifolds. Moreover, there exists a canonical transformation to action-angle coordinates $(\mathcal{I}_i, \phi_i)$ such that the Hamiltonian can be written independently of coordinates ϕ_i. The action variables are chosen to be $\mathcal{I}, J, W$, where $\mathcal{I}$ is the u-p_u action variable. That is $H = H(\mathcal{I}, J, W)$.*

Locally on the $H = 0$ submanifold, the $(\mathcal{I}_i, \phi_i)$ action-angle coordinates satisfy Hamilton's equations with respect to Hamiltonian $H = H(\mathcal{I}, J, W)$. The equations of motion are, $\dot{\mathcal{I}}_i = -\frac{\partial H}{\partial \phi_i} = 0$ and $\dot{\phi}_i = \frac{\partial H}{\partial \mathcal{I}_i}$. Thus, the actions $\mathcal{I}_i$ are constants, and so must be the frequencies $\omega_i := \dot{\phi}_i$.

Taking the ratio of two frequencies gives an invariant quantity known as the rotation number, $R_i^j = \omega_i/\omega_j$. Qualitatively, this corresponds to the change in one angle-coordinate with respect to another. In general, the ϕ_i coordinates lie on a 1-torus, that is they 'warp around' the interval $[0, 2\pi)$. In ϕ_i-ϕ_j phase space, the orbits follow linear lines that return to initial state after finite or infinite time, and thus the rotation number dictates whether the orbit is periodic or quasi-periodic. If the rotation number is rational, the coordinates returns to initial state in finite time and the system is periodic. Otherwise, if it is irrational, the orbit is dense in $[0, 2\pi) \times [0, 2\pi)$, and is known as quasi-periodic.

As the solution curves of the $H = 0$ subsystem are non-compact, one of the coordinates must lie on the real-line (not a torus), and so is not a true angle. In this case, it is still possible to define an analogous form of a 'rotation number', however when these rotation numbers are non-zero, the motion will be neither periodic or quasi-periodic. Hence, the system only exhibits periodic motion when the corresponding non-compact rotation number is zero.

3 Results

3.1 Action Integral

The action $\mathcal{I}$ is constructable by 'integrating out' the time dependence of p_u and u. We construct the action integral as the path integral along the u-p_u orbit,

$$\mathcal{I} = \frac{1}{2\pi} \oint p_u \, du$$

The action $\mathcal{I}$ corresponds to the area enclosed by the u-p_u orbit. From equation (2) this is bounded for all $J \neq 0$, as otherwise $p_u \to \pm\infty$ as $u \to \pm\frac{\pi}{2}$, which corresponds to collision with the z-axis.

3.1.1 ξ Coordinate Transform

We seek to parametrise p_u and u such that we reduce the trigonometric expressions in equation (2) to polynomial terms. We introduce the bijective coordinate transform, $\psi : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$, given by $\xi := \psi(u) = \tan\frac{u}{2}$.

Proposition 4. *General equations of motion in terms of ξ are,*

$$\begin{aligned} \dot{s} &= \frac{1}{1+\xi^2}\left(p_s(1-\xi^2) + 2J\xi\right) \\ \dot{\theta} &= \frac{1}{1-\xi^4}\left(2p_s\xi(1-\xi^2) + J(1+\xi^2)^2 + 2p_u(1-\xi^2)^2\right) \\ \dot{u} &=: \frac{2}{1+\xi^2}\,\dot{\xi} \end{aligned}$$

Likewise, we retain $\dot{H} = 0$, $\dot{p}_\theta = 0$, and $\dot{p}_s = 2H$.

Notice, by considering the $\dot{u}$ equation, one finds, $\frac{1}{2}J+p_u = \frac{\dot{\xi}}{2(1-\xi^2)}$. Hence, rewriting equation (2) in terms of ξ and $\dot{\xi}$,

$$\dot{\xi}^2 = 2(e^{2s}E - U)(1-\xi^4) - \big(p_s(1-\xi^2) + 2J\xi\big)^2 \tag{3}$$

The sign of $\dot{\xi}^2$ is critical for understanding the motion of the system. Solution curves are permitted only if $\dot{u}$, and hence $\dot{\xi}$, is real. Notice $\dot{\xi}^2$ can be expressed as a single variable quartic polynomial. Strictly speaking we consider $E = 0$, and hence $p_s = W$ is constant. Let $\mathcal{K} := e^{2s}E - U \equiv \frac{1}{8\pi}$, for $E = 0$. We introduce, $P(\xi) := \dot{\xi}^2$.

$$P(\xi) := (2\mathcal{K} - p_s^2) - 4p_sJ\xi + (2p_s^2 - 4J^2)\xi^2 + 4p_sJ\xi^3 - (2\mathcal{K} + p_s^2)\xi^4 \tag{4}$$

Thus, in terms of ξ, p_u obeys the relation,

$$\big(p_u + \tfrac{1}{2}J\big)^2 = \frac{P(\xi)}{4(1-\xi^2)^2} \tag{5}$$

By introducing the polynomial $P(\xi)$ we only need to consider the roots abstractly. The quartic degree allows us to readily classify the roots via the discriminant, and by the fundamental theorem of algebra, that there exists four not necessarily distinct roots. We employ this abstraction to compute the action integral.

Proposition 5. *Let a, b be roots of the quartic polynomial $P(\xi)$ such that $P(\xi) \geq 0$ for all $\xi \in (b, a)$. Then, the u-p_u action integral is given by,*

$$\mathcal{I} = \frac{1}{\pi}\int_b^a \frac{\sqrt{P(\xi)}}{1-\xi^4}\, d\xi \tag{6}$$

Proof. We construct the action integral $\mathcal{I}$ by, $\mathcal{I} = \frac{1}{2\pi}\oint p_u\, du$. From equation 5, it's clear that $\big(p_u + \frac{1}{2}J\big)^2 = Q(u) = \frac{P(\tan u/2)}{4(1-\tan^2 u/2)^2}$. That is, offset by some constant $-\frac{J}{2}$, p_u^2 can be written as a single variable function of u, $Q(u)$. The solution curves in u, p_u space are symmetric about $p_u = -\frac{1}{2}J$.

We consider the path integral over the top and bottom curves, and parametrised in terms of u we have, $p_u^{\pm} = -\frac{1}{2}J \pm \sqrt{Q(u)}$. Clearly for p_u to be real, we require $Q(u)$ to be non-negative. We notate the endpoints as α, β, where $Q(\alpha) = Q(\beta) = 0$. Expanding the action integral,

$$\begin{aligned}\oint p_u\, du &= \int_\beta^\alpha p_u^+\, du + \int_\alpha^\beta p_u^-\, du \\ &= \int_\beta^\alpha -\tfrac{1}{2}J + \sqrt{Q(u)}\, du - \int_\beta^\alpha -\tfrac{1}{2}J - \sqrt{Q(u)}\, du \\ &= 2\int_\beta^\alpha \sqrt{Q(u)}\, du\end{aligned}$$

The integral is changed to be expressed in terms of the polynomial ξ. We substitute $\xi = \tan\frac{u}{2}$, and use $\frac{du}{d\xi} = \frac{2}{1+\xi^2}$. By construction, $\sqrt{Q(u)} = \frac{\sqrt{P(\xi)}}{2(1-\xi^2)}$. Root a of $P(\xi)$ corresponds to root β of $Q(u)$, and likewise b for β. Performing the substitution thus gives, $\int_\beta^\alpha \sqrt{Q(u)}\, du = \int_b^a \frac{\sqrt{P(\xi)}}{1-\xi^4} \frac{du}{d\xi}\, d\xi$. Thus, $\frac{1}{2\pi} \oint p_u\, du = \frac{1}{\pi} \int_b^a \frac{\sqrt{P(\xi)}}{1-\xi^4}\, d\xi$ as required.

□

3.2 Period and Rotation Numbers

The existence of action-angle coordinates in the zero Hamiltonian sub-system allows us to greatly simplify the analysis of the system. Recall, by the Liouville-Arnold theorem, there exists canonical transformation to action-angle coordinates, with action integrals as $\mathcal{I}, J, W$. Moreover, the Hamiltonian can be implicitly written as, $H = H(I, J, W)$. By construction we have explicitly $\mathcal{I} = \mathcal{I}(H, J, W)$.

Abstractly, let the conjugate coordinates to actions $\mathcal{I}, J, W$ be $\phi_\mathcal{I}, \phi_J, \phi_W$ respectively. Recall for action-angle coordinates, the actions $\mathcal{I}_i$ and frequencies $\omega_i = \dot{\phi}_i = \frac{\partial H}{\partial \mathcal{I}_i}$ are constants. We introduce two rotation numbers of intrest. The rotation number $\mathcal{R} := \omega_J/\omega_\mathcal{I}$ and the dilation number $\mathcal{D} := \omega_W/\omega_\mathcal{I}$. The remaining rotation number is simply the ratio of the other rotation numbers, $\omega_J/\omega_W = \mathcal{R}/\mathcal{D}$, however this does not provide any additional information.

Likewise, by computing the frequencies, it is possible to find the period of each angle coordinate. We consider the conjugate angle $\phi_\mathcal{I}$ to action $\mathcal{I}$. The angular frequency is $\omega_\mathcal{I} = \frac{\partial H}{\partial \mathcal{I}}$, and thus period $T = 2\pi \frac{\partial \mathcal{I}}{\partial H}\big|_{H=0}$. For the τ-time period, we replace the Hamiltonian H with pseudo-Hamiltonian $G = e^{2s} H$. Note, $G = \mathcal{K} + U$, where U is a constant, hence $\frac{\partial}{\partial G} = \frac{\partial}{\partial \mathcal{K}}$.

Rather than explicitly computing the partials of the Hamiltonian in action-angle coordinates, we utilise the implicit nature of the Hamiltonian and appeal to the implicit function theorem.

Lemma 1. *The rotation and dilation numbers are given by the following partial derivatives,*

$$\mathcal{R} = -\frac{\partial \mathcal{I}}{\partial J} \qquad \text{and} \qquad \mathcal{D} = -\frac{\partial \mathcal{I}}{\partial W}$$

Proof. Recall the Implicit function theorem. Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be a smooth function, and consider the level set $f(x, y) = 0$, where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$. Suppose $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ satisfy $f(a, b) = 0$. Then, if $\frac{\partial f}{\partial y}(a, b) \neq 0$, then there locally exists $g : \mathbb{R}^n \to \mathbb{R}$ such that $f(x, g(x)) = 0$ is satisfied. Moreover, $\frac{\partial g}{\partial x_i} = -\left(\frac{\partial f}{\partial y}\right)^{-1} \frac{\partial f}{\partial x_i}$.

For the Hamiltonian subsystem, we claim the Hamiltonian $H : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is locally smooth away from the zero discriminant curve. Note that by construction $\mathcal{I} = \mathcal{I}(J, W)$ is an implicit function satisfying $H(J, W, \mathcal{I}(J, W)) = 0$. Applying the partial derivative result, $\frac{\partial \mathcal{I}}{\partial \mathcal{I}_i} = -\left(\frac{\partial H}{\partial \mathcal{I}}\right)^{-1} \frac{\partial H}{\partial \mathcal{I}_i}$. Rearranging terms, we thus have, $\mathcal{R}_i = \frac{\omega_i}{\omega_\mathcal{I}} = \frac{\partial H/\partial \mathcal{I}_i}{\partial H/\partial \mathcal{I}} = -\frac{\partial \mathcal{I}}{\partial \mathcal{I}_i}$ as required.

□

Proposition 6. *Let $b < a$ be roots of the quartic polynomial $P(\xi)$ such that $P(\xi) \geq 0$ for all $\xi \in (b, a)$. Then the τ-time period, rotation number, and dilation number are given by the following integrals.*

Period τ-time,

$$\mathcal{T} = 2\int_b^a \frac{1}{\sqrt{P(\xi)}}\,d\xi \tag{7}$$

Rotation Number,

$$\mathcal{R} = \frac{1}{\pi}\int_b^a \frac{2\xi}{1-\xi^4}\big(W(1-\xi^2)+2J\xi\big)\frac{d\xi}{\sqrt{P(\xi)}} \tag{8}$$

Dilation Number,

$$\mathcal{D} = \frac{1}{\pi}\int_b^a \frac{1}{1+\xi^2}\big(W(1-\xi^2)+2J\xi\big)\frac{d\xi}{\sqrt{P(\xi)}} \tag{9}$$

Proof. Recall we have action integral, $\mathcal{I} = \frac{1}{\pi}\int_b^a \frac{\sqrt{P(\xi)}}{1-\xi^4}\,d\xi$. As discussed earlier, the period in τ-time is given by $\frac{\partial \mathcal{I}}{\partial G} = \frac{\partial \mathcal{I}}{\partial \mathcal{K}}$. The rotation and dilation numbers are given by $-\frac{\partial \mathcal{I}}{\partial J}$ and $-\frac{\partial \mathcal{I}}{\partial W}$ respectively.

Compute $\frac{\partial \mathcal{I}}{\partial X}$, where $X = \mathcal{K}, J, W$.

$$\frac{\partial \mathcal{I}}{\partial X} = \frac{1}{\pi}\int_b^a \frac{1}{2\sqrt{P(\xi)}}\frac{\partial P}{\partial X}\frac{d\xi}{1-\xi 4} = \frac{1}{2\pi}\int_b^a \frac{1}{\sqrt{P(\xi)}}\frac{\partial P}{\partial X}\frac{d\xi}{1-\xi 4}$$

Taking the partial derivatives of the $P(\xi)$ polynomial, $\frac{\partial P}{\partial \mathcal{K}} = 2(1-\xi^4)$, $\frac{\partial P}{\partial J} = -4\xi\big(W(1-\xi^2)+2J\xi\big)$, and $\frac{\partial P}{\partial W} = -2(1-\xi^2)\big(W(1-\xi^2)+2J\xi\big)$. Simplifying gives the result as required. □

3.3 Integral Evaluation

The integrals we wish to solve are known in the Mathematical literature as elliptic integrals. In general, no simple closed form solution exists in terms of elementary functions such as polynomial or logarithmic expressions. We consult a standard table of integrals [8]. Separating our integrals by partial fraction decomposition allows us to readily read off the solutions to the integral equations in terms of elliptic functions.

Lemma 2. *Let*

$$I_p := \int_b^a \frac{1}{\xi - p}\frac{d\xi}{\sqrt{P(\xi)}}$$

Then the rotation and dilation numbers can be expressed as,

$$\mathcal{R} = \frac{1}{\pi}\left(JI_{-1} - JI_1 + (W+iJ)I_i + (W-iJ)I_{-i}\right) \tag{10}$$

and

$$\mathcal{D} = \frac{1}{\pi}\left(-\tfrac{1}{2}W\mathcal{T} + (J-iW)I_i + (J+iW)I_{-i}\right) \tag{11}$$

Proof. We separate the rotation number and dilation number integrals into W and J components. We find,

$$\mathcal{R} = \frac{W}{\pi}\int_b^a \frac{2\xi}{1+\xi^2}\frac{d\xi}{\sqrt{P(\xi)}} + \frac{J}{\pi}\int_b^a \frac{4\xi^2}{1-\xi^4}\frac{d\xi}{\sqrt{P(\xi)}}$$
$$\mathcal{D} = \frac{W}{\pi}\int_b^a \frac{1-\xi^2}{1+\xi^2}\frac{d\xi}{\sqrt{P(\xi)}} + \frac{J}{\pi}\int_b^a \frac{2\xi}{1+\xi^2}\frac{d\xi}{\sqrt{P(\xi)}}$$

Applying partial fraction decomposition, we find,

$$\frac{2\xi}{1+\xi^2} = \frac{1}{\xi+i} + \frac{1}{\xi-i}$$
$$\frac{4\xi^2}{1-\xi^4} = \frac{1}{\xi+1} - \frac{1}{\xi-1} + \frac{i}{\xi-i} - \frac{i}{\xi+i}$$
$$\frac{1-\xi^2}{1+\xi^2} = \frac{i}{\xi+i} - \frac{i}{\xi-i} - 1$$

Note $\frac{W}{\pi}\int_b^a \frac{d\xi}{\sqrt{P(\xi)}} = \frac{W}{2\pi}\mathcal{T}$. Expanding the above integrals and applying the definition of I_p we find the corresponding results above.

□

3.3.1 Discriminant

The domain over which the integral is computed is determined entirely by the region which the polynomial $P(\xi)$ is non-negative. To this end, it is necessarily to classify the roots of the polynomial depending on parameters J and W.

The polynomial discriminant provides a thorough method of classifying the roots of an arbitrary polynomial. In general, the discriminant is zero whenever two roots coincide. In the case of the quartic polynomial, the discriminant is positive when the roots are distinct and all real or all non-real, and negative otherwise. For real coefficient polynomials, as all complex roots appear in complex conjugate pairs, when the discriminant is negative we have two distinct real roots and a non-real complex conjugate pair of roots.

The explicit expression for the discriminant is,

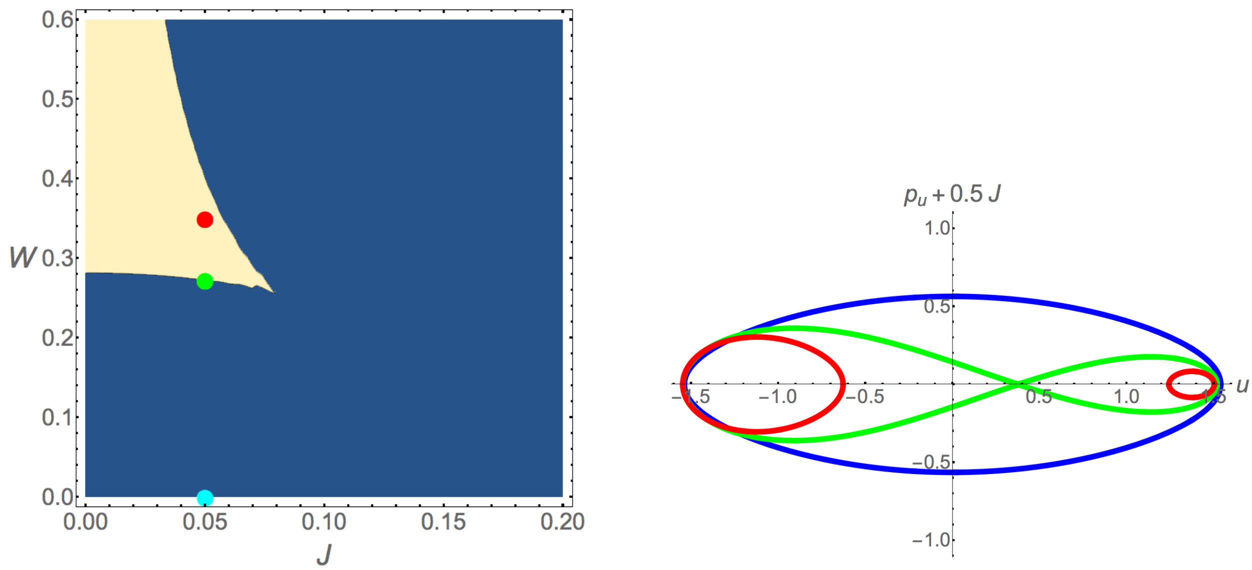
$$-4096\left(4J^8\mathcal{K}^2 + 8J^4\mathcal{K}^4 + 4\mathcal{K}^6 + 12J^6\mathcal{K}^2W^2 - 20J^2\mathcal{K}^4W^2 + 12J^4\mathcal{K}^2W^4 - \mathcal{K}^4W^4 + 4J^2\mathcal{K}^2W^6\right) \quad (12)$$

3.3.2 Period Integrals

Proposition 7. *Consider J, W where the polynomial $P(\xi)$ has positive discriminant. The roots are real distinct. Let them be $d < c < b < a$.*

Then the τ-time period integrals evaluate to,

$$\mathcal{T} = 2\int_d^c \frac{d\xi}{\sqrt{P(\xi)}} = 2\int_b^a \frac{d\xi}{\sqrt{P(\xi)}} = \frac{2g}{\sqrt{2\mathcal{K}+W^2}}K(k^2) \quad (13)$$



(a) Discriminant function plotted over $J, W \geq 0$. In the yellow and blue regions, the discriminant is positive and negative, respectively. The zero discriminant curve occurs at the boundary of the positive and negative regions. The coloured dots correspond to choice of (J, W) parameters used to form the orbits in figure 1b.

(b) Orbits in u-p_u phase space. Blue, green, and red curve occur in negative, zero, and positive discriminant region respectively. The orbits satisfy equation (2).

Figure 1: Discriminant of polynomial $P(\xi)$ and example orbits in u-p_u phase space.

Where K is the complete elliptic integral of the first kind, and $g = 2/\sqrt{(a-c)(b-d)}$ and $k^2 = \frac{(a-b)(c-d)}{(a-c)(b-d)}$.

Proof. We consult the elliptic integral handbook, [8], and refer to it's pages and integral identity numbering scheme. We assume J, W are chosen such that the polynomial $P(\xi)$ is in the positive discriminant region.

Consider the integral $\int \frac{d\xi}{\sqrt{P(\xi)}}$. Abstractly we can expression the polynomial as $P(\xi) = -(2\mathcal{K} + W^2)(\xi - a)(\xi - b)(\xi - c)(\xi - d)$, where $d < c < b < a$. Thus, we have $\mathcal{T} = \frac{2}{\sqrt{2\mathcal{K}+W^2}}\tilde{I}$, where, $\tilde{I} = \int \frac{d\xi}{\sqrt{-(\xi-a)(\xi-b)(\xi-c)(\xi-d)}}$.

Consider the left lobe. From page 103, integral identity 252.00, we find, $\tilde{I} = gF(\varphi, k^2)$, where $g = 2/\sqrt{(a-c)(b-d)}$, $k^2 = \frac{(a-b)(c-d)}{(a-c)(b-d)}$, and $\varphi = \frac{\pi}{2}$. F is an elliptic integral of the first kind. However, as $\varphi = \frac{\pi}{2}$ the elliptic integral is complete and can write $F(\varphi, k^2) = K(k^2)$. Thus, $\tilde{I} = gK(k^2)$.

Consider the right lobe, from page 120, integral identity 256.00, we find the same results as for the left lobe. Hence the periods are equal.

□

Proposition 8. *Consider J, W where the polynomial $P(\xi)$ has negative discriminant. Let $b < a$ be the corresponding real roots and $c, d = \bar{c}$ the non-real complex conjugate root.*

Then the period integral evaluates to,

$$\mathcal{T} = \int_b^a \frac{d\xi}{\sqrt{P(\xi)}} = \frac{4g}{\sqrt{2\mathcal{K} + W^2}} K(k^2) \tag{14}$$

Where K is the complete elliptic integrals of the first kind, and $A^2 = (a - \Re(c))^2 + \Im(c)^2$, $B^2 = (b - \Re(c))^2 + \Im(c)^2$, $g = 1/\sqrt{AB}$, and $k^2 = \frac{(a-b)^2-(A-B)^2}{4AB}$.

Proof. Assume J, W chosen such that the polynomial $P(\xi)$ is in the negative discriminant region. Let the roots be notated by $b < a, c, d = \bar{c}$ respectively. As in the negative discriminant region, we compute $\mathcal{T} = \frac{2}{\sqrt{2\mathcal{K}+W^2}} \tilde{I}$, where $\tilde{I} = \frac{d\xi}{(a-\xi)(\xi-b)(\xi-c)(\xi-d)}$.

From page 133, integral identity 259.00, $\tilde{I} = gF(\varphi, k^2)$, where $\varphi = \pi$, $A^2 = (a - \Re(c))^2 + \Im(c)^2$, $B^2 = (b - \Re(c))^2 + \Im(c)^2$, $g = 1/\sqrt{AB}$, and $k^2 = \frac{(a-b)^2-(A-B)^2}{4AB}$. However, we have the known identity, $F(\pi, k^2) = 2K(k^2)$. Thus, $\tilde{I} = 2gK(k^2)$. □

3.3.3 Rotation Number Integrals

As discussed earlier, to evaluate the integrals analytically, we first evaluate the integral I_p for $p = 1, -1, i, -i$, and use this to construct the appropriate rotation number integrals. See the appendix for the thorough working detail of deriving the analytic expressions for the integrals by looking up the appropriate integral identities in the elliptic integral handbook, [8].

Proposition 9. *Consider J, W where the polynomial $P(\xi)$ has positive discriminant. The roots are real distinct. Let them be $d < c < b < a$.*

Then the integral,

$$I_p = \int_d^c \frac{1}{\xi - p} \frac{d\xi}{\sqrt{P(\xi)}}$$

Is given by $I_p = \frac{1}{\sqrt{2\mathcal{K}-W^2}} \tilde{I}_p$, where,

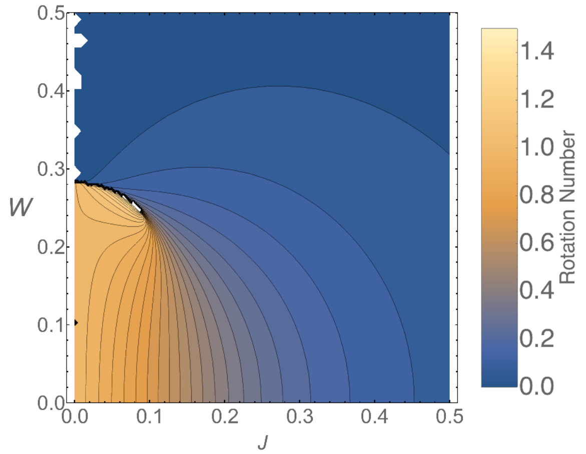
$$\tilde{I}_p = -\frac{g}{p-a} K(k^2) - \frac{g(a-d)}{(a-p)(p-d)} \Pi(\beta^2, k^2) \tag{15}$$

K and Π are complete elliptic integrals of the first and third kind respectively. $g = 2/\sqrt{(a-c)(b-d)}$, $k^2 = \frac{(a-b)(c-d)}{(a-c)(b-d)}$, and $\beta^2 = \frac{(d-c)(p-a)}{(a-c)(p-d)}$.

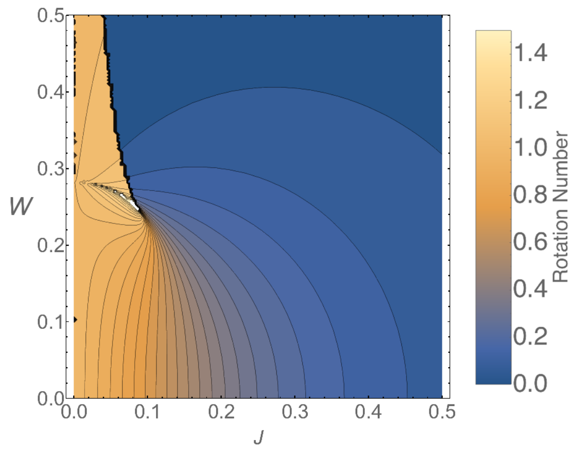
Proposition 10. *Consider J, W where the polynomial $P(\xi)$ has positive discriminant. The roots are real distinct. Let them be $d < c < b < a$.*

Then the integral,

$$I_p = \int_b^a \frac{1}{\xi - p} \frac{d\xi}{\sqrt{P(\xi)}}$$



(a) Contour plot of Rotation Number $\mathcal{R}$ for the 'left lobe'.



(b) Contour plot of Rotation Number $\mathcal{R}$ for the 'right lobe'.

Figure 2: Contour plots of left and right lobes of Rotation Number functions. Rotation number values for left and right lobe are equal modulo one. The contours are in 0.05 step increments.

Is given by $I_p = \frac{1}{\sqrt{2\mathcal{K} - W^2}} \tilde{I}_p$, *where,*

$$\tilde{I}_p \;=\; -\frac{g}{p-c}\, K(k^2) \,+\, \frac{g(b-c)}{(b-p)(p-c)}\, \Pi(\beta^2, k^2) \tag{16}$$

K and Π are complete elliptic integrals of the first and third kind respectively. $g = 2/\sqrt{(a-c)(b-d)}$, $k^2 = \frac{(a-b)(c-d)}{(a-c)(b-d)}$, *and* $\beta^2 = \frac{(a-b)(p-c)}{(a-c)(p-b)}$.

Proposition 11. *Consider J, W where the polynomial $P(\xi)$ has negative discriminant. Let the corresponding real roots be $b < a$ and non-real complex conjugate roots be $c, d = \bar{c}$.*

Then the integral,

$$I_p = \int_b^a \frac{1}{\xi - p} \frac{d\xi}{\sqrt{P(\xi)}}$$

Is given by $I_p = \frac{1}{\sqrt{2\mathcal{K} - W^2}} \tilde{I}_p$, *where,*

$$\tilde{I}_p \;=\; \frac{2g(A-B)}{A(b-p) - B(a-p)}\, K(k^2) - \frac{g(A+B)(a-b)}{(a-p)(b-p)} \left(\frac{A(b-p) + B(a-p)}{A(b-p) - B(a-p)} \right) \Pi(\alpha^2, k^2) \tag{17}$$

K and Π are complete elliptic integrals of the first and third kind respectively.

$A^2 = (a - \Re(c))^2 + \Im(c)^2$, $B^2 = (b - \Re(c))^2 + \Im(c)^2$, $g = 1/\sqrt{AB}$, $k^2 = \frac{(a-b)^2 - (A-B)^2}{4AB}$, *and* $\alpha^2 = \frac{\left(A(b-p) - B(a-p)\right)^2}{4AB(a-p)(b-p)}$.

3.3.4 Properties of Rotation and Dilation Number Functions

Evaluating the analytic expressions for the period and integrals I_p, we are able to compute the rotation and dilation numbers for all $J, W \in \mathbb{R}$, where the discriminant of $P(\xi)$ is non-zero.

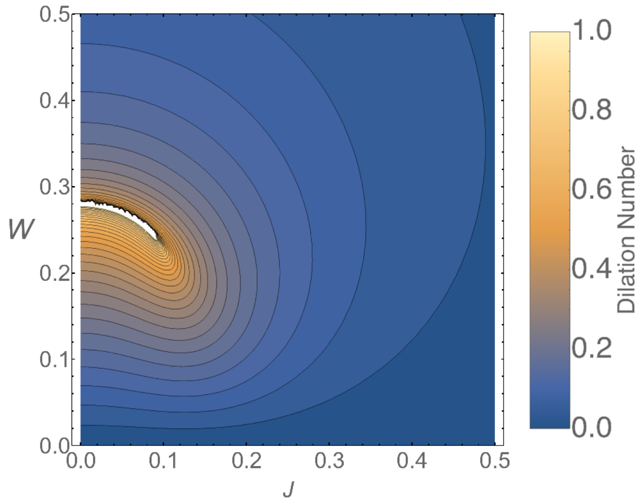


Figure 3: Contour plot of Dilation Number function. Left and right lobes numerically have equal values, hence only the left lobe is shown. The contours are in 0.033 step increments.

The rotation number for $W, J \geq 0$ is shown in figure 2. Within the positive discriminant region, the rotation numbers for the left and right lobes are equivalent modulo one. Qualitatively, the left lobe corresponds to motion with contractible orbits in θ-$\dot{\theta}$ phase space, while the right lobe lies on T^1 is not contractible.

The special case of the rotation number for when $W = 0$ is shown in figure 4. By inspection we surmise that $\mathcal{R} \to 1$ as $J \to 0$ and $\mathcal{R} \to 0$ as $J \to \infty$, however we have not formally proved this. We claim this function is monotonically decreasing, and hence forms a bijection between $[0, \infty]$ and $[0, 1]$. Figure 3 shows the dilation number for $W, J \geq 0$. In fact, numerically the dilation numbers for the left and right lobes appear to be equal. However, we have not formally proved this.

From the analytical expression, we find the following symmetries of the rotation and dilation numbers. For $\mathcal{R}$, $\mathcal{R}(-J, 0) = -\mathcal{R}(J, 0)$, $\mathcal{R}_{\text{left}}(-J, W) = -\mathcal{R}_{\text{right}}(J, W)$, and $\mathcal{R}(-J, -W) = -\mathcal{R}(J, W)$. Likewise, for $\mathcal{D}$, $\mathcal{D}(-J, W) = \mathcal{D}(J, W)$ and $\mathcal{D}(J, -W) = \mathcal{D}(J, W)$. Together with the analytical expression, we conclude that $\mathcal{D} < 0$ for $W < 0$, $\mathcal{D} = 0$ for $W = 0$, and $\mathcal{D} > 0$ for $W > 0$. This corroborates with the results of [5].

4 Orbit Classification

4.1 Rotation Numbers and Self-Similarity

In the Kepler-Heisenberg $H = 0$ subsystem, the rotation and dilation numbers $\mathcal{R}$ and $\mathcal{D}$ correspond to the amount the system rotates and dilates respectively.

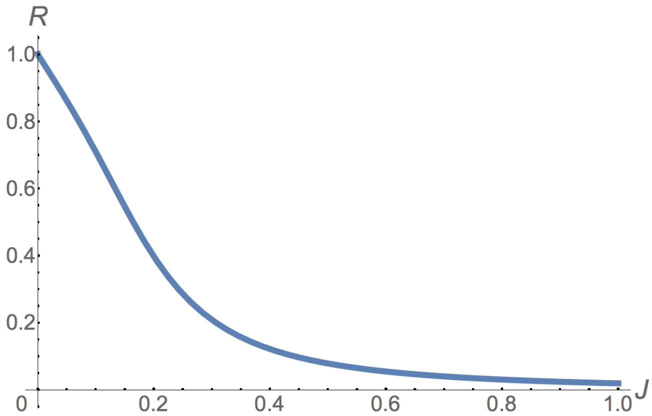


Figure 4: Rotation number function $\mathcal{R}(J)$ for $\mathcal{H} = 0$ and $W = 0$ subsystem. Plot is shown for positive angular momentum J, however $\mathcal{R}(J)$ has odd symmetry, $\mathcal{R}(-J) = -\mathcal{R}(J)$. By inspection we notice that as $J \to 0$, $\mathcal{R} \to 1$, and as $J \to \infty$, $\mathcal{R} \to 0$.

Lemma 3. *The rotation and dilation numbers have the following physical correspondence,*

$$\mathcal{R} = \frac{\Delta\theta}{2\pi} \quad \text{and} \quad \mathcal{D} = \frac{\Delta s}{2\pi}$$

where $\Delta\theta$ and Δs are the change in coordinate θ and s after the system completes an orbit in u-p_u phase space.

We leave the details to the appendix. We take the ratios of the τ-time derivatives of θ and s with u, and integrate over u.

Proposition 12. *The inverse point transformation $(s, \theta, u) \mapsto (x, y, z)$ is given by,*

$$x = e^s\sqrt{\cos u}\,\cos\theta, \quad y = e^s\sqrt{\cos u}\,\sin\theta, \quad \text{and } z = \tfrac{1}{4}e^{2s}\sin u$$

After each complete orbit in u-p_u phase space, we have the rotation and dilation mapping,

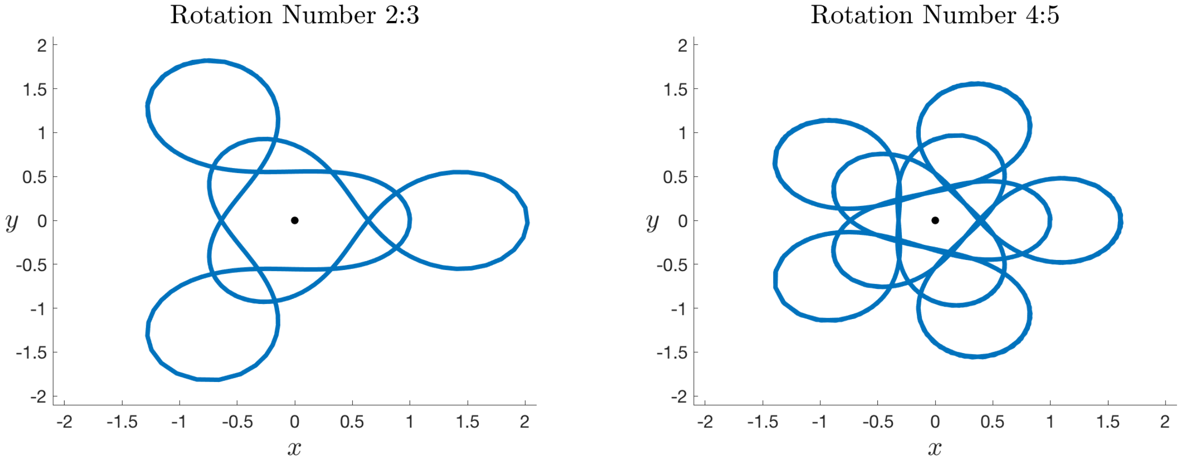
$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto e^{2\pi\mathcal{D}} \begin{bmatrix} \cos 2\pi\mathcal{R} & -\sin 2\pi\mathcal{R} \\ \sin 2\pi\mathcal{R} & \cos 2\pi\mathcal{R} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{and} \qquad z \mapsto e^{4\pi\mathcal{D}}z$$

Proof. The inverse transformation readily follows from inverting the $(x, y, z) \mapsto (s, \theta, u)$ coordinates. After a complete orbit, $u \mapsto u$, $\theta \mapsto \theta + \Delta\theta = \theta + 2\pi\mathcal{R}$, and $s \mapsto s + \Delta s = s + 2\pi\mathcal{D}$. Evaluating x, y, z at these updated s, θ, u gives the results as required.

□

Corollary 1. *Consider a curve $\gamma(\tau) = (x, y, z, p_x, p_y, p_z)$ in phase space satisfying the Hamiltonian $H(\gamma) = 0$, with rotation and dilation numbers $\mathcal{R}$ and $\mathcal{D}$. Suppose $\gamma_u(\tau) = (u, p_u)$ is γ projected into u-p_u phase space, and let $\mathcal{T}$ be the τ-time period. That is, the smallest $\mathcal{T}$ such that $\gamma_u(\tau + \mathcal{T}) = \gamma_u(\tau)$. Notate $\mathbf{x}(\tau) := [x(\tau),\, y(\tau)]^T$*

Then, the coordinates satisfy $\mathbf{x}(\tau + \mathcal{T}) = e^{2\pi\mathcal{D}}R_{2\pi\mathcal{R}}\mathbf{x}(\tau)$ and $z(\tau + \mathcal{T}) = e^{4\pi\mathcal{D}}z(\tau)$. Where R_θ is the rotation matrix by angle θ.



(a) Rotation number $j/k = 2/3$. Corresponding $J \approx 0.11308$, $W = 0$.

(b) Rotation number $j/k = 4/5$. Corresponding $J \approx 0.0709672$, $W = 0$.

Figure 5: Periodic orbits of the system projected into the x, y plane. Each orbit has rational rotation number $\mathcal{R} = j : k$. Solutions lie on the $H = 0$ subsystem, and initial conditions are $W = 0$, $J = \mathcal{R}^{-1}(j/k)$, and $s = \theta = u = 0$. p_u is chosen such that $H = 0$ is satisfied.

In other words, whenever the $\mathcal{R}$ and $\mathcal{D}$ rotation and dilation numbers exist, and the u-p_u phase space orbits are periodic, the orbits are self similar. The self-similarity is illustrated in figure 6a. By equation (2) and Hamilton's equations, When $H = 0$ one can show $\dot{u} = \dot{u}(u, J, W)$. As such, the u τ-time period exists (and hence p_u τ-time period), and the $H = 0$ subsystem exhibits self-similarity.

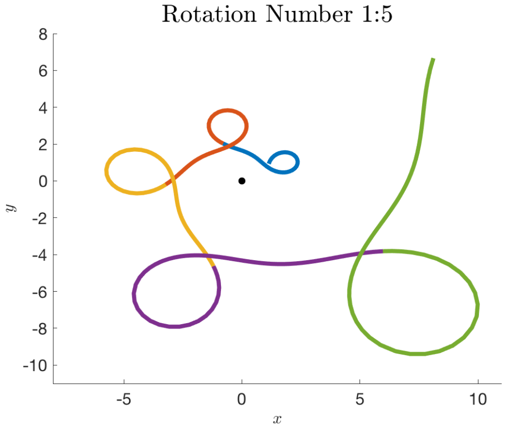
4.2 Periodic Orbits

If we consider the action-angle coordinate system, the actions $\mathcal{I}_i$ are constants, and motion depends entirely on linearly evolving 'angle'-coordinates ϕ_i. The orbits of the system are completely characterised by rotation and dilation numbers. When $W = 0$, the dilation number is zero, $\mathcal{D} = 0$, and the orbits are non-dilating. The orbits are thus characterised by the rotation number $\mathcal{R}(J)$. This allows us to classify the periodic orbits of the $H = 0$ subsystem, for $J \neq 0$.

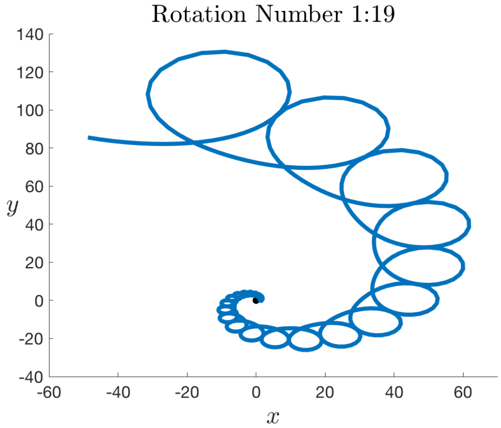
Theorem 2. *Consider the $H = 0$ subsystem of the Kepler-Heisenberg Problem. For every real number $r \in (-1, 1) \setminus \{0\}$, there exists a unique orbit with rotation number r and dilation number $\mathcal{D} = 0$. These orbits occur when $W = p_s = 0$, and are periodic for rational r.*

Proof. We employ the result that $\mathcal{D} = 0 \iff W = 0$. From our conjecture of the rotation number function for $W = 0$, $\mathcal{R}$ form a bijection between $(0, \infty)$ and $(0, 1)$. The odd symmetry of $\mathcal{R}(J, W = 0)$ thus gives a bijection between $\mathbb{R} \setminus \{0\}$ and $(-1, 1) \setminus \{0\}$. $\square$

Example plots of orbits for $H = W = 0$ are shown in figure 5. For rational rotation number $\mathcal{R} = j : k$, j corresponds to the number of times the solution curve 'rotates' around the z-axis



(a) Blue coloured curve shows motion within one 'u-p_u complete cycle'. The red, yellow, purple, and green curves are dilated and rotated according to proposition 12. The rotation number is $\mathcal{R} = 1/5$ and dilation number is $\mathcal{D} = 0.0625$. $J \approx 0.287$ and $W = 0.1$.



(b) Rotation number $\mathcal{R} = 1/19$, and dilation number $\mathcal{D} = 0.029$. Initial conditions, $J = 0.502789, W = 0.28$.

Figure 6: Example Future unbounded orbits ($W > 0$). Rotation numbers are chosen as $\mathcal{R} = 1 : k$, and appropriate J parameter is chosen by inverting the $\mathcal{R}(J, W)$ for a particular W. Ration rotation numbers $\mathcal{R} = j : k$ are also possible, however, the dilating properties of the orbits means the symmetry becomes obfuscated.

every k periods of coordinate u in u-p_u phase space. The orbit classification agrees with the numeric classification of $H = 0$ orbits discovered by Dods and Shanbrom [6].

4.3 General case

When $W \neq 0$ it's known that the orbits are either contracting ($W, \mathcal{D} < 0$) or expanding ($W, \mathcal{D} > 0$). Example unbounded orbits are shown in figure 6. The self-similar nature of the orbits is exemplified in figure 6a. The space of rotation and dilation numbers over J, W parameter space is not yet rigorously understood. However, the contour plot in figure 7 gives a qualitative understanding. The contour lines in general are not orthogonal to one another, and there exists tangential contour lines that restricts $(\mathcal{R}, \mathcal{D})$ to some subset of $\mathbb{R}^2$. This is especially obvious near the zero discriminant curve. We would like to formally understand the range of $\mathcal{R}$ and $\mathcal{D}$, and determine whether the values are infinite or limit to a finite value along the zero-discriminant ridge.

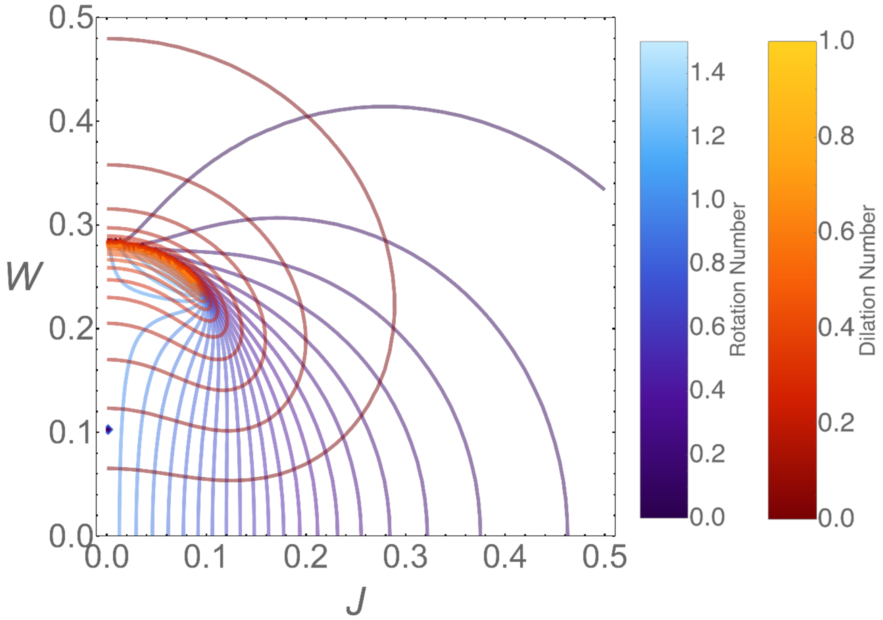


Figure 7: Contour plot of Rotation and Dilation number functions for left-lobe. Contours are separated by incremental values of 0.025. Each contour line corresponds to constant rotation or dilation number values.

5 Summary

Conclusion We have computed analytical expressions for the rotation and dilation numbers for the Hamiltonian subsystem $H = 0$. Our analysis holds for the positive and negative discriminant region of the polynomial $P(\xi)$. The rotation and dilation number invariant quantities allow us to completely classify the orbits of the $H = 0$ subsystem.

The rotation number $\mathcal{R}$ corresponds to rotation of the orbits about the z-axis. The dilation number $\mathcal{D}$ corresponds to dilational expansion (positive $\mathcal{D}$) or contraction (negative $\mathcal{D}$) of the orbits; future unbounded or future collision respectively. The sign of $\mathcal{D}$ is given by the sign of dilational momentum parameter W. As such, periodic orbits occur only when $W = 0$. The rotation and dilation number invariant quantities give rise to the self-similar nature of the $H = 0$ orbits. They describe how Hamiltonian solution curve segments map to the global solution curve under periodic τ-time translation, via rotation and dilation.

From observation of figure 4 we conjecture that $\mathcal{R}(J)$ reaches every rotation number $r \in (0, 1]$ uniquely for $J \in [0, \infty)$. Even symmetry of $\mathcal{R}(J)$ implies similar results for rotation numbers $r \in (-1, 0]$. However we have not yet been able to formally prove this. We have verified that for the $H = 0$ subsystem, closed periodic orbits exist when $W = 0$. For periodic orbits the rotation number $\mathcal{R}(J)$ is rational.

Future Work On the $W = 0$ line, the rotation number $\mathcal{R}$ is ill-defined. In finite time, the system reaches the z-axis ($u \to \pm\frac{\pi}{2}$), which is known to be singular in the Heisenberg geometry [5]. Moreover, below the zero discriminant curve in (J, W) parameter space, as $W \to 0$ from the positive side, the rotation number function appears to limit to 1, while it appears to limit -1 as $W \to 0$ from the negative side. We would like to prove these limits formally.

Similarly, we would like to extend our analysis of the rotation and dilation number integrals to the zero-discriminant curves. As this would require a double root in polynomial $P(\xi)$, the integrals are no longer be elliptic and would be solvable in terms of elementary functions. The contour lines directed into the zero discriminant curve in figure 2 appears to imply the rotation number may be finite along the lower zero discriminant curve. However, from 3, we expect the dilation number to diverge along this curve. Similarly, we would like to parameterise the zero discriminant curves, which can be done by assuming the double root function form of the polynomial $P(\xi)$ and comparing coefficients.

We would also like to show that the dilation number for left and right lobes are equal. Qualitatively, this appears to be true. Likewise, we would like to find an analytical argument to show the rotation numbers of the left and right lobe differ by 1.

We would also like to attempt to analyse the $H \neq 0$ case. In this setting, the system is at least partially integrable, with H, J constant, however, it is not clear whether there exists a third integral of motion.

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A Proof of Lemma 3

Suppose the system has some initial condition $(s_0, \theta_0, u_0, p_{s,0}, p_{\theta,0}, p_{u,0})$ at $t = 0$. Let $T \neq 0$ be the first time where u and p_u return to their initial values. We compute the corresponding change in θ and s.

For θ, $\Delta\theta = \theta_T - \theta_0 = \int_0^T \frac{d\theta}{d\tau}\,d\tau$. We perform change of variables to integrate in terms of u and find, $\Delta\theta = \oint \frac{d\theta}{d\tau}\frac{d\tau}{du}\,du$. Likewise for s, $\Delta s = \oint \frac{ds}{d\tau}\frac{d\tau}{du}\,du$.

Computing the ratio of τ-time derivatives, we find, $\dot\theta/\dot u = \frac{2W\xi(1-\xi^2)+J(1+\xi^2)^2+2p_u(1-\xi^2)}{2\dot\xi(1-\xi^2)}$ and $\dot s/\dot u = \frac{W(1-\xi^2)+2J\xi}{2\dot\xi}$. Recall, $2p_u = \frac{\dot\xi}{1-\xi^2} - J$. The first expression simplifies, $\dot\theta/\dot u = \frac{1}{2\dot\xi(1-\xi^2)}\left(2W\xi(1-\xi^2) + J(1+\xi^2)^2 - J(1-\xi^2)^2 + \dot\xi(1-\xi^2)\right)$. Notice, $(1+\xi^2)^2 - (1-\xi^2)^2 = 4\xi^2$. Thus expanding, $\dot\theta/\dot u = \frac{W\xi}{\dot\xi} + \frac{2J\xi^2}{\dot\xi(1-\xi^2)} + \frac{1}{2}$.

Writing out the $\Delta\theta$ integral, $\Delta\theta = \oint \frac{W\xi(1-\xi^2)+2J\xi^2}{\dot\xi(1-\xi^2)}\,du + \frac{1}{2}\oint du$. The second term is identically zero.

Similar to the $\mathcal{I}$ action integral, we integrate over the positive and negative symmetric halves of the $\dot\theta/\dot u$ and $\dot s/\dot u$ integrands, from $u = \alpha$ to $u = \beta$. Thus, compute $\Delta\theta = 2\int_\alpha^\beta \dot\theta/\dot u\,du$ and $\Delta s = 2\int_\alpha^\beta \dot s/\dot u\,du$. We now perform change of variables to integrate over ξ. Let $a = \tan\frac{\alpha}{2}$ and $b = \tan\frac{\beta}{2}$. Recall, $\frac{du}{d\xi} = \frac{2}{1+\xi^2}$, and the definition of polynomial $P(\xi) := \xi^2$. Thus, $\Delta\theta = 2\int_a^b \frac{W\xi(1-\xi^2)+2J\xi^2}{(1-\xi^4)\sqrt{P(\xi)}}\,d\xi$. Thus,

$$\Delta\theta = 2\int_b^a \frac{2\xi}{1-\xi^4}\big(W(1-\xi^2) + 2J\xi\big)\frac{d\xi}{\sqrt{P(\xi)}}$$

Likewise, for δs, the integral becomes,

$$\Delta s = 2\int_b^a \frac{1}{1-\xi^4}\big(W(1-\xi^2) + 2J\xi)\big)\frac{d\xi}{\sqrt{P(\xi)}}$$

hence, $\Delta\theta = 2\pi\mathcal{R}$ and $\Delta s = 2\pi\mathcal{D}$ as required.

B Elliptic Integral Evaluation

Consider the integral,

$$I_p = \int \frac{1}{\xi - p} \frac{d\xi}{\sqrt{P(\xi)}}$$

Where $P(\xi) = -(\xi - a)(\xi - b)(\xi - c)(\xi - d)$ is a quartic polynomial with real coefficients, and $a, b, c, d \in \mathbb{C}$ are all distinct roots.

We consider two cases of the polynomial, real discriminant ($d < c < b < d$), and negative discriminant ($b < a$ real and c, d non-real complex conjugate). Necessarily $P(\xi) \geq 0$ for $\xi \in [d, c] \cup [b, a]$ for positive discriminant case, and $P(\xi) \geq 0$ for $\xi \in [b, a]$ for negative discriminant case.

We consult a standard handbook of elliptic integrals, [8], to evaluate the integral for the cases of negative and position polynomial discriminant. 'Pages' refer to page from the handbook, and 'integral identity' corresponds to the numbering scheme of the result consulted.

B.1 Positive Discriminant

In the positive discriminant case we consider two pairs of integration bounds. The 'left lobe' d, c, and the 'right lobe' b, a.

B.1.1 Left Lobe

From Page 107, integral identity 252.39, we find the integral $I_p = \int_d^c \frac{1}{p-\xi} \frac{d\xi}{\sqrt{P(\xi)}}$ can be written as,

$$I_p = \frac{g}{p - d} \int_0^{u'} \frac{1 - \alpha^2 \text{sn}^2 u}{1 - \beta^2 \text{sn}^2 u} \, du$$

where, $g = 2/\sqrt{(a - c)(b - d)}$, $\alpha^2 = \frac{d-c}{a-c}$, and $\beta^2 = \alpha^2 \frac{p-a}{p-d}$. The elliptic modulus is $k^2 = \frac{(a-b)(c-d)}{(a-c)(b-d)}$. Moreover, choice of integration bounds gives the elliptic angle to be $\varphi = \frac{\pi}{2}$.

Likewise on page 205, integral identity 340.04, the integral simplifies,

$$\int_0^{u'} \frac{1 - \alpha^2 \text{sn}^2 u}{1 - \beta^2 \text{sn}^2 u} \, du = \frac{1}{\beta^2} \left((\beta^2 - \alpha^2) \Pi(\varphi, \beta^2, k^2) + \alpha^2 F(\varphi, k^2) \right)$$

Where F and Π are elliptic integrals of the first and third kind respectively. However as $\varphi = \frac{\pi}{2}$, the elliptic integrals are complete.

One can show that the coefficient of the elliptic integral of the third kind simplifies to,

$$\frac{\beta^2 - \alpha^2}{\beta^2} = \frac{a - d}{a - p}$$

Likewise for the coefficient of the elliptic integral of the first kind,

$$\frac{\alpha^2}{\beta^2} = \frac{p - d}{p - a}$$

Notice, $\frac{1}{p-d}\frac{p-d}{p-a} = \frac{1}{p-a}$.

Thus, we find,

$$I_p = \frac{g}{p-a}K(k^2) + \frac{g(a-d)}{(a-p)(p-d)}\Pi(\beta^2, k^2) \tag{18}$$

B.1.2 Right Lobe

From Page 124, integral identity 256.39, , we find the integral $I_p = \int_b^a \frac{1}{\xi - p}\frac{d\xi}{\sqrt{P(\xi)}}$ can be written as,

$$I_p = \frac{g}{b-p}\int_0^{u'} \frac{1-\alpha^2\mathrm{sn}^2 u}{1-\beta^2\mathrm{sn}^2 u}\,du$$

where, $g = 2/\sqrt{(a-c)(b-d)}$, $\alpha^2 = \frac{a-b}{a-c}$, and $\beta^2 = \alpha^2\frac{p-c}{p-b}$. The elliptic modulus is $k^2 = \frac{(a-b)(c-d)}{(a-c)(b-d)}$. The elliptic angle is $\varphi = \frac{\pi}{2}$.

Likewise on page 205, integral identity 340.04, the integral simplifies,

$$\int_0^{u'} \frac{1-\alpha^2\mathrm{sn}^2 u}{1-\beta^2\mathrm{sn}^2 u}\,du = \frac{1}{\beta^2}\left((\beta^2-\alpha^2)\Pi(\varphi, \beta^2, k^2) + \alpha^2 F(\varphi, k^2)\right)$$

Where F and Π are elliptic integrals of the first and third kind respectively, and are complete as $\varphi = \frac{\pi}{2}$.

One can show that the coefficient of the elliptic integral of the third kind simplifies to,

$$\frac{\beta^2 - \alpha^2}{\beta^2} = \frac{b-c}{p-c}$$

Likewise for the coefficient of the elliptic integral of the first kind,

$$\frac{\alpha^2}{\beta^2} = \frac{p-b}{p-c}$$

Thus, we find,

$$I_p = -\frac{g}{p-c}K(k^2) + \frac{g(b-c)}{(b-p)(p-c)}\Pi(\beta^2, k^2) \tag{19}$$

B.2 Negative Discriminant

In the negative discriminant case we take integration bounds $b < a$. Notate the complex conjugate roots of $P(\xi)$ by c and $\bar{c}$. Consider the integral $I_p = \int_b^a \frac{1}{\xi - p}\frac{d\xi}{\sqrt{P(\xi)}}$. From page 133, integral identity 259.04, we find the integral can be written as,

$$I_p = \frac{g(A+B)}{A(b-p) - B(a-p)}\left(\beta\int_0^{u'} du + (\alpha - \beta)\int_0^{u'} \frac{du}{1+\alpha\mathrm{cn}u}\right)$$

where, $A^2 = (a - \Re(c))^2 + \Im(c)^2$, $B^2 = (b - \Re(c))^2 + \Im(c)^2$, $g = 1/\sqrt{AB}$, $\alpha = \frac{A(b-p)-B(a-p)}{A(b-p)+B(a-p)}$, and $\beta = \frac{A-B}{A+B}$.

From page 205, integral identity 341.02, it holds that $\int_0^{u'} du = F(\varphi, k^2)$, where F is an elliptic integral of the first kind, $\varphi = \pi$ (choice of integration bounds), and $k^2 = \frac{(a-b)^2-(A-B)^2}{4AB}$.

Likewise, integral identity 341.03 gives,

$$\int_0^{u'} \frac{du}{1+\alpha \mathrm{cn}u} = \frac{1}{1-\alpha^2}\left(\Pi(\varphi, \tfrac{\alpha^2}{\alpha^2-1}, k^2) - \alpha f\right)$$

where Π is an elliptic integral of the third kind, and f is given by (page 215, identity 361.54),

$$f = \begin{cases} \sqrt{\frac{1-\alpha^2}{k^2+(1-k^2)\alpha^2}} \arctan\left(\sqrt{\frac{k^2+(1-k^2)\alpha^2}{1-\alpha^2}}\,\mathrm{sd}u\right), & \frac{\alpha^2}{\alpha^2-1} < k^2 \\ \mathrm{sd}u, & \frac{\alpha^2}{\alpha^2-1} = k^2 \\ \frac{1}{2}\sqrt{\frac{\alpha^2-1}{k^2+(1-k^2)}} \ln\left(\frac{\sqrt{k^2+(1-k^2)\alpha^2}\,\mathrm{dn}u+\sqrt{\alpha^2-1}\,\mathrm{sn}u}{\sqrt{k^2+(1-k^2)\alpha^2}\,\mathrm{dn}u-\sqrt{\alpha^2-1}\,\mathrm{sn}u}\right), & \frac{\alpha^2}{\alpha^2-1} > k^2 \end{cases}$$

The elliptic functions sd$u$, dn$u$, and sn$u$, are obey the relations, $\mathrm{dn}u = \sqrt{1-k^2\sin^2\varphi}$, $\mathrm{sn}u = \sin\varphi$, and $\mathrm{sd}u = \frac{\mathrm{sn}u}{\mathrm{dn}u}$, where k and φ are implicitly defined by the elliptic integrals.

One can show, for the particular values of k and α, $\frac{k^2+(1-k^2)\alpha^2}{\alpha^2-1}$ is non-zero and finite. Moreover as $\varphi = \pi$, $\mathrm{sn}u = 0$, and thus $\mathrm{sd}u = 0$. Thus we find $f \equiv 0$. Hence, $\int_0^{u'} \frac{du}{1+\alpha\mathrm{cn}u} = \frac{1}{1-\alpha^2}\Pi(\varphi, \frac{\alpha^2}{\alpha^2-1}, k^2)$.

Simplifying the prefactor of the elliptic integral of the third kind, we first find, $\alpha - \beta = \frac{2AB(b-a)}{(A+B)(A(b-p)+B(a-p)}$ and $\frac{1}{1-\alpha^2} = \frac{(A(b-p)+B(a-p))^2}{4AB(b-p)(a-p)}$. Thus

$$\frac{\alpha-\beta}{1-\alpha^2} = -\frac{(a-b)(A(b-p)+B(a-p))}{2(b-p)(a-p)(A+B)}$$

The elliptic integral argument $\frac{\alpha^2}{1-\alpha^2}$ simplifies to,

$$\frac{\alpha^2}{1-\alpha^2} = \frac{\big(A(b-p)-B(a-p)\big)^2}{4AB(a-p)(b-p)}$$

Finally, we note that $F(\pi, k^2) = 2K(k^2)$ and $\Pi(\pi, \alpha^2, k^2) = 2\Pi(\alpha^2, k^2)$ where K and Π are complete elliptic integrals of the first and third kind respectively.

Thus, find,

$$I_p = \frac{2g(A-B)}{A(b-p)-B(a-p)}K(k^2) - \frac{g(A+B)(a-b)}{(a-p)(b-p)}\left(\frac{A(b-p)+B(a-p)}{A(b-p)-B(a-p)}\right)\Pi(\tfrac{\alpha^2}{1-\alpha^2}, k^2) \tag{20}$$

C $W = 0$ Rotation Numbers

We will compute the rotation and dilation numbers for when $W = 0$.

The polynomial $P(\xi)$ simplifies to quadratic in ξ^2,

$$P(\xi) = 2\mathcal{K} - 4J^2\xi^2 - 2\mathcal{K}\xi^4$$

Solving for the roots, we can rewrite $P(\xi)$ as $P(\xi) = 2\mathcal{K}(a^2 - \xi^2)(\xi^2 + b^2)$, with roots, $\pm a, \pm ib$, where,

$$a^2 = -J^2/\mathcal{K} + \sqrt{1 + J^4/\mathcal{K}^2}$$
$$b^2 = +J^2/\mathcal{K} + \sqrt{1 + J^4/\mathcal{K}^2}$$

To ensure $P(\xi) \geq 0$, we thus have $\xi \in [-a, a]$. The integration bounds are now $-a, a$.

C.1 Dilation Number

Notice when $W = 0$, the dilation number integral simplifies to,

$$\mathcal{D} = \frac{2J}{\pi} \int_{-a}^{a} \frac{\xi}{1+\xi^2} \frac{d\xi}{\sqrt{P(\xi)}}$$

However, as $P(\xi)$ is quadratic in ξ^2, $1/\sqrt{P(\xi)}$ is even. Thus, as $\frac{\xi}{1+\xi^2}$ is odd, the dilation number integrand is odd. Consequently, $\mathcal{D} \equiv 0$.

C.2 Rotation Number

Proposition 13. *The $W = 0$ rotation number, is given by the analytical expression,*

$$\mathcal{R}(J) = \frac{\sqrt{\mathcal{K}}}{\pi J} \left(1 + J^4/\mathcal{K}^2\right)^{-\frac{1}{4}} \left(K(k^2) - (1 - b^2)\Pi(\alpha^2, k^2) - (1 + b^2)\Pi(\beta^2, k^2)\right) \tag{21}$$

Where, K and Π are complete elliptic integrals of the first and third kind respectively. The constants are, $b^2 = J^2/\mathcal{K} + \sqrt{1 + J^4/\mathcal{K}}$, $k^2 = \frac{1}{2} - \frac{1}{2}\frac{J^2/\mathcal{K}}{\sqrt{1+J^4/\mathcal{K}^4}}$, $\alpha^2 = \frac{1}{2}\left(1 + \frac{J^2/\mathcal{K}+1}{\sqrt{1+J^4/\mathcal{K}^2}}\right)$, and $\beta^2 = \frac{1}{2}\left(1 + \frac{J^2/\mathcal{K}-1}{\sqrt{1+J^4/\mathcal{K}^2}}\right)$. Moreover, the function is continuous on $J \in \mathbb{R} \setminus \{0\}$.

Proof. When $W = 0$, the rotation number is given by, $\mathcal{R} = \frac{J}{\pi}\int_{-a}^{a} \frac{4\xi^2}{1-\xi^4}\frac{d\xi}{\sqrt{P(\xi)}}$. Note that $P(\xi)$ is an even function when $W = 0$. Hence, we have, $\mathcal{R} = \frac{4J}{\pi\sqrt{2\mathcal{K}}}\tilde{I}$, where

$$\tilde{I} = \int_0^a \frac{2\xi^2}{1-\xi^4} \frac{d\xi}{\sqrt{(a^2-\xi^2)(b^2+\xi^2)}}$$

By partial fraction decomposition, we find $\frac{2\xi^2}{1-\xi^2} = \frac{1}{1-\xi^2} + \frac{1}{-1-\xi^2}$. Thus, $\tilde{I} = I_1 + I_{-1}$, where, $I_p = \int_0^a \frac{d\xi}{(p-\xi^2)\sqrt{(a^2-\xi^2)(b^2+\xi^2)}}$. The integral can be solved by comparison to known tabulated elliptical integral identities. It can be shown (see appendix subsection),

$$\int_0^a \frac{d\xi}{(p-\xi^2)\sqrt{(a^2-\xi^2)(b^2+\xi^2)}} = \frac{g}{b^2+p}\left(pK(k^2) + b^2\Pi(\alpha^2, k^2)\right)$$

where K and Π are complete elliptic integrals of the first and third kind respectively, and $g = \frac{1}{\sqrt{a^2+b^2}}$, $k = ag$, and $\alpha^2 = \frac{p+b^2}{p}k^2$.

Using the expressions for a^2 and b^2, it follows that, $g^2 = \frac{1}{2}(1 + J^4/\mathcal{K}^2)^{-1/2}$ and $k^2 = \frac{1}{2} - \frac{1}{2}\frac{J^2/\mathcal{K}}{\sqrt{1+J^4/\mathcal{K}^2}}$. There are two cases of α. For $p = +1$, we have $\alpha_+^2 = (1 + b^2)k^2$, and for $p = -1$, $\alpha_-^2 = (1 - b^2)k^2$. Thus, $\alpha_+^2 = \frac{1}{2}\left(1 + \frac{J^2/\mathcal{K}+1}{\sqrt{1+J^4/\mathcal{K}^2}}\right)$, and $\alpha_-^2 = \frac{1}{2}\left(1 + \frac{J^2/\mathcal{K}-1}{\sqrt{1+J^4/\mathcal{K}^2}}\right)$.

Thus, evaluating the integrals,

$$\tilde{I} = \frac{g}{b^2+1}\left(K(k^2) + b^2\Pi(\alpha_+^2, k^2)\right) + \frac{g}{b^2-1}\left(K(k^2) - b^2\Pi(\alpha_-^2, k^2)\right)$$

Simplifying this expression it can be shown,
$\tilde{I} = \frac{gb^2}{b^4-1}\left(K(k^2) - (1-b^2)\Pi(\alpha_+^2, k^2) - (1+b^2)\Pi(\alpha_-^2, k^2)\right)$. Moreover, from the expression of b^2, we find, $\frac{b^2}{b^4-1} = \frac{\mathcal{K}}{2J^2}$.

Thus,

$$\mathcal{R} = \frac{\sqrt{\mathcal{K}}}{\pi J}\left(1 + J^4/\mathcal{K}^2\right)^{-\frac{1}{4}}\left(K(k^2) - (1-b^2)\Pi(\alpha_+^2, k^2) - (1+b^2)\Pi(\alpha_-^2, k^2)\right)$$

Finally, continuity of the prefactors and the elliptic functions, gives that the rotation number function is continuous, as required.

□

C.2.1 Elliptic Integral Evaluation

Consider the Elliptic Integral,

$$I_p = \int_0^a \frac{d\xi}{(p-\xi^2)\sqrt{(a^2-\xi^2)(b^2+\xi^2)}}$$

Where, $p \neq 0$. From page 51, integral identity 214.13, and page 203, integral indentity 339.01, [8], we find that this reduces to,

$$I = \frac{g}{p\alpha^2}\left(k^2F(\varphi, k^2) + (\alpha^2 - k^2)\Pi(\varphi, \alpha^2, k^2)\right)$$

Where F and Π are elliptic integrals of the first and third kind respectively, $g = \frac{1}{\sqrt{a^2+b^2}}$, $k^2 = \frac{a^2}{a^2+b^2}$, and $\alpha^2 = \frac{(p+b^2)a^2}{p(a^2+b^2)}$. $\varphi = \frac{\pi}{2}$, and so the elliptic integrals are complete.

Computing the ratio of k^2 and α^2, we find, $\frac{k^2}{\alpha^2} = \frac{p}{p+b^2}$, and likewise $\frac{\alpha^2-k^2}{\alpha^2} = \frac{b^2}{p+b^2}$.

Thus, we have the integral identity,

$$\int_0^a \frac{d\xi}{(p-\xi^2)\sqrt{(a^2-\xi^2)(b^2+\xi^2)}} = \frac{g}{p+b^2}\left(pK(k^2) + b^2\Pi(\alpha^2, k^2)\right)$$