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Complex Analytic Aspects of Differential Equations

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Contents

Abstract

Second-order ordinary differential equations lie at the heart of many physical phenomena and are crucial to describing the nature and dynamics of our world. A class of these, with low-order polynomial coefficients, permit polynomial solutions given by Rodrigues' Formula. The solutions are summed as a power series to give a generating function, and several results of complex analysis are applied to manipulate and close this expression. In doing so, contour integral representations of the Hermite, Legendre and Laguerre polynomials are also observed.

1 Introduction

The study of differential equations dates back to the 17^{th} century, when the greats Newton and Liebniz independently published their respective theories of calculus, building upon the work of many other renowned mathematicians of the era. The discovery of this branch of mathematics led to many physical advances, including Newton's Theory of Motion and exact calculations of geometric areas and volumes. Over the coming centuries, mathematicians such as Cauchy, Riemann and Weierstrass developed and rigorised Calculus using a formal treatment with limits. At this stage, many techniques were being introduced and problems solved. In the mid 19^{th} century, Charles-François Sturm and Joseph Liouville investigated a general class of second-order, linear, homogenous ordinary differential equations (ODEs) and developed what is known as Sturm-Liouville Theory. Interestingly, many of the examples in this class gave rise to polynomial solutions when particular boundary conditions were asserted.

The formalisation of complex numbers came much after Newton and Liebniz, and the advent of complex analysis was not until the early 19^{th} century, when the French mathematician Augustin-Louis Cauchy described complex integration. This theory too found its way into the realm of physics, aiding the understanding of potential theory, fluid flow, general relativity and more recently string theory.

Complex analysis also allowed for new ways of solving and representing solutions to differential equations, which will be explored in this paper. We begin by introducing Sturm-Liouville systems and discussing several important distinctions and results in this class. We proceed by reviewing complex analysis and presenting some important theorems in the field. In Section 5, we present Rodrigues' Theorem for which a proof is included in Appendix A. With these tools in hand, we explore contour integral representations of solutions to particular SL systems. These representations are used to develop closed forms for the generating functions of the Hermite, Legendre and Laguerre polynomials. We extend this technique to some general classes of second-order, linear, homogenous ODEs.

2 Statement of Authorship

In this report, Sections 3, 4 and 5 explore a number of theorems, formulas and methods surrounding Sturm-Liouville systems and complex analysis. This theory has been developed, written about and published by many mathematicians, and the author of this paper does not take credit for this. The method used in Section 6 to find the generating functions for the Hermite and Legendre DEs has also been taken from existing literature. The treatment of the Laguerre system and the two consequent generalisations has been conducted independently by the author, in conjunction with his supervisor, Gregory Markowsky. While no publication on these generalisations was found, it is acknowledged that these too possibly exist in literature.

3 Sturm-Liouville Systems

Sturm-Liouville (SL) systems consist of a Sturm-Liouville differential equation together with some boundary conditions over a specified domain [1].

Definition 3.1 (Sturm-Liouville Differential Equation) *A second-order, homogenous, linear ODE of the form*

$$D\left[p(x)y'(x)\right] + \left(\lambda r(x) - q(x)\right)y(x) = 0, \tag{3.1}$$

where p, q, r are real functions of x over an interval $I = [a, b]$ (possibly infinite or semi-infinite), $p, r > 0$, except possibly at their endpoints, and λ is a real parameter.

To ensure solutions exist, we generally assume that $p \in \mathcal{C}^1$ and $q, r \in \mathcal{C}^0$, except possibly in the singular case. Given some boundary conditions, the solutions are called *eigenfunctions*, and the corresponding values of λ are the *eigenvalues*. A *regular* SL equation has I finite, p, q, r bounded and p, r positive over I. There are three classifications of SL systems - regular, periodic or singular.

Definition 3.2 (Regular SL System) *A regular SL system consists of a regular SL equation and boundary conditions of the form*

$$\alpha y(a) + \beta y'(a) = \gamma y(b) + \delta y'(b) = 0,$$

where neither (α, β) nor (γ, δ) are $(0, 0)$.

Definition 3.3 (Periodic SL System) *A periodic SL system has p, q, r being periodic functions with $b - a$ as a period, and boundary conditions of the form*

$$y(a) = y(b) \qquad y'(a) = y'(b).$$

Definition 3.4 (Singular SL System) *An SL system is singular if*

(i) I is semi-infinite or infinite, or

(ii) p or r vanish at an endpoint, or

(iii) q is discontinuous.

Boundary conditions may include homogenous statements such as y being bounded.

There are many fascinating results concerning the eigenfunctions of SL systems. From standard DE theory, for a regular SL equation we are guaranteed two linearly independent eigenfunctions for each eigenvalue λ. Interestingly, eigenfunctions belonging to distinct eigenvalues are pairwise orthogonal, in the manner described in Theorem 3.1.

Theorem 3.1 (Orthogonality of Eigenfunctions) *Suppose u and v are eigenfunctions of a regular or periodic SL system with distinct eigenvalues. Then*

$$\int_a^b r(x)u(x)v(x)dx = 0.$$

We say that u and v are orthogonal with respect to the weight function r (from 3.1) over $I = [a, b]$.

In Section 6, we study the properties of the Hermite, Legendre and Laguerre DEs, which are all examples of Sturm-Liouville systems.

4 Integration in the Complex Plane

There are a number of significant differences between integration over real and complex domains. In this section, we review a number of important results developed by Cauchy in the early 19$^{\text{th}}$ century. These will be used in Section 6 to develop the generating functions of several sets of polynomials. The results in this section can be found in Snider's *Fundamentals of Complex Analysis* [2].

Theorem 4.1 (Cauchy's Integral Theorem) *If f is an analytic function in a simply connected domain $\mathcal{D}$, and $\mathcal{C}$ is any closed loop in $\mathcal{D}$, then*

$$\oint_{\mathcal{C}} f(z)dz = 0.$$

Complex analysis also offers a useful way to express derivatives of a function using the generalised Cauchy Integral Formula.

Theorem 4.2 (Generalised Cauchy Integral Formula) *If f is an analytic function in a simply connected domain $\mathcal{D}$, $\mathcal{C}$ is any closed loop in $\mathcal{D}$, and x is a point inside $\mathcal{C}$, then*

$$f^{(n)}(x) = \frac{n!}{2\pi i}\oint_{\mathcal{C}} \frac{f(z)}{(z-x)^{n+1}}dz.$$

The case of $n = 0$ in the above is common and is know simply as the Cauchy Integral Formula. Finally, the Residue Theorem provides a useful way to calculate the contour integral of a function around a singularity.

Theorem 4.3 (Residue Theorem) *Suppose f is analytic in a simply connected domain $\mathcal{D}$ except at z_0, where there is an isolated singularity. Then f permits a Laurent expansion about z_0 of the form*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

The coefficient a_{-1} is called the residue of f at z_0, denoted by $res(f, z_0)$. If $\mathcal{C}$ is any closed loop around z_0 in $\mathcal{D}$, then

$$\oint_{\mathcal{C}} f(z)dz = 2\pi i \cdot res(f, z_0)$$

5 Rodrigues' Theorem

We now introduce a formula which provides polynomial solutions to a large class of second-order, linear, homogenous ODEs including the Hermite, Legendre, Laguerre and Chebyshev equations. We will denote by $w(x)$ the integrating factor of an ODE of the form

$$p(x)y''(x) + q(x)y'(x) + \lambda y(x) = 0. \tag{5.1}$$

Multiplying by this integrating factor would enable us to write the DE in self-adjoint form, and hence identify a Sturm-Liouville system. The integrating factor must satisfy

$$D\left[w(x)p(x)\right] = w(x)q(x) \tag{5.2}$$

and can therefore be given by the formula

$$w(x) = \frac{1}{p(x)} \exp\left(\int \frac{q(x)}{p(x)} dx\right). \tag{5.3}$$

We now arrive at the fascinating result that is Rodrigues' Theorem [3].

Theorem 5.1 (Rodrigues' Theorem) *If p is quadratic and q is linear, then (5.1) has polynomial solutions of degree n for each $n \in \mathbb{N} \cup \{0\}$. The eigenvalues are $\lambda_n = \frac{-n(n-1)}{2} p''(x) - nq'(x)$ and the eigenfunctions are given by Rodrigues' Formula:*

$$y_n(x) = \frac{1}{w(x)} D^n \left[w(x)p(x)^n\right]. \tag{5.4}$$

The proof of this formula is an arduous journey in algebra, and is included in Appendix A for the interested reader. Our focus is on the interplay between Rodrigues' Formula and complex analysis. In particular, we will later transform the D^n condition using Theorem 4.2 to express the polynomial solutions as contour integrals.

6 Applications to Generating Functions

6.1 Introduction

Rodrigues' Formula provides a succinct way of expressing the polynomial solutions to SL systems of the form (5.1). Letting $Q_n(x)$ be the polynomial solution of degree n, we can define the generating function in the context of solutions to SL systems as follows.

Definition 6.1 (Generating Function) *This is the infinite power series in t with the eigenfunctions as its coefficients, given by*

$$G(x,t) = \sum_{n=0}^{\infty} Q_n(x)t^n. \tag{6.1}$$

We also define the exponential generating function as another means of succinctly capturing a sequence of functions.

Definition 6.2 (Exponential Generating Function) *This is the infinite power series in t with the scaled eigenfunctions as its coefficients, given by*

$$G(x,t) = \sum_{n=0}^{\infty} Q_n(x)\frac{t^n}{n!}. \tag{6.2}$$

While we predominantly seek the standard generating function, in the Hermite DE it will be more suitable to find the exponential generating function. Instead of an infinite power series, the generating function can often be written in closed form, which captures all the coefficients of every eigenfunction into a single, short expression. With the aid of complex analysis, we will show how this can be done for the Hermite, Legendre and Laguerre DEs. We conclude this section with two generalisations of the techniques we have used.

6.2 Hermite Differential Equation

We begin by treating the Hermite DE, which was studied in detail by Pafnuty Chebyshev in the mid 19$^{\text{th}}$ century. Among the solutions to this DE are the Hermite Polynomials, which arise in many fields including quantum mechanics, Brownian motion, signal processing and combinatorics [4].

Definition 6.3 (Hermite Differential Equation)

$$y''(x) - 2xy'(x) + \lambda y(x) = 0$$

The Hermite DE satisfies the conditions for Rodrigues' Theorem, with $p(x) = 1$ and $q(x) = -2x$. Using (5.3), the integrating factor is $w(x) = e^{-x^2}$. Rodrigues' Formula tells us that the polynomial solutions to Hermite's DE are given by

$$y_n = e^{x^2} D^n \left[e^{-x^2}\right],$$

with eigenvalues $\lambda_n = 2n$. Due to the homogeneity of SL systems, we may scale these polynomials arbitrarily and they will remain a solution. The Hermite polynomials are scaled to have leading coefficient 2^n, which requires the above polynomials to be multiplied by $(-1)^n$. By Theorem 4.2, they can therefore be written in contour integral form as

$$H_n(x) = (-1)^n e^{x^2} \frac{n!}{2\pi i} \oint_{\mathcal{C}} \frac{e^{-z^2}}{(z-x)^{n+1}} dz. \tag{6.3}$$

For context, we list the first few Hermite polynomials.

$$\begin{aligned} &H_0(x) = 1 && \lambda_0 = 0 \\ &H_1(x) = 2x && \lambda_1 = 2 \\ &H_2(x) = 4x^2 - 2 && \lambda_2 = 4 \end{aligned}$$

Substituting (6.3) into (6.2) gives the generating function as

$$G(x,t) = \sum_{n=0}^{\infty} (-1)^n e^{x^2} \frac{n!}{2\pi i} \oint_{\mathcal{C}} \frac{e^{-z^2}}{(z-x)^{n+1}} dz \frac{t^n}{n!}.$$

We will collect the terms with exponents n and sum the resulting geometric series. We will then apply the generalised Cauchy Integral Formula to evaluate the contour integral.

$$\begin{aligned} G(x,t) &= \frac{e^{x^2}}{2\pi i} \oint_{\mathcal{C}} \sum_{n=0}^{\infty} \frac{e^{-z^2}(-t)^n}{(z-x)^{n+1}} dz \\ &= \frac{e^{x^2}}{2\pi i} \oint_{\mathcal{C}} \frac{e^{-z^2}}{z-x} \sum_{n=0}^{\infty} \left(\frac{-t}{z-x} \right)^n dz \\ &= \frac{e^{x^2}}{2\pi i} \oint_{\mathcal{C}} \frac{e^{-z^2}}{z-x} \cdot \frac{1}{1+\frac{t}{z-x}} dw \\ &= \frac{e^{x^2}}{2\pi i} \oint_{\mathcal{C}} \frac{e^{-z^2}}{z-x+t} dz \\ &= \frac{e^{x^2}}{2\pi i} \cdot 2\pi i e^{-(x-t)^2} && \text{(By Thm. 4.2)} \\ &= e^{2xt-t^2} \end{aligned}$$

Thus we have established a closed form of the exponential generating function for the Hermite polynomials. We notice that in the expansion of $e^{2xt-t^2} \cdot t^{-(n+1)}$, the coefficient of t^{-1} is $\frac{H_n(x)}{n!}$. By Residue Theorem, this gives us another way to express the Hermite polynomials.

Formula 6.1 (Contour integral representation of Hermite polynomials)

$$H_n(x) = \frac{n!}{2\pi i} \oint_{\mathcal{C}} \frac{e^{2xt-t^2}}{t^{n+1}} dt$$

6.3 Legendre Differential Equation

We now turn our attention to another famous differential equation - the Legendre DE. This equation is encountered in many branches of physics, engineering and mathematics. It arises in the determination of wave functions of electrons, nuclear reactor physics, neutron scattering calculations [5] and some trigonometric identities.

Definition 6.4 (Legendre Differential Equation)

$$(1-x^2)y''(x) - 2xy'(x) + \lambda y(x) = 0$$

The equation is self-adjoint as an integrating factor is not required; it is simply $w(x) = 1$. Hence, the Legendre DE defines an SL system, and we are guaranteed an orthogonal set of solutions by Theorem 3.1. Furthermore, this satisfies the conditions for Rodrigues' Formula, with $p(x) = 1 - x^2$ and $q(x) = -2x$. The solutions are given by the formula as

$$y_n = D^n\left[(1-x^2)^n\right]$$

with eigenvalues $\lambda_n = n(n+1)$. The Legendre polynomials are scaled to satisfy $P(1) = 1$, so that the sum of the coefficients is 1. The scaling factor is $\frac{(-1)^n}{2^n n!}$, so we have

$$\begin{aligned} P_n(x) &= \frac{(-1)^n}{2^n n!} D^n\left[(1-x^2)^n\right] \\ &= \frac{1}{2^n n!} D^n\left[(x^2-1)^n\right] \end{aligned}$$

This gives us our first contour integral representation of the Legendre polynomials. The formula was discovered by and named after Ludwig Schläfli in 1881 [6].

Formula 6.2 (Schläfli integral representation of Legendre polynomials)

$$P_n(x) = \frac{1}{2^{n+1}\pi i} \oint_{\mathcal{C}} \frac{(z^2-1)^n}{(z-x)^{n+1}} dz.$$

The first few Legendre polynomials are

$$\begin{aligned} P_0(x) &= 1 & \lambda_0 &= 0 \\ P_1(x) &= x & \lambda_1 &= 2 \\ P_2(x) &= \frac{1}{2}(3x^2-1) & \lambda_2 &= 6 \end{aligned}$$

We now turn our attention to the generating function of the Legendre polynomials. We follow the method presented by Evans in his lecture notes [7]. Substituting Formula 6.2 into (6.1) and following steps similar to the Hermite case gives

$$\begin{aligned} G(x,t) &= \sum_{n=0}^{\infty} \frac{1}{2^{n+1}\pi i} \oint_{\mathcal{C}} \frac{(z^2-1)^n}{(z-x)^{n+1}} dz \cdot t^n \\ &= \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{1}{z-x} \sum_{n=0}^{\infty} \left(\frac{t(z^2-1)}{2(z-x)}\right)^n dz \\ &= \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{1}{z-x} \frac{1}{1 - \frac{t(z^2-1)}{2(z-x)}} dz \\ &= \frac{-1}{\pi i} \oint_{\mathcal{C}} \frac{1}{-2(z-x) + t(z^2-1)} dz. \end{aligned}$$

The denominator is a quadratic in z, with roots $z_\pm = \frac{1}{t}(1 \pm \sqrt{1-2xt+t^2})$. We desire convergence of $G(x,t)$, at the least when $|t|$ is small. As $t \to 0$, $z_+ \to \infty$ whereas $z_- \to x$. Hence, for t sufficiently small, the pole at z_-

will lie inside the contour $\mathcal{C}$ (recalling that $\mathcal{C}$ was any contour around x) and so by Cauchy Integral Formula,

$$\begin{aligned} G(x,t) &= \frac{-1}{\pi i}\oint_{\mathcal{C}} \frac{1}{t(z-z_-)(z-z_+)}dz \\ &= \frac{-1}{\pi i}\oint_{\mathcal{C}} \frac{\left(\frac{1}{t(z-z_+)}\right)}{z-z_-}dz \\ &= \frac{-1}{\pi i}\frac{2\pi i}{t(z_- - z_+)} \\ &= \frac{-2}{t(\frac{-2}{t}\sqrt{1-2xt+t^2})} \\ &= \frac{1}{\sqrt{1-2xt+t^2}}. \end{aligned}$$

Hence, we have established the generating function for the Legendre polynomials. By Residue Theorem, this gives us another way to express the Legendre polynomials.

Formula 6.3 (Contour integral representation of Legendre polynomials)

$$P_n(x) = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{1}{t^{n+1}\sqrt{1-2xt+t^2}}dt.$$

6.4 Laguerre Differential Equation

The final equation we explore in this section is the Laguerre DE, which has applications in quantum mechanics and optics [8].

Definition 6.5 (Laguerre Differential Equation)

$$xy''(x) + (1-x)y'(x) + \lambda y(x) = 0$$

Again, this satisfies the conditions for Rodrigues' Theorem, with $p(x) = x$ and $q(x) = 1 - x$. The integrating factor, calculated using Formula 5.3, is $w(x) = e^{-x}$. The solutions are then given by Rodrigues' Formula as

$$y_n = e^x D^n \left[e^{-x}x^n\right]$$

with eigenvalues $\lambda_n = n$. The Laguerre polynomials are scaled so that the constant term is 1, giving

$$L_n(x) = \frac{e^x}{n!}D^n\left[e^{-x}x^n\right].$$

Using (4.2), this gives us our first contour integral representation of the Laguerre Polynomials as

$$L_n(x) = \frac{e^x}{2\pi i}\oint_{\mathcal{C}} \frac{e^{-z}z^n}{(z-x)^{n+1}}dz. \tag{6.4}$$

The first few Laguerre polynomials are

$$\begin{aligned} L_0(x) &= 1 & \lambda_0 &= 0 \\ L_1(x) &= 1 - x & \lambda_1 &= 1 \\ L_2(x) &= \frac{1}{2}(x^2 - 4x + 2) & \lambda_2 &= 2 \end{aligned}$$

Substituting (6.4) into (6.1) gives

$$G(x,t) = \sum_{n=0}^{\infty} \frac{e^x}{2\pi i} \oint_{\mathcal{C}} \frac{e^{-z} z^n}{(z-x)^{n+1}} dz t^n.$$

We deal with this expression in a similar way to the Hermite case.

$$\begin{aligned} G(x,t) &= \frac{e^x}{2\pi i} \oint_{\mathcal{C}} \frac{e^{-z}}{z-x} \sum_{n=0}^{\infty} \left(\frac{zt}{z-x} \right)^n dz \\ &= \frac{e^x}{2\pi i} \oint_{\mathcal{C}} \frac{e^{-z}}{z-x} \frac{1}{1-\frac{zt}{z-x}} dz \\ &= \frac{e^x}{2\pi i} \oint_{\mathcal{C}} \frac{e^{-z}}{z-x-zt} dz \\ &= \frac{e^x}{2\pi i(1-t)} \oint_{\mathcal{C}} \frac{e^{-z}}{z-\frac{x}{1-t}} dz \\ &= \frac{e^x}{2\pi i(1-t)} \cdot 2\pi i e^{-\frac{x}{1-t}} && \text{(By Thm. 4.2)} \\ &= \frac{e^{\frac{-xt}{1-t}}}{1-t} \end{aligned}$$

Hence we have established the generating function for the Laguerre polynomials. By Residue Theorem, this gives us another way to express the Laguerre polynomials.

Formula 6.4 (Contour integral representation of Laguerre polynomials)

$$L_n(x) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{e^{\frac{-xt}{1-t}}}{(1-t)t^{n+1}} dt.$$

6.5 Generalisation for p linear

We now present a generalisation of the results for the Hermite and Laguerre DEs. Suppose $A_n(x)$ is a solution associated with eigenvalue λ_n of the second-order, linear, homogenous ODE with linear coefficients, taking the form

$$(bx+c)y'' + (dx+e)y' + \lambda y = 0. \tag{6.5}$$

By (5.3), the integrating factor is then given by

$$w(x) = \frac{1}{bx+c} \exp\left(\int \frac{dx+e}{bx+c} dx \right).$$

We can evaluate the integral in the exponential for $b \neq 0$ as

$$\begin{aligned} \int \frac{dx+e}{bx+c} dx &= \int \frac{d}{b} + \frac{e - \frac{cd}{b}}{bx+c} dx \\ &= \frac{d}{b} x + \frac{be-cd}{b^2} \ln(bx+c). \end{aligned}$$

The integrating factor for $b \neq 0$ can therefore be expressed as

$$w(x) = (bx+c)^{\frac{be-cd}{b^2} - 1} e^{\frac{d}{b} x}. \tag{6.6}$$

For $b = 0$, the integrating factor is

$$w_0(x) = \frac{1}{c} e^{\frac{(dx+e)^2}{2cd}} \tag{6.7}$$

Applying Rodrigues' Formula, and scaling by $\frac{1}{n!}$, the polynomial solutions to this ODE are

$$\begin{aligned} A_n(x) &= \frac{1}{n! w(x)} D^n \left[w(x) p(x)^n \right] \\ &= \frac{1}{2\pi i w(x)} \oint_{\mathcal{C}} \frac{w(z) p(z)^n}{(z-x)^{n+1}} dz. && \text{(By Thm. 4.2)} \end{aligned}$$

Plugging this into (6.1), we see that

$$\begin{aligned} G(x,t) &= \sum_{n=0}^{\infty} \frac{1}{2\pi i w(x)} \oint_{\mathcal{C}} \frac{w(z) p(z)^n}{(z-x)^{n+1}} dz \ t^n \\ &= \frac{1}{2\pi i w(x)} \oint_{\mathcal{C}} \frac{w(z)}{z-x} \sum_{n=0}^{\infty} \left(\frac{p(z) t}{z-x} \right)^n dz. \end{aligned}$$

Summing the geometric series gives

$$\begin{aligned} G(x,t) &= \frac{1}{2\pi i w(x)} \oint_{\mathcal{C}} \frac{w(z)}{z-x} \frac{1}{1 - \frac{p(z)t}{z-x}} dz \\ &= \frac{1}{2\pi i w(x)} \oint_{\mathcal{C}} \frac{w(z)}{z - x - (bz+c)t} dz \\ &= \frac{1}{2\pi i w(x)(1-bt)} \oint_{\mathcal{C}} \frac{w(z)}{z - \frac{x+ct}{1-bt}} dz \\ &= \frac{1}{2\pi i w(x)(1-bt)} 2\pi i w \left(\frac{x+ct}{1-bt} \right) \\ &= \frac{w\left(\frac{x+ct}{1-bt}\right)}{w(x)(1-bt)}. \end{aligned}$$

We hence arrive at a succinct expression for the generating function of the polynomial solutions to (6.5). By substituting (6.6) into the above equation, we find the following formula for the generating function.

Formula 6.5 (Generating Function when p is linear) *For $b \neq 0$, the generating function is*

$$G(x,t) = \frac{e^{\frac{dt(bx+c)}{b(1-bt)}}}{(1-bt)^{\frac{be-cd}{b^2}}}$$

For $b = 0$, this becomes

$$G(x,t) = e^{t(dx+e) + \frac{cdt^2}{2}}$$

Once again, applying Residue Theorem, we can express the solutions $A_n(x)$ in the following way.

Formula 6.6 (Contour integral representation of polynomial solutions) *For $b \neq 0$, the polynomials solutions to (6.5) can be written as*

$$A_n(x) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{e^{\frac{dt(bx+c)}{b(1-bt)}}}{(1-bt)^{\frac{be-cd}{b^2}} t^{n+1}} dt.$$

6.6 Generalisation for p quadratic

We now explore a further generalisation, for when p has degree 2. Suppose $A_n(x)$ is a solution associated with eigenvalue λ_n of the ODE with form

$$(ax^2+bx+c)y''+(dx+e)y'+\lambda y=0. \tag{6.8}$$

By (5.3), the integrating factor is now given by

$$w(x)=\frac{1}{ax^2+bx+c}\exp\left(\int\frac{dx+e}{ax^2+bx+c}dx\right).$$

As in Section 6.5, by applying Rodrigues' Theorem, scaling by $\frac{1}{n!}$ and applying generalised Cauchy Integral Formula, we find that

$$A_n(x)=\frac{1}{2\pi i w(x)}\oint_{\mathcal{C}}\frac{w(z)p(z)^n}{(z-x)^{n+1}}dz.$$

We again plug this result into (6.1). We have skipped the first two steps as they are identical to Section 6.5.

$$\begin{aligned}G(x,t)&=\frac{1}{2\pi i w(x)}\oint_{\mathcal{C}}\frac{w(z)}{z-x}\frac{1}{1-\frac{p(z)t}{z-x}}dz\\&=\frac{1}{2\pi i w(x)}\oint_{\mathcal{C}}\frac{w(z)}{z-x-(az^2+bz+c)t}dz\\&=\frac{1}{-2\pi i a t w(x)}\oint_{\mathcal{C}}\frac{w(z)}{z^2+\frac{bt-1}{at}z+\frac{x+ct}{at}}dz\end{aligned}$$

Applying the quadratic formula, the roots of the denominator are

$$\begin{aligned}z_\pm&=\frac{1}{2at}\left(1-bt\pm\sqrt{(bt-1)^2-4at(x+ct)}\right)\\&=\frac{1}{2at}\left(1-bt\pm\sqrt{1-2bt+b^2t^2-4atx-4act^2}\right)\\&\approx\frac{1}{2at}\left(1-bt\pm\left(1+\frac{1}{2}\left(-2bt+b^2t^2-4atx-4act^2\right)\right)\right)\\&=\frac{1}{2at}\left(1-bt\pm\left(1-bt+\frac{1}{2}b^2t^2-2atx-2act^2\right)\right).\end{aligned}$$

In the last step, we have applied the approximation $\sqrt{1+k}\approx 1+\frac{1}{2}k$ which holds true for small k, seen by expanding the square root as a Taylor Series. Simplifying, we find that as $t\to 0$, we have

$$\begin{aligned}z_+&=\frac{1}{2at}\left(2-2bt+\frac{1}{2}b^2t^2-2atx-2act^2\right)\to\infty\\z_-&=\frac{1}{2at}\left(1-bt-1+bt-\frac{1}{2}b^2t^2+2atx+2act^2\right)\\&=\frac{-\frac{1}{2}b^2t^2+2atx+2act^2}{2at}\\&=-\frac{b^2t}{4a}+x+ct\to x.\end{aligned}$$

We recall that the contour $\mathcal{C}$ need only enclose x. Hence for t sufficiently small, $\mathcal{C}$ will enclose z_-, so we may apply the Cauchy Integral Formula to find that

$$\begin{aligned} G(x,t) &= \frac{-1}{2\pi i a t w(x)} \oint_{\mathcal{C}} \frac{w(z)}{(z-z_-)(z-z_+)} dz \\ &= \frac{-1}{2\pi i a t w(x)} \cdot 2\pi i \frac{w(z_-)}{z_- - z_+} \\ &= \frac{w(z_-)}{w(x)\sqrt{(bt-1)^2 - 4at(x+ct)}} \end{aligned}$$

where $z_- = x - \frac{b^2-4ac}{4a} t$ and $w(x)$ were found earlier.

7 Conclusion

In this report, we looked at several famous ODEs which could be brought to Sturm-Liouville form with an integrating factor. These form part of a large subset of second-order differential equations which permit polynomial solutions given by Rodrigues' Formula. Further, we have seen that there are polynomial solutions for each non-negative degree, and these can be placed as coefficients in a power series to create a generating function. With the aid of several theorems of complex analysis, we have seen how the generating functions of the Hermite, Legendre and Laguerre polynomials can be expressed in closed form. Finally, we have studied the generalisations of these examples and presented the resultant formulas. These could be used to easily find the polynomial solutions of ODEs that arise in physics and nature, or to verify the correctness of existing solutions.

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Appendix A

A proof of Rodrigues' Theorem, provided by Harris [3]. We start by using (5.2) above to re-express the following derivative.

$$\begin{aligned} pD[wp^n] &= pD[(wp)\cdot p^{n-1}] \\ &= p\left(D[wp]p^{n-1} + wpD[p^{n-1}]\right) \\ &= p\left(wq\cdot p^{n-1} + wp\cdot(n-1)p^{n-2}D[p]\right). \\ \therefore pD[wp^n] &= wp^n\left(q + (n-1)D[p]\right). \end{aligned}$$

We then differentiate the above equation $n+1$ times to obtain

$$\sum_{i=0}^{n+1}\binom{n+1}{i}D^i[p]D^{n+2-i}[wp^n] = \sum_{i=0}^{n+1}\binom{n+1}{i}D^i[wp^n]D^{n+1-i}[q+(n-1)D[p]].$$

Many of these derivatives vanish as p is quadratic and $q+(n-1)D[p]$ is linear. Omitting these and dividing by w gives

$$\frac{p}{w}D^{n+2}[wp^n] + \frac{(n+1)D[p]}{w}D^{n+1}[wp^n] + \frac{n(n+1)D^2[p]}{2w}D^n[wp^n] =$$
$$\frac{q+(n-1)D[p]}{w}D^{n+1}[wp^n] + \frac{(n+1)(D[q]+(n-1)D^2[p])}{w}D^n[wp^n].$$

We use (5.4) above to place y_n wherever we can, and collect the terms to the left side.

$$\frac{p}{w}D^{n+2}[wp^n] + \frac{2D[p]-q}{w}D^{n+1}[wp^n] + \left(\frac{n(n+1)D^2[p]}{2} - (n+1)D[q] - (n+1)(n-1)D^2[p]\right)y_n = 0.$$

$$\therefore \frac{p}{w}D^{n+2}[wp^n] + \frac{2D[p]-q}{w}D^{n+1}[wp^n] - \left(\frac{n^2-n-2}{2}D^2[p] + (n+1)D[q]\right)y_n = 0. \tag{7.1}$$

We will develop a useful identity to represent the terms of (7.1) as derivatives of y_n. Firstly,

$$pD^2\left[\frac{1}{w}D^n[wp^n]\right] = p\frac{1}{w}D^{n+2}[wp^n] + 2pD\left[\frac{1}{w}\right]D^{n+1}[wp^n] + pD^2\left[\frac{1}{w}\right]D^n[wp^n].$$

Hence we may re-express

$$\frac{p}{w}D^{n+2}[wp^n] = pD^2[y_n] - 2pD\left[\frac{1}{w}\right]D^{n+1}[wp^n] - wpD^2\left[\frac{1}{w}\right]y_n. \tag{7.2}$$

Notice that since $D[wp] = wq$ (5.2), we have by product rule that

$$D[w]p + wD[p] = wq \Rightarrow D[w] = \frac{w(q-D[p])}{p}.$$

Hence,

$$D\left[\frac{1}{w}\right] = \frac{-1}{w^2}D[w] = \frac{D[p]-q}{pw}. \tag{7.3}$$

and

$$\begin{aligned} D^2\left[\frac{1}{w}\right] &= D\left[\frac{D[p]-q}{pw}\right] \\ &= \frac{(D^2[p]-D[q])pw - (D[p]-q)D[pw]}{p^2w^2} \\ &= \frac{D^2[p]-D[q]}{pw} - \frac{(D[p]-q)q}{p^2w}. \end{aligned}$$

Substituting these into 7.2 gives

$$\begin{aligned}\frac{p}{w}D^{n+2}[wp^n] &= py_n'' - 2p\frac{D[p]-q}{pw}D^{n+1}[wp^n] - wp\left(\frac{D^2[p]-D[q]}{pw} - \frac{(D[p]-q)q}{p^2w}\right)y_n \\ &= py_n'' - 2\frac{D[p]-q}{w}D^{n+1}[wp^n] - \left(D^2[p]-D[q]-\frac{(D[p]-q)q}{p}\right)y_n.\end{aligned}$$

Hence, (7.1) becomes

$$\begin{aligned}py_n'' - 2\frac{D[p]-q}{w}D^{n+1}[wp^n] - \left(D^2[p]-D[q]-\frac{(D[p]-q)q}{p}\right)y_n + \frac{2D[p]-q}{w}D^{n+1}[wp^n] \\ - \left(\frac{n^2-n-2}{2}D^2[p] + (n+1)D[q]\right)y_n = 0. \\ \Rightarrow py_n'' + \frac{q}{w}D^{n+1}[wp^n] - \left(\frac{n^2-n}{2}D^2[p] + nD[q] - \frac{(D[p]-q)q}{p}\right)y_n = 0.\end{aligned}$$

Note that the middle term can be rewritten as $qy_n' + \frac{q(q-D[p])}{p}y_n$ since

$$\begin{aligned}qy_n' + \frac{q(q-D[p])}{p}y_n &= qD\left[\frac{1}{w}D^n[wp^n]\right] + \frac{q(q-D[p])}{p}\frac{1}{w}D^n[wp^n] \\ &= qD\left[\frac{1}{w}\right]D^n[wp^n] + \frac{q}{w}D^{n+1}[wp^n] + \frac{q(q-D[p])}{pw}D^n[wp^n] \\ &= \frac{q}{w}D^{n+1}[wp^n].\end{aligned}$$

The last step is due to (7.3). Our equation becomes

$$\begin{aligned}py_n'' + qy_n' + \frac{q(q-D[p])}{p}y_n - \left(\frac{n^2-n}{2}D^2[p] + nD[q] - \frac{(D[p]-q)q}{p}\right)y_n = 0 \\ \Rightarrow py_n'' + qy_n' - \left(\frac{n^2-n}{2}D^2[p] + nD[q]\right)y_n = 0\end{aligned}$$

We see now that y_n satisfies the differential equation (5.1) with eigenvalue $\lambda_n = -\left(\frac{n^2-n}{2}D^2[p] + nD[q]\right)$, which we note is a constant since p is at most quadratic and q is at most linear.