AMSI VACATION RESEARCH SCHOLARSHIPS 2021–22

Get a taste for Research this Summer

The Representation Theory of Hecke Algebras through the Knizhnik-Zamolodchikov Functor

Muhammad Haris Rao

Supervised by Prof. Arun Ram and Dr. Yaping Yang

The University of Melbourne

AMSI AUSTRALIAN MATHEMATICAL SCIENCES INSTITUTE

Abstract

Associated to groups of transformations of a complex vector space generated by reflections are two algebras called the rational Cherednik algebra and the Hecke algebra. In this project, we studied the connection between the representation theories of these two structures through the Knizhnik-Zamolodchikov functor. The differential equations which determine the Hecke algebra modules outputted by the functor are expressed explicitly in several examples, and in the simplest case of a cyclic group, the solution is computed to specify the Hecke algebra representation explicitly.

Acknowledgements

I would like to express my gratitude to my advisors Prof. Arun Ram and Dr. Yaping Yang, without whom this project would not have been possible. I am also grateful to the Australian mathematical sciences institute for providing the opportunity to take part in their vacation research program and produce this work. This project was closely related to that of another AMSI scholar, Yifan Guo, and my discussions with her and her supervisor Dr. Ting Xue also enabled much of this project.

Statement of Authorship

This whole report was written by me, and reviewed by Prof. Arun Ram and Dr. Yaping Yang.

A lot of the discussion in the Hecke algebra section was my own elaboration of the the general topological definition given in [EM10], and the definition in terms of generators and relations given in [AR21], but there are no new results.

The algebraic manipulations to obtain the differential for the dihedral groups and symmetric groups worked out in the section on Monodromy Representations were almost completely done by me, with some special cases being cross checked with the computations of Yifan Guo.

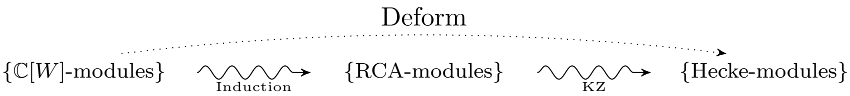
Background information on topology, group theory, and representation theory was taken from standard texts on the subject.

Contents

1 Introduction

Associated to any complex reflection group are two algebras called the rational Cherednik algebra and Hecke algebra, each specified by some complex parameters. In the 2003 paper titled "On the category $\mathcal{O}$ of the rational Cherednik algebra" [GGOR03], the authors Victor Ginzburg, Nicholas Guay, Eric Opdam and Raphaël Rouquier establised a connection between the representation theories of these two algebras through the so called Knizhnik-Zamolodchikov functor.

Roughly speaking, any representation of a complex reflection group induces a representations of its rational Cherednik algebra, and can also be deformed to specify representations of its Hecke algebra. The remaining link between the rational Cherednik algebra and Hecke algebra is the Knizhnik-Zamolodchikov functor.



In this project, we intend on studying the deformation that produces representations of the Hecke algebra, and briefly describe how the parameters of the rational Cherednik algebra defines such a deformation to give us the functor.

2 Hecke Algebras

Here, we will follow the definition of the Hecke algebra given by X. Ma and P. Etingof in [EM10].

Associated to any complex reflection group W on a complex vector space V with refelction hyperplanes $\mathcal{A}$, we define

$$V^{\text{reg}} = V - \bigcup_{H \in \mathcal{A}} H$$

It can be shown that V^{reg} is path connected. Given $x, y \in V^{\text{reg}}$, say that $x \sim y$ if there is $w \in W$ such that $y = wx$. Clearly, this is an equivalence relation, so defines a quotient map

$$q : V^{\text{reg}} \longrightarrow V^{\text{reg}}/W$$

which induces a topology on the quotient space V^{reg}/W of equivalence classes. The braid group B_W based at $\mathfrak{a}_0 \in V^{\text{reg}}/W$ is

$$B_W = \pi_1 \left(V^{\text{reg}}/W, \mathfrak{a}_0\right)$$

There is a continuous small loop around $q(H) \subseteq V^{\text{reg}}/W$ for each hyperplane $H \in \mathcal{A}$. More precisely, the pointwise stabiliser of H is a cyclic subgroup of W of some order m_H. So let $s \in W$ be the reflection element reflecting across H whose non-unit eigenvalue is $\zeta = e^{2\pi \mathbf{i}/m_H}$. Let $v_s^{\perp} \in V$ be a non-zero eigenvector of s, with assocaited eigenvalue $\zeta \neq 1$. Let $v_s \in H$, $v_s \neq 0$. Then we have a path $\gamma_H : [0, 1] \longrightarrow V^{\text{reg}}$ defined by

$$\gamma_H(t) = v_s + \varepsilon \zeta^t v_s^{\perp}$$

For some small enough $\varepsilon > 0$, this will be a path inside V^{reg}. Composing with the quotient map gives the desired loop γ_H around the image of H in V^{reg}/W. It can be checked that picking a different $v_s, v_s^{\perp}$ and different $\varepsilon > 0$ (provided ε is sufficiently small) will give homotopic loops in V^{reg}/W. Moreover, the loop obtained in this way at a different reflection hyperplane conjugate to the one we just worked with will also give a homotopic loop in V^{reg}/W.

From elementary algebraic topology, it is known that two loops in V^{reg}/W based at $\mathfrak{a}_0$ are conjugate in B_W if and only if they are homotopic as loops in V^{reg}/W without fixed basepoints. So this loop γ_H defines a conjugacy class of the braid group B_W. Let T_H be a representative of this conjugacy class.

Now for each hyperplane $H \in \mathcal{A}$, choose complex parameters $\{q_{j,H}\}_{j=1}^{m_H - 1}$ such that whenever $H' = wH$, also $q_{j,H} = q_{j,H'}$. If $H' = wH$ then it is easily checked that the pointwise stabilisers of each hyperplane have the same order so that this restriction makes sense.

The Hecke algebra $\mathcal{H}_q(W)$ is the quotient

$$\mathcal{H}_q(W) = \mathbb{C}[B_W] \Big/ \Big\langle (T_H - 1) \prod_{j=1}^{m_H - 1} \left(T_H - e^{2j\pi \mathbf{i}/m_H} q_{j,H} \right) \text{ , for all } H \in \mathcal{A} \Big\rangle$$

From now, write $q^*_{j,H} = e^{2j\pi \mathbf{i}/m_H} q_{j,H}$. Recall that the T_H was chosen as a representative of the conjugacy class defined by a small loop around the image of H in the orbit space V^{reg}/W. The definition above is independent of this choice because if $T'_H = \gamma T_H \gamma^{-1}$, then the relation for T_H holds if and only if it holds for T'_H since

$$\begin{aligned}
(T'_H - 1) \prod_{j=1}^{m_{H'}-1} (T'_H - q^*_{j,H'}) &= \left(\gamma T_H \gamma^{-1} - 1\gamma\gamma^{-1}\right) \left(\gamma T_H \gamma^{-1} - q^*_{1,H}\gamma\gamma^{-1}\right) \cdots \left(\gamma T_H \gamma^{-1} - q^*_{m_H-1,H}\gamma\gamma^{-1}\right) \\
&= \left[\gamma (T_1 - 1) \gamma^{-1}\right] \left[\gamma \left(T_H - q^*_{1,H}\right) \gamma^{-1}\right] \cdots \left[\gamma \left(T_H - q^*_{m_H-1,H}\right) \gamma^{-1}\right] \\
&= \gamma \left[(T_H - 1) \prod_{j=1}^{m_H-1} \left(T_H - q^*_{j,H}\right) \right] \gamma^{-1}
\end{aligned}$$

2.1 Cyclic Group Case $W = G(1,1,r)$

Here, $V^{\mathrm{reg}} = \mathbb{C} - \{0\}$. Given an equivalence class $p \in V^{\mathrm{reg}}$, it is easily seen that the orbit of p is

$$[p] = \left\{ e^{2k\pi \mathbf{i}/r} p \mid 0 \le k < r \right\}$$

Using the convention that the principal argument of a complex number is in $[0, 2\pi)$, there is always precisely one point p_0 in the orbit $[p]$ whose argument is in $[0, 2\pi/r)$. Define

$$\begin{aligned}
\phi : V^{\mathrm{reg}}/W &\longrightarrow V^{\mathrm{reg}} \\
[p] &\longmapsto |p_0| e^{r\arg(p_0) i}
\end{aligned}$$

The map ϕ defines a homeomorphism $V^{\mathrm{reg}}/W \cong V^{\mathrm{reg}}$, and so

$$B_W = \pi\left(V^{\mathrm{reg}}/W, \mathfrak{a}_0\right) \cong \pi\left(V^{\mathrm{reg}}, \phi\left(\mathfrak{a}_0\right)\right) = \pi\left(\mathbb{C} - \{0\}\right) \cong \mathbb{Z}$$

The homomorphism between $\pi\left(V^{\mathrm{reg}}/W, \mathfrak{a}_0\right)$ and $\pi\left(V^{\mathrm{reg}}, \phi\left(\mathfrak{a}_0\right)\right)$ is the one induced by the homeomorphism ϕ. The group $\pi\left(V^{\mathrm{reg}}, \phi(\mathfrak{a}_0)\right)$ is generated by the loop γ based at $\phi\left(\mathfrak{a}_0\right)$ which goes around the origin anticlockwise once. Thus, the corresponding loop in V^{reg}/W based at $\mathfrak{a}_0$ obtained by pushing through ϕ^{-1} is the generator of the braid group $\pi_1(V^{\text{reg}}/W, \mathfrak{a}_0)$. That is, $\pi_1(V^{\text{reg}}/W, \mathfrak{a}_0)$ is generated by the loop

$$T_1 = \phi^{-1} \circ \gamma : [0,1] \longrightarrow V^{\text{reg}}/W$$

This means that the group algebra of the braid group $\mathbb{C}[B_W]$ is the free algebra generated by T_1. Morevoer, notice that T_1 is a satisfactory choice of a small loop around the single hyperplane in V fixed by W, as described in the definition of the Hecke algebra. Thus, given complex parameters $q = (q_1, q_2, \cdots, q_{r-1}) \subseteq \mathbb{C}$, the Hecke algebra has the following presentation in terms of generators and relations

$$\begin{aligned}\mathcal{H}_q(W) &= \mathbb{C}[B_W] \Big/ \Big\langle (T_1 - 1) \prod_{j=1}^{r-1} \left(T_1 - e^{2j\pi \mathbf{i}/r} q_j\right) \Big\rangle \\ &= \Big\langle T_1 \mid (T_1 - 1) \prod_{j=1}^{r-1} \left(T_1 - e^{2j\pi \mathbf{i}/r} q_j\right) \Big\rangle\end{aligned}$$

2.2 Coxeter Group Case

Let $S = \{s_1, s_2, \cdots, s_N\}$ be any finite set, $M = [m_{ij}]_{i,j=1}^{N}$ a Coxeter matrix, and W_M to the Coxeter group with these specifications.

Then W_M acts as a complex reflection group on $\mathfrak{h} = \mathbb{C}^N$. Then the braid group is (see background section on Artin groups)

$$B_{W_M} = \pi_1(\mathfrak{h}^{\text{reg}}/W) \cong A_M = \Big\langle T_1, T_2, \cdots, T_N \mid \underbrace{T_iT_jT_i\cdots}_{m_{i,j}\text{ factors}} = \underbrace{T_jT_iT_j\cdots}_{m_{i,j}\text{ factors}}, \text{ where } i \neq j \Big\rangle$$

Moreover, from the discussion on the relationship between the braid group and Artin group in the background section, T_i corresponds to the image in $\mathfrak{h}^{\text{reg}}/W$ of a path in $\mathfrak{h}^{\text{reg}}$ from the basepoint to its reflection by the action of s_i which just avoids the hyperplane by circling around. This path is in the conjugacy class defined by a small circle around the hyperplane.

Since we have a real reflection group, the pointwise stabilisers of each hyperplane are cyclic of order 2, so there is only one complex parameter for each hyperplane.

Then, the Hecke algebra is

$$\mathcal{H}_q(W_M) = \mathbb{C}[B_{W_M}] \Big/ \Big\langle (T_i - 1)(T_i + q_i), 1 \leq i \leq N \Big\rangle$$

where $q = \{q_i \mid 1 \leq i \leq N\} \subseteq \mathbb{C}$ are the parameters.

This is the algebra with generators $\{T_i \mid 1 \leq i \leq N\}$ and relations

$$(T_i - 1)\,(T_1 + q_i) = 0 \qquad \underbrace{T_iT_jT_i\cdots}_{m_{i,j}\text{ factors}} = \underbrace{T_jT_iT_j\cdots}_{m_{i,j}\text{ factors}}$$

for $1 \leq i, j \leq N$ and $i \neq j$.

Dihedral Group Case $D_n = G(n, n, 2)$

Recall that the dihedral group D_n of order $2n$ as a Coxeter group is

$$D_n \cong \langle s_1, s_2 \mid s_1^2 = (s_1s_2)^n = (s_2s_1)^n = s_2^2 = 1\rangle$$

Then the Hecke algebra with parameters $q_1, q_2 \in \mathbb{C}$ is the algebra with generators T_1, T_2 and relations

$$(T_1 - 1)\,(T_1 + q_1) = (T_2 - 1)\,(T_2 + q_2) = 0$$
$$\underbrace{T_iT_jT_i\cdots}_{n\text{ factors}} = \underbrace{T_jT_iT_j\cdots}_{n\text{ factors}}$$

If n is odd, we require $q_1 = q_2$.

There is another way to specify the Hecke algebra for D_n in an isomorphic way. Make a choice of square roots $q_1^{1/2}$ and $q_2^{1/2}$ with $q_1^{1/2} = q_2^{1/2}$ when n is odd, and write $q_i^{k/2} = (q_i^{1/2})^k$ for any integer $k \in \mathbb{Z}$. Set $T_i^* = q_i^{-1/2}T_i$. In the free algebra generated by T_1, T_2, we have for any ideal $\mathcal{I}$

$$\begin{aligned}(T_1 - 1)\,(T_1 + q_1) \in \mathcal{I} \text{ if and only if } & q_1\left(q_1^{-1/2}T_1 - q_1^{-1/2}\right)\left(q_1^{-1/2}T_2 + q_1^{1/2}\right) \in \mathcal{I} \\ \text{if and only if } & \left(q_1^{-1/2}T_1 - q_1^{-1/2}\right)\left(q_1^{-1/2}T_2 + q_1^{1/2}\right) \in \mathcal{I} \\ \text{if and only if } & \left(T_1^* - q_1^{-1/2}\right)\left(T_2^* + q_1^{1/2}\right) \in \mathcal{I}\end{aligned}$$

Likewise, we have for the other generator

$$(T_2 - 1)(T_2 + q_2) \in \mathcal{I} \text{ if and only if } \left(T_2^* - q_2^{-1/2}\right)\left(T_2^* + q_2^{1/2}\right) \in \mathcal{I}$$

By treating the cases of even and odd n spearately, it is also true that

$$\underbrace{T_1T_2T_1\cdots}_{n\text{ factors}} - \underbrace{T_2T_1T_2\cdots}_{n\text{ factors}} \in \mathcal{I} \text{ if and only if } \underbrace{T_1^*T_2^*T_1^*\cdots}_{n\text{ factors}} - \underbrace{T_2^*T_1^*T_2^*\cdots}_{n\text{ factors}} \in \mathcal{I}$$

This means that the ideal generated by the relations

$$(T_1 - 1)(T_1 + q_1) = 0 \qquad (T_2 - 1)(T_2 + q_2) = 0 \qquad \underbrace{T_1 T_2 T_1 \cdots}_{n \text{ factors}} = \underbrace{T_2 T_1 T_2 \cdots}_{n \text{ factors}}$$

is the same as the ideal generated by the relations

$$\left(T_1^* - q_1^{-1/2}\right)\left(T_1^* + q_1^{1/2}\right) = 0 \qquad \left(T_2^* - q_2^{-1/2}\right)\left(T_2^* + q_2^{1/2}\right) = 0 \qquad \underbrace{T_1^* T_2^* T_1^* \cdots}_{n \text{ factors}} = \underbrace{T_2^* T_1^* T_2^* \cdots}_{n \text{ factors}}$$

Obviously, the algebra generated by T_1^*, T_2^* is the same as that by T_1, T_2. Sending T_i to T_i^* and setting $p = q_1^{-1/2}$, $q = q_2^{-1/2}$, we have that the Hecke algebra for D_n with parameters q_1, q_2 is isomorphic to the algebra with generators T_1, T_2 and relations

$$(T_1 - p)\left(T_1 + p^{-1}\right) = 0 \qquad (T_2 - q)\left(T_2 + q^{-1}\right) = 0 \qquad \underbrace{T_1 T_2 T_1 \cdots}_{n \text{ factors}} = \underbrace{T_2 T_1 T_2 \cdots}_{n \text{ factors}}$$

Symmetric Group Case $\mathfrak{S}_n = G(1,1,n)$

The symmetric group as a Coxeter group is the group generated by $\{s_1, s_2, \cdots, s_{n-1}\}$ with relations

$$s_i^2 = 1 \qquad (s_j s_{j+1})^3 = 1 \qquad (s_i s_j)^2 = 1$$

where $j < n-1$ and $|i-j| > 1$. $\mathfrak{S}_n$ acts on $\mathfrak{h} = \mathbb{C}^n$ as a complex reflection group by permuting coordinates, so the complement of the hyperplane arrangement $\mathfrak{h}^{\text{reg}}$ is the points all of whose coordinates differ. The braid group is

$$B_{\mathfrak{S}_n} = \pi_1\left(\mathfrak{h}^{\text{reg}}/\mathfrak{S}_n\right) \cong \left\langle T_1, T_2, \cdots, T_{n-1} \mid T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, T_i T_j = T_j T_i \text{ where } |i-j| > 1,\ i < n-1 \right\rangle$$

The complex reflections are the transpositions, which are all conjugate. So there is a single complex parameter $q \in \mathbb{C}$ needed to specify the Hecke algebra. The Hecke algebra is generated by $T_1, T_2, \cdots, T_{n-1}$ with relations

$$(T_i - 1)(T_i + q) = 0 \qquad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \qquad T_i T_j = T_j T_i$$

where $|i-j| > 1$ and $i < n-1$.

In the same way as in the dihedral group case, we can reparametrise by taking $q_* = q^{-1/2}$ so that the Hecke algebra is isomorphic to the algebra generated by $T_1, T_2, \cdots, T_{n-1}$ with relations

$$(T_i - q_*)\left(T_i + q_*^{-1}\right) = 0 \qquad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \qquad T_i T_j = T_j T_i$$

3 Monodromy Representations

Now we describe the method of going from a representation of a complex reflection group to the corresponding Hecke module. Let $W \subseteq GL(\mathfrak{h})$ be a complex reflection group acting on some complex vector space $\mathfrak{h}$ of dimension n with basis $\{\varepsilon_i^\vee \mid 1 \leq i \leq n\}$, let $\mathfrak{h}^*$ be the dual space with dual basis $\{\varepsilon_i \mid 1 \leq i \leq n\}$, let $T \subseteq W$ be the reflection elements, and for each $s \in T$, let $H_s \subseteq \mathfrak{h}$ be the reflection hyperplane fixed by the action of s. Let (ρ, E) be any $\mathbb{C}[W]$-module, where E is a complex vector space of some dimension d, and $\rho : W \longrightarrow GL(E)$ is a group homomorphism. For each $s \in T$, choose $\alpha_s \in \mathfrak{h}^*$ such that $\ker \alpha_s = H_s$. This choice is unique up to a constant multiple. Set

$$\mathfrak{h}^{\mathrm{reg}} = \mathfrak{h} - \bigcup_{s \in T} H_s$$

and fix $\mathfrak{a}_0 \in \mathfrak{h}^{\mathrm{reg}}$.

As defined in [EM10], the rational Cherednik algebra of W is specified by some complex parameters $\{c_s \mid s \in T\}$ where $c_s = c_t$ when s, t are conjugate, and for each $p \in E$, this gives rise to a differential equation with initial condition [AR21]

$$\frac{\partial f}{\partial x_{\lambda^\vee}} = \sum_{s \in T} \frac{c_s \langle \alpha_s, \lambda^\vee \rangle}{\langle \alpha_s, x \rangle} \left(-f + \rho\left(s\right) f\right) \qquad f\left(\mathfrak{a}_0\right) = p$$

where $\partial / \partial x_{\lambda^\vee}$ denotes the partial derivative in the direction of $\lambda^\vee \in \mathfrak{h}$. A solution to this system of partial differential equations in a neighborhood $\mathfrak{a}_0 \in U \subseteq \mathfrak{h}^{\mathrm{reg}}$ is called horizontal section and is a function of the form

$$f_p : U \longrightarrow E, f_p\left(\mathfrak{a}_0\right) = p$$

This allows us to define a representation of the braid group $B_W = \pi_1\left(\mathfrak{h}^{\mathrm{reg}}/W, [\mathfrak{a}_0]\right)$ as follows.

The braid group B_W is generated by paths around the reflection hyperplanes [BMR98, Thm. 2.17]. To compute the monodromy of B_W, it suffices to compute the monodromy of those generators. These paths in $\mathfrak{h}^{\mathrm{reg}}/W$ can be obtained by choosing a particular path γ_s in $\mathfrak{h}^{\mathrm{reg}}$ between $\mathfrak{a}_0$ and $s\mathfrak{a}_0$ and pushing through the quotient map for each $s \in T$. Let

$$\tilde{f}_p^s : U_s \longrightarrow E$$

be the analytic continuation of f_p along the path γ_s to a neighborhood U_s of $s\mathfrak{a}_0$.

Set $E_q = E$ and define $T_s : E_q \longrightarrow E_q$ by the rule [AR21]

$$T_s^{-1} p = \rho\left(s^{-1}\right) \tilde{f}_p^s\left(s\mathfrak{a}_0\right)$$

This defines an action of the generators of the monodromies which together generate the braid group B_W. It turns out that not only does this action satisfy the Artin braid relations and extend to a representation of B_W, and hence of $\mathbb{C}\left[B_W\right]$, but by the main result [GGOR03, Thm. 5.13], also satisfies the Hecke algebra relation

$$(T_H - 1) \prod_{j=1}^{m_H - 1} \left(T_H - e^{2j\pi \mathbf{i}/m_H} q_{j,H}\right) = 0$$

where the parameters $\{q_{j,H}\}_{H\in\mathcal{A}}^{1\leq j<m_H}$ depend on the parameters $\{c_s \mid s \in T\}$ for the rational Cherednik algebra [EM10, Thm. 6.4].

In other words, E_q becomes a representation of the Hecke algebra of W whose parameters are controlled by $\{c_s \mid s \in T\}$.

If we make a choice of basis $\mathfrak{h} = \operatorname{span}\{\varepsilon_i^\vee \mid 1 \leq i \leq n\}$ and $E = \operatorname{span}\{e_i \mid 1 \leq i \leq d\}$, and let $[\rho(s)]$ be the matrix representation of $\rho(s) \in GL(E)$ for some $s \in W$ with respect to the chosen basis, then we may instead write

$$\frac{\partial f_i}{\partial x_k} = \sum_{s\in T} \frac{c_s\langle \alpha_s, \varepsilon_k^\vee\rangle}{\langle \alpha_s, x\rangle} \left(-f_i + \sum_{j=1}^{d} \left[\rho(s)\right]_{i,j} f_j\right) \qquad p = \sum_{j=1}^{d} f_j\left(\mathfrak{a}_0\right) e_j$$

for $i \in \{1, 2, \cdots, d\}$, $k \in \{1, 2, \cdots, n\}$, $\partial/\partial x_k = \partial/\partial x_{\varepsilon_k^\vee}$, and the $\{f_i\}_{i=1}^d$ are the complex valued component functions of f with respect to the basis of E. We will also write x_i to be the ith componenent of x in the basis of $\mathfrak{h}$.

Our goal is to write down these differential equation for several combinations of W, $\mathfrak{h}$ and E.

3.1 Cyclic Group Case $W = G(n, 1, 1)$

The group W acts on $\mathfrak{h} = \mathbb{C}$ by multiplication by the nth roots of unity. Pick the basepoint $\mathfrak{a}_0 = 1 \in \mathfrak{h}^{\text{reg}}$, and let E be the irreducible module of W on which the generator acts by multiplication by ζ^r. Pick $p = 1 \in E$.

So here, the single differential equation to be solved is

$$\begin{aligned}\frac{\partial f_1}{\partial x_1} &= \sum_{\ell=0}^{n-1} \frac{c_\ell \langle \alpha_\ell, \varepsilon_1^\vee \rangle}{x_{\alpha_\ell}} \left(-f_1 + \zeta^\ell f_1\right) \\ &= \sum_{\ell=0}^{n-1} \frac{c_\ell (1-\zeta^{-\ell})}{(1-\zeta^{-\ell}) x_1} (\zeta^\ell - 1) f_1 \\ &= \sum_{\ell=0}^{n-1} \frac{c_\ell}{x_1} (\zeta^\ell - 1) f_1 \\ &= \left[\sum_{\ell=0}^{n-1} c_\ell (\zeta^\ell - 1)\right] \frac{f_1}{x_1}\end{aligned}$$

with initial condition $f(1) = 1$. So really, $c = 0$. The action of T_1 is then

$$T_1 p = \rho\left(t_1^{-1}\right) \exp\left(\frac{2\pi \mathbf{i}}{n} \sum_{\ell=0}^{n-1} c_\ell \left(\zeta^{r\ell}. - 1\right)\right) = e^{-2\pi \mathbf{i}/r} \exp\left(\frac{2\pi \mathbf{i}}{n} \sum_{\ell=0}^{n-1} c_\ell \left(\zeta^{r\ell}. - 1\right)\right)$$

3.2 Odd Dihedral Group Case $D_n = G(n, n, 2)$

For a detailed study of the differential equations we are about to write down, see [Dun98].

Let $\mathfrak{h} = \mathbb{C}^2$ have basis $\{\varepsilon_1^\vee, \varepsilon_2^\vee\}$, and let $\mathfrak{h}^*$ be the dual space with dual basis $\{\varepsilon_1, \varepsilon_2\}$. Let $n \geq 3$ be odd. The reflection elements $s_0, \cdots, s_{n-1} \in D_n$ acts on $\mathfrak{h}$ as a complex reflection group by the matrix

$$s_k = \begin{pmatrix} 0 & \zeta^k \\ \zeta^{-k} & 0 \end{pmatrix}$$

By inspection, the hyperplane fixed by s_k is the span of th single vector $\zeta^k \varepsilon_1^\vee + \varepsilon_2^\vee$. Then we can choose

$$\alpha_k = \alpha_{s_k} = \varepsilon_1 - \zeta^k \varepsilon_2$$

We will deal with the irreducible representations of dimension 2. They are $\{(\pi_j, E)\}_{j=1}^{(n-1)/2}$ with $E = \mathbb{C}^2$ and

$$\pi_j\left(s_\ell\right) = \begin{pmatrix} 0 & \zeta^{j\ell} \\ \zeta^{-j\ell} & 0 \end{pmatrix}$$

The equations are then

$$\begin{aligned}
\frac{\partial f_1}{\partial x_1} &= c_0 \sum_{\ell=0}^{n-1} \frac{1}{x_1 - \zeta^\ell x_2} \left(-f_1 + \zeta^{j\ell} f_2\right) \\
\frac{\partial f_1}{\partial x_2} &= c_0 \sum_{\ell=0}^{n-1} \frac{-\zeta^\ell}{x_1 - \zeta^\ell x_2} \left(-f_1 + \zeta^{j\ell} f_2\right) \\
\frac{\partial f_2}{\partial x_1} &= c_0 \sum_{\ell=0}^{n-1} \frac{1}{x_1 - \zeta^\ell x_2} \left(-f_2 + \zeta^{-j\ell} f_1\right) \\
\frac{\partial f_2}{\partial x_2} &= c_0 \sum_{\ell=0}^{n-1} \frac{-\zeta^\ell}{x_1 - \zeta^\ell x_2} \left(-f_2 + \zeta^{-j\ell} f_1\right)
\end{aligned}$$

Now we work on reducing these into a form without a summation. The following identity will be useful:

$$\sum_{\ell=0}^{n-1} \zeta^{j\ell} \prod_{r=0, r\neq \ell}^{n-1} (x_1 - \zeta^r x_2) = \left(\frac{x_1}{x_2}\right)^{j-1} \left(n x_2^{n-1}\right)$$

where $j > 0$. This is true because

$$\begin{aligned}
\sum_{\ell=0}^{n-1} \zeta^{j\ell} \prod_{r=0, r\neq \ell}^{n-1} (x_1 - \zeta^r x_2) &= -\frac{1}{x_2} \sum_{\ell=0}^{n-1} \zeta^{(j-1)\ell} \left(x_1 - \zeta^\ell x_2 - x_1\right) \prod_{r=0, r\neq \ell}^{n-1} (x_1 - \zeta^r x_2) \\
&= -\frac{x_1^n - x_2^n}{x_2} \sum_{\ell=0}^{n-1} \zeta^{(j-1)\ell} + \frac{x_1}{x_2} \sum_{\ell=0}^{n-1} \zeta^{(j-1)\ell} \prod_{r=0, r\neq \ell}^{n-1} (x_1 - \zeta^r x_2) \\
&= \frac{x_1}{x_2} \sum_{\ell=0}^{n-1} \zeta^{(j-1)\ell} \prod_{r=0, r\neq \ell}^{n-1} (x_1 - \zeta^r x_2)
\end{aligned}$$

This gives a recursive way to get from j to $j-1$, and so on downwards. So

$$\begin{aligned}
\sum_{\ell=0}^{n-1} \zeta^{j\ell} \prod_{r=0, r\neq \ell}^{n-1} \zeta^{j\ell} (x_1 - \zeta^r x_2) &= \frac{x_1}{x_2} \sum_{\ell=0}^{n-1} \zeta^{(j-1)\ell} \prod_{r=0, r\neq \ell}^{n-1} (x_1 - \zeta^r x_2) \\
&= \left(\frac{x_1}{x_2}\right)^2 \sum_{\ell=0}^{n-1} \zeta^{(j-2)\ell} \prod_{r=0, r\neq \ell}^{n-1} (x_1 - \zeta^r x_2) \\
&\vdots \\
&= \left(\frac{x_1}{x_2}\right)^{j-1} \sum_{\ell=0}^{n-1} \zeta^{\ell} \prod_{r=0, r\neq \ell}^{n-1} (x_1 - \zeta^r x_2) \\
&= \left(\frac{x_1}{x_2}\right)^{j-1} \left(\frac{\partial}{\partial x_2} \prod_{r=0}^{n-1} (x_1 - \zeta^r x_2)\right) \\
&= n x_2^{n-1} \left(\frac{x_1}{x_2}\right)^{j-1}
\end{aligned}$$

The last equality follows from the fact that the sum over roots of unity is 0. In the case where we have $-j$, (where $j \geq 0$) we have an analogous result:

$$\sum_{\ell=0}^{n-1} \zeta^{-j\ell} \prod_{r=0, r\neq\ell}^{n-1} (x_1 - \zeta^r x_2) = n x_1^{n-1} \left(\frac{x_2}{x_1}\right)^j$$

Now to write down the differential equations.

Equation 1: We have

$$\begin{aligned}
\frac{\partial f_1}{\partial x_1} &= c_0 \sum_{\ell=0}^{r-1} \frac{1}{x_1 - \zeta^\ell x_2} \left(-f_1 + \zeta^{j\ell} f_2\right) \\
&= -\frac{c_0 f_1}{x_1^n - x_2^n} \sum_{\ell=0}^{n-1} \prod_{r=0, r\neq\ell}^{n-1} (x_1 - \zeta^r x_2) + \frac{c_0 f_2}{x_1^n - x_2^n} \sum_{\ell=0}^{n-1} \zeta^{j\ell} \prod_{r=0, r\neq\ell}^{n-1} (x_1 - \zeta^r x_2) \\
&= -\frac{c_0 f_1}{x_1^n - x_2^n} \left(n x_1^{n-1}\right) + n x_2^{n-1} \left(\frac{x_1}{x_2}\right)^{j-1} \\
&= \frac{n c_0}{x_1^n - x_2^n} \left(-x_1^{n-1} f_1 + x_2^{n-1} \left(\frac{x_1}{x_2}\right)^{j-1} f_2\right)
\end{aligned}$$

Equation 2: The second is

$$\begin{aligned}
\frac{\partial f_1}{\partial x_2} &= c_0 \sum_{\ell=0}^{r-1} \frac{-\zeta^\ell}{x_1 - \zeta^\ell x_2} \left(-f_1 + \zeta^{j\ell} f_2\right) \\
&= \frac{c_0 f_1}{x_1^n - x_2^n} \sum_{\ell=0}^{n-1} \zeta^\ell \prod_{r=0, r\neq\ell}^{n-1} (x_1 - \zeta^r x_2) - \frac{c_0 f_2}{x_1^n - x_2^n} \sum_{\ell=0}^{n-1} \zeta^{(j+1)\ell} \prod_{r=0, r\neq\ell}^{n-1} (x_1 - \zeta^r x_2) \\
&= \frac{c_0 f_1}{x_1^n - x_2^n} \left(n x_2^{n-1}\right) - \frac{c_0 f_2}{x_1^n - x_2^n} \left(\frac{x_1}{x_2}\right)^j \left(n x_2^{n-1}\right) \\
&= \frac{n c_0 x_2^{n-1}}{x_1^n - x_2^n} \left(f_1 - \left(\frac{x_1}{x_2}\right)^j f_2\right)
\end{aligned}$$

Equation 3: And the third,

$$\begin{aligned}
\frac{\partial f_2}{\partial x_1} &= c_0 \sum_{\ell=0}^{r-1} \frac{1}{x_1 - \zeta^\ell x_2} \left(-f_2 + \zeta^{-j\ell} f_1\right) \\
&= -\frac{c_0 f_2}{x_1^n - x_2^n} \sum_{\ell=0}^{n-1} \prod_{r=0, r\neq\ell}^{n-1} (x_1 - \zeta^r x_2) + \frac{c_0 f_1}{x_1^n - x_2^n} \sum_{\ell=0}^{n-1} \zeta^{-j\ell} \prod_{r=0, r\neq\ell}^{n-1} (x_1 - \zeta^r x_2) \\
&= -\frac{c_0 f_2}{x_1^n - x_2^n} \left(n x_1^{n-1}\right) + \frac{c_0 f_1}{x_1^n - x_2^n} \left(\frac{x_2}{x_1}\right)^j \left(n x_1^{n-1}\right) \\
&= \frac{n c_0 x_1^{n-1}}{x_1^n - x_2^n} \left(\left(\frac{x_2}{x_1}\right)^j f_1 - f_2\right)
\end{aligned}$$

Equation 4: And finally the last one is

$$\begin{aligned}
\frac{\partial f_2}{\partial x_2} &= c_0 \sum_{\ell=0}^{r-1} \frac{-\zeta^\ell}{x_1 - \zeta^\ell x_2} \left(-f_2 + \zeta^{-j\ell} f_1\right) \\
&= \frac{c_0 f_2}{x_1^n - x_2^n} \sum_{\ell=0}^{n-1} \zeta^\ell \prod_{r=0, r\neq \ell}^{n-1} (x_1 - \zeta^r x_2) - \frac{c_0 f_1}{x_1^n - x_2^n} \sum_{\ell=0}^{n-1} \zeta^{-(j-1)\ell} \prod_{r=0, r\neq \ell}^{n-1} (x_1 - \zeta^r x_2) \\
&= \frac{c_0 f_2}{x_1^n - x_2^n} \left(n x_2^{n-1}\right) - \frac{c_0 f_1}{x_1^n - x_2^n} \left(\frac{x_2}{x_1}\right)^{j-1} \left(n x_1^{n-1}\right) \\
&= \frac{n c_0}{x_1^n - x_2^n} \left(-x_1^{n-1} \left(\frac{x_2}{x_1}\right)^{j-1} f_1 + x_2^{n-1} f_2\right)
\end{aligned}$$

So together we have the system

$$\begin{aligned}
\frac{\partial f_1}{\partial x_1} &= \frac{n c_0}{x_1^n - x_2^n} \left(-x_1^{n-1} f_1 + x_2^{n-1} \left(\frac{x_1}{x_2}\right)^{j-1} f_2\right) \\
\frac{\partial f_1}{\partial x_2} &= \frac{n c_0 x_2^{n-1}}{x_1^n - x_2^n} \left(f_1 - \left(\frac{x_1}{x_2}\right)^{j} f_2\right) \\
\frac{\partial f_2}{\partial x_1} &= \frac{n c_0 x_1^{n-1}}{x_1^n - x_2^n} \left(\left(\frac{x_2}{x_1}\right)^{j} f_1 - f_2\right) \\
\frac{\partial f_2}{\partial x_2} &= \frac{n c_0}{x_1^n - x_2^n} \left(-x_1^{n-1} \left(\frac{x_2}{x_1}\right)^{j-1} f_1 + x_2^{n-1} f_2\right)
\end{aligned}$$

3.3 Even Dihedral Group Case $D_n = G(n, n, 2)$

Now for the case of even $n \geq 4$. Let $\mathfrak{h}, \mathfrak{h}^*$, $\{\varepsilon_1^\vee, \varepsilon_2^\vee\}$ and $\{\varepsilon_1, \varepsilon_2\}$ be as before in the odd case. Set $m = n/2$, and let the two conjugacy classes of reflections be $\{s_0, s_2, \cdots, s_{m-1}\}$ and $\{t_0, t_2, \cdots, t_{m-1}\}$. Let $\xi = \zeta^2$, so that the even roots of unity of order n are $1, \xi, \xi^2, \xi^3, \cdots$ and the odd roots are $\xi\zeta^{-1}, \xi^2\zeta^{-1}, \xi^3\zeta^{-1}, \cdots$. The irreducible representations of dimension 2 are of course $\{(\pi_j, E)\}_{j=1}^{n/2-1}$, with $E = \mathbb{C}^2$, and group actions

$$\pi_j(t_\ell) = \begin{pmatrix} 0 & \xi^{j\ell} \\ \xi^{-j\ell} & 0 \end{pmatrix} \qquad \pi_j(s_\ell) = \begin{pmatrix} 0 & \xi^{j\ell}\zeta \\ \xi^{-j\ell}\zeta^{-1} & 0 \end{pmatrix}$$

Now say $\alpha_{t_\ell} = \alpha_\ell^{(1)}$ and $\alpha_{s_\ell} = \alpha_\ell^{(2)}$ and take

$$\begin{aligned}
\alpha_\ell^{(1)} &= \varepsilon_1 - \xi^\ell \varepsilon_2 \\
\alpha_\ell^{(2)} &= \varepsilon_1 - \xi^\ell \zeta \varepsilon_2
\end{aligned}$$

If c_0 is the parameter for the conjugacy class $\{t_0, t_2, \cdots, t_{m-1}\}$ and c_1 for $\{s_0, s_2, \cdots, s_{m-1}\}$, then the equation is

$$\begin{aligned}\frac{\partial f_i}{\partial x_k} &= \sum_{s\in T} \frac{c_s\langle \alpha_s, \varepsilon_k^\vee\rangle}{x_{\alpha_s}}\left(-f_i + \sum_{t=1}^{2}[\pi_j(s)]_{it} f_t\right)\\ &= c_0 \sum_{\ell=0}^{m-1} \frac{\langle \alpha_\ell^{(1)}, \varepsilon_k^\vee\rangle}{x_{\alpha_\ell^{(1)}}}\left(-f_i + \sum_{t=1}^{2}\left[\begin{pmatrix} 0 & \xi^{j\ell} \\ \xi^{-j\ell} & 0\end{pmatrix}\right]_{it} f_t\right) + c_1 \sum_{\ell=0}^{m-1} \frac{\langle \alpha_\ell^{(2)}, \varepsilon_k^\vee\rangle}{x_{\alpha_\ell^{(2)}}}\left(-f_i + \sum_{t=1}^{2}\left[\begin{pmatrix} 0 & \xi^{j\ell}\zeta \\ \xi^{-j\ell}\zeta^{-1} & 0\end{pmatrix}\right]_{it} f_t\right)\end{aligned}$$

We will work on each of the two summations over $\ell \in \{0, 1, \cdots, m-1\}$ for each of the four differential equations separately. The sequence of steps to simplify each equation is similar: first, we factor out $(x_1^m - x_2^m)^{-1} = \prod_{r=0}^{m-1}(x_1 - \xi^r)^{-1}$, and this will leave us with the summation with a product inside. Then, we apply the identity

$$\sum_{\ell=0}^{m-1} \xi^{j\ell} \prod_{r=0, r\neq \ell}^{m-1} (x_1 - \xi^r x_2) = m x_2^{m-1}\left(\frac{x_1}{x_2}\right)^{j-1}$$

for $j > 0$, and the analogous result for $-j$ ($j \geq 0$). Finally we will factorise.

The algebraic manipulations are long and arduous, and so have been moved to the appendices. In the end, the system of differential equations together is

$$\begin{aligned}\frac{\partial f_1}{\partial x_1} &= \frac{(n/2)c_0}{x_1^{n/2} - x_2^{n/2}}\left[-x_1^{n/2-1} f_1 + x_2^{n/2-1}\left(\frac{x_1}{x_2}\right)^{j-1} f_2\right] + \frac{(n/2)c_1}{x_1^{n/2} + x_2^{n/2}}\left[-x_1^{n/2-1} f_1 - \zeta^{1-j} x_2^{n/2-1}\left(\frac{x_1}{x_2}\right)^{j-1} f_2\right]\\ \frac{\partial f_1}{\partial x_2} &= \frac{(n/2)c_0 x_2^{n/2-1}}{x_1^{n/2} - x_2^{n/2}}\left[f_1 - \left(\frac{x_1}{x_2}\right)^{j} f_2\right] + \frac{(n/2)c_1 x_2^{n/2-1}}{x_1^{n/2} + x_2^{n/2}}\left[-f_1 + \zeta^{1-j}\left(\frac{x_1}{x_2}\right)^{j} f_2\right]\\ \frac{\partial f_2}{\partial x_1} &= \frac{(n/2)c_0 x_1^{n/2-1}}{x_1^{n/2} - x_2^{n/2}}\left[\left(\frac{x_2}{x_1}\right)^{j} f_1 - f_2\right] + \frac{(n/2)c_1 x_1^{n/2-1}}{x_1^{n/2} + x_2^{n/2}}\left[\zeta^{j-1}\left(\frac{x_2}{x_1}\right)^{j} f_1 - f_2\right]\\ \frac{\partial f_2}{\partial x_2} &= \frac{(n/2)c_0}{x_1^{n/2} - x_2^{n/2}}\left[-x_1^{n/2-1}\left(\frac{x_2}{x_1}\right)^{j-1} f_1 + x_2^{n/2-1} f_2\right] + \frac{(n/2)c_1}{x_1^{n/2} + x_2^{n/2}}\left[-\zeta^{j-1} x_1^{n/2-1}\left(\frac{x_2}{x_1}\right)^{j-1} f_1 - x_2^{n/2-1} f_2\right]\end{aligned}$$

3.4 Symmetric Group Case $\mathfrak{S}_n = G(1, 1, n)$

Let $\mathfrak{h} = \mathbb{C}^n$ with basis $\{\varepsilon_1^\vee, \varepsilon_2^\vee, \cdots, \varepsilon_n^\vee\}$. Then $\mathfrak{S}_n$ acts on $\mathfrak{h}$ by permuting coordinates as a complex reflection group. For each transposition $s = (q, r)$, its action $\tau_s = \tau_{q,r} : \mathfrak{h} \longrightarrow \mathfrak{h}$ is by exchanging $\varepsilon_q^\vee, \varepsilon_r^\vee$, and fixing all the other basis vectors. This means that the hyperplane fixed by the action of s is

$$H_s = H_{q,r} = \operatorname{span}\left\{\varepsilon_i^\vee, \varepsilon_q^\vee + \varepsilon_r^\vee \mid i \neq q, r\right\}$$

We will choose $\alpha_s = \alpha_{q,r} \in \mathfrak{h}^*$ to then be

$$\alpha_{q,r} = \varepsilon_r - \varepsilon_q$$

Although we will always write $\alpha_{q,r} = \alpha_{r,q}$, for the above choice of $\alpha_{q,r}$ we assume $q < r$.

Now to deform $\mathbb{C}[W]$-modules. We will focus on only the sign, permutation, and regular representations of $\mathfrak{S}_n$.

Sign Representation

We have the sign representation (ρ, E) with $E = \mathbb{C}$, and $\rho : W \longrightarrow GL(E)$ defined by

$$\rho(\sigma) : E \longrightarrow E, z \longmapsto \operatorname{sgn}(\sigma) z$$

Pick any basis $e_1 \in E$, and let $[\rho(s)]$ be the matrix of the action of s on E. Then,

$$\begin{aligned}
\frac{\partial f}{\partial x_k} &= c_0 \sum_{s\in T} \frac{\langle \alpha_s, \varepsilon_k^\vee\rangle}{\langle \alpha_s, x\rangle} \left(-f + \operatorname{sgn}(s) f\right) \\
&= c_0 \sum_{1\le q<r\le n} \frac{\langle \varepsilon_r - \varepsilon_q, \varepsilon_k^\vee\rangle}{\langle \varepsilon_r - \varepsilon_q, x\rangle} (-2f) \\
&= -2c_0 f \left(\sum_{1\le q<k} \frac{\langle \varepsilon_k - \varepsilon_q, \varepsilon_k^\vee\rangle}{\langle \varepsilon_k - \varepsilon_q, x\rangle} + \sum_{k<r\le n} \frac{\langle \varepsilon_r - \varepsilon_k, \varepsilon_k^\vee\rangle}{\langle \varepsilon_r - \varepsilon_k, x\rangle} \right) \\
&= -2c_0 f \left(\sum_{1\le q<k} \frac{1}{x_k - x_q} + \sum_{k<r\le n} \frac{1}{x_k - x_r} \right)
\end{aligned}$$

So the horizontal sections are the solutions to

$$\frac{\partial f}{\partial x_k} = 2c_0 f \sum_{\ell\ne k} \frac{1}{x_\ell - x_k}$$

Permutation Representation

Next, we deal with the permutation representation. Set $E = \mathbb{C}^n$ with basis $\{e_1, e_2, \cdots, e_d\}$, and let $\rho : \mathfrak{S}_n \longrightarrow GL(E)$ be defined on any $\sigma \in \mathfrak{S}_n$ by

$$\rho(\sigma) : E \longrightarrow E, e_k \longmapsto e_{\sigma(k)}$$

Then for any transposition $(q,r) \in \mathfrak{S}_n$, the matrix representation $[\rho((q,r))]$ of the $\mathfrak{S}_n$-action on E with respect to the specified basis is

$$[\rho((q,r))]_{ij} = \begin{cases} 1 & \text{, if } i = j \notin \{q,r\} \\ 1 & \text{, if } i = q, j = r \text{ or } i = r, j = q \\ 0 & \text{, otherwise} \end{cases}$$

Now to get the differential equations. If $\partial/\partial x_k$ is the directional derivative operator in the direction of the vector $\varepsilon_k \in \mathfrak{h}$, then at $x \in \mathfrak{h}$ the derivative of the ith components of our horizontal sections in the direction of ε_k will be

$$\begin{aligned}
\frac{\partial f_i}{\partial x_k} &= \sum_{s \in T} \frac{c_0 \langle \alpha_s, \varepsilon_k^\vee \rangle}{\langle \alpha_s, x \rangle} \left(-f_i + \sum_{j=1}^n [\rho(s)]_{ij} f_j \right) \\
&= c_0 \sum_{1 \le q < r \le n} \frac{\langle \alpha_{q,r}, \varepsilon_k^\vee \rangle}{\langle \alpha_{q,r}, x \rangle} \left(-f_i + \sum_{j=1}^n [\rho((q,r))]_{ij} f_j \right) \\
&= c_0 \sum_{1 \le q < r \le n} \frac{\langle \varepsilon_r - \varepsilon_q, \varepsilon_k^\vee \rangle}{\langle \varepsilon_r - \varepsilon_q, x \rangle} \left(-f_i + \sum_{j=1}^n [\rho((q,r))]_{ij} f_j \right) \\
&= c_0 \sum_{1 \le q < k} \frac{\langle \varepsilon_k - \varepsilon_q, \varepsilon_k^\vee \rangle}{\langle \varepsilon_k - \varepsilon_q, x \rangle} \left(-f_i + \sum_{j=1}^n [\rho((q,k))]_{ij} f_j \right) + c_0 \sum_{k < r \le n} \frac{\langle \varepsilon_r - \varepsilon_k, \varepsilon_k^\vee \rangle}{\langle \varepsilon_r - \varepsilon_k, x \rangle} \left(-f_i + \sum_{j=1}^n [\rho((k,r))]_{ij} f_j \right) \\
&= c_0 \sum_{1 \le q < k} \frac{1}{x_k - x_q} \left(-f_i + \sum_{j=1}^n [\rho((q,k))]_{ij} f_j \right) + c_0 \sum_{k < r \le n} \frac{1}{x_k - x_r} \left(-f_i + \sum_{j=1}^n [\rho((k,r))]_{ij} f_j \right) \\
&= c_0 \sum_{\ell \neq k} \frac{1}{x_k - x_\ell} \left(-f_i + \sum_{j=1}^n [\rho((\ell,k))]_{ij} f_j \right)
\end{aligned}$$

We will evaluate the inner sums by conditioning on the value of i, k. If $i = k$,

$$-f_i + \sum_{j=1}^n [\rho((\ell,k))]_{ij} f_j = -f_k + \sum_{j=1}^n [\rho((\ell,k))]_{kj} f_j = -f_k + f_\ell$$

If $i \neq k$, then

$$-f_i + \sum_{j=1}^n [\rho((\ell,k))]_{ij} f_j = \begin{cases} -f_i + f_i = 0 & \text{, if } \ell \neq i \\ -f_i + f_k & \text{, if } \ell = i \end{cases}$$

So the system of partial differential equations is

$$\frac{\partial f_i}{\partial x_k} = \begin{cases} \dfrac{c_0}{x_k - x_i}\left(-f_i + f_k\right) & \text{, if } i \neq k \\ c_0 \displaystyle\sum_{\ell \neq k} \frac{1}{x_k - x_\ell}\left(-f_k + f_\ell\right) & \text{, if } i = k \end{cases}$$

Regular Representation

Now, let $E = \mathbb{C}[\mathfrak{S}_n]$ be the group algebra with basis $\{e_\sigma \mid \sigma \in \mathfrak{S}_n\}$, with the representation $\rho : \mathfrak{S}_n \longrightarrow GL(E)$ at some $\tau \in \mathfrak{S}_n$ defined by

$$\rho(\tau) : E \longrightarrow E, e_\sigma \longmapsto e_{\tau\sigma}$$

For any transposition (q, r), the matrix $[\rho((q, r))]$ of the action on E with respect to the specified has a row and column for each element of the symmetric group, so we will index them by $\sigma, \tau \in \mathfrak{S}_n$. We have

$$[\rho((q,r))]_{\sigma,\tau} = \begin{cases} 1 & \text{, if } \sigma = (q,r)\tau \\ 0 & \text{, otherwise} \end{cases}$$

The horizontal sections will have one complex valued component for each element of $\mathfrak{S}_n$. So we will index them by $\sigma \in \mathfrak{S}_n$. The horizontal sections are then the solutions of

$$\begin{aligned} \frac{\partial f_\sigma}{\partial x_k} &= c_0 \sum_{s \in T} \frac{\langle \alpha_s, \varepsilon_k^\vee \rangle}{\langle \alpha_s, x \rangle}\left(-f_\sigma + \sum_{\tau \in W} [\rho(s)]_{\sigma,\tau} f_\tau\right) \\ &= c_0 \sum_{1 \leq q < r \leq n} \frac{\langle \varepsilon_r - \varepsilon_q, \varepsilon_k^\vee \rangle}{\langle \varepsilon_r - \varepsilon_q, x \rangle}\left(-f_\sigma + f_{(q,r)\sigma}\right) \\ &= c_0 \sum_{1 \leq q < k} \frac{\langle \varepsilon_k - \varepsilon_q, \varepsilon_k^\vee \rangle}{\langle \varepsilon_k - \varepsilon_q, x \rangle}\left(-f_\sigma + f_{(q,k)\sigma}\right) + c_0 \sum_{k < r \leq n} \frac{\langle \varepsilon_r - \varepsilon_k, \varepsilon_k^\vee \rangle}{\langle \varepsilon_r - \varepsilon_k, x \rangle}\left(-f_\sigma + f_{(k,r)\sigma}\right) \\ &= c_0 \sum_{1 \leq q < k} \frac{1}{x_k - x_q}\left(-f_\sigma + f_{(q,k)\sigma}\right) + c_0 \sum_{k < r \leq n} \frac{1}{x_k - x_r}\left(-f_\sigma + f_{(k,r)\sigma}\right) \end{aligned}$$

This is

$$\frac{\partial f_\sigma}{\partial x_k} = c_0 \sum_{\ell \neq k} \frac{1}{x_k - x_\ell}\left(-f_\sigma + f_{(\ell,k)\sigma}\right)$$

4 Discussion and Conclusion

In this project, we studied the relationship between representations of a complex reflection group and its Hecke algebra, via some partial differential equations which arise from the rational Cherednik algbra of the group.

The information passed in to get the differential equations is a representation of the complex reflection group, and the parameters of the rational Cherednik algebra. The solutions of the differential equations specify monodromy representations of a Hecke algebra of the group. The parameters of this Hecke algebra determining exactly which Hecke algebra we have obtained are controlled by the parameters of the rational Cherednik algebra.

The representation of the complex reflection group we started with also induces a module over the rational Cherednik algebra, and the correspondence between this module and the Hecke algebra representation we obtained is the Knizhnik-Zamolodchikov functor.

Initially, we hoped that we may be able to solve some of the differential equations to explicitly compute the monodromy representations, but this was not something we got to. A continuation of this project could be to extract more information about the differential equation through analytic means, or even numerically compute approximate solutions as this may further elucidate the connection between representation theories of the rational Cherednik algebra and Hecke algebras of a complex reflection group.

References

[GGOR03] V. Ginzburg, N. Guay, E. Opdam, R. Rouquier: *On the category O for rational Cherednik algebras.* Invent. Math. 154 (2003), no. 3, 617–651.

[BMR98] M. Broué, G. Malle, R. Rouquier: *Complex reflection groups, braid groups, Hecke algebras.* J. Reine

[Dun98] C. F. Dunkl: *Differential-difference operators associated to reflection groups.* Trans. Amer. Math. Soc. 311 (1989), 167–183. Angew. Math. 500 (1998), 127-190.

[Hu90] J. E. Humphreys: *Reflection Groups and Coxeter Groups.* Cambridge Studies in Advanced. Mathematics, 29, Cambridge University Press (1990).

[FH91] W. Fulton, J. Harris: *Representation Theory.* Graduate Texts in Mathematics, 129, Springer-Verlag, New York, (1991).

[Hat00] A. Hatcher: *Algebraic Topology.* Cambridge Univ. Press, Cambridge, (2000).

[Bri71] E. Brieskorn: *Die Fundamentalgruppe des Raumes der regul aren Orbits einer endlichen komplexen Spiegelungsgruppe.* Inventiones math. 12, 57- 61(1971).

[EM10] P. Etingof, X. Ma. *Lecture Notes of Cherednik Algebras.* `https://arxiv.org/abs/1001.0432` (2010)

[ST54] G. C. Shephard, A. J. Todd. *Finite unitary reflection groups.* Canadian Journal of Mathematics, Canadian Mathematical Society, 6: 274–304 (1954)

[AR21] A. Ram. *Reflection Groups and the KZ functor.* `http://math.soimeme.org/~arunram/presentations.html`(2021)

A Manipulation of Differential Equations

Equation 1, Summation 1: This is

$$\begin{aligned} c_0 \sum_{\ell=0}^{m-1} \frac{\langle \varepsilon_1 - \xi^\ell \varepsilon_2, \varepsilon_1^\vee \rangle}{x_1 - \xi^\ell x_2} \left[-f_1 + \xi^{j\ell} f_2 \right] &= -c_0 f_1 \sum_{\ell=0}^{m-1} \frac{1}{x_1 - \xi^\ell x_2} + c_0 f_2 \sum_{\ell=0}^{m-1} \frac{\xi^{j\ell}}{x_1 - \xi^\ell x_2} \\ &= -\frac{c_0 f_1}{x_1^m - x_2^m} \sum_{\ell=0}^{m-1} \prod_{r=0, r\neq \ell}^{m-1} (x_1 - \xi^r x_2) \\ &\quad + \frac{c_0 f_2}{x_1^m - x_2^m} \sum_{\ell=0}^{m-1} \xi^{j\ell} \prod_{r=0, r\neq\ell}^{m-1} (x_1 - \xi^r x_2) \\ &= -\frac{c_0 f_1 \left(m x_1^{m-1} \right)}{x_1^m - x_2^m} + \frac{c_0 f_2}{x_1^m - x_2^m} \left(\frac{x_1}{x_2} \right)^{j-1} \left(m x_2^{m-1} \right) \\ &= \frac{m c_0}{x_1^m - x_2^m} \left[-x_1^{m-1} f_1 + x_2^{m-1} \left(\frac{x_1}{x_2} \right)^{j-1} f_2 \right] \end{aligned}$$

Equation 1, Summation 2: From now, we define $z_2 = \zeta x_2$. We have

$$\begin{aligned} c_1 \sum_{\ell=0}^{m-1} \frac{\langle \varepsilon_1 - \xi^\ell \zeta \varepsilon_2, \varepsilon_1^\vee \rangle}{x_1 - \xi^\ell z_2} \left[-f_1 + \xi^{j\ell} \zeta f_2 \right] &= -c_1 f_1 \sum_{\ell=0}^{m-1} \frac{1}{x_1 - \xi^\ell z_2} + c_1 \zeta f_2 \sum_{\ell=0}^{m-1} \frac{\xi^{j\ell}}{x_1 - \zeta^\ell z_2} \\ &= -\frac{c_1 f_1}{x_1^m - z_2^m} \sum_{\ell=0}^{m-1} \prod_{r=0, r\neq\ell}^{m-1} (x_1 - \xi^\ell z_2) \\ &\quad + \frac{c_1 \zeta f_2}{x_1^m - z_2^m} \sum_{\ell=0}^{m-1} \xi^{j\ell} \prod_{r=0, r\neq\ell}^{m-1} (x_1 - \xi^r z_2) \\ &= -\frac{c_1 f_1 \left(m x_1^{m-1} \right)}{x_1^m + x_2^m} + \frac{c_1 \zeta f_2}{x_1^m + x_2^m} \left(\frac{x_1}{z_2} \right)^{j-1} \left(m z_2^{m-1} \right) \\ &= -\frac{c_1 f_1 \left(m x_1^{m-1} \right)}{x_1^m + x_2^m} - \frac{c_1 \zeta f_2 \zeta^{1-j}}{x_1^m + x_2^m} \left(\frac{x_1}{x_2} \right)^{j-1} \left(m x_2^{m-1} \right) \zeta^{-1} \\ &= \frac{m c_1}{x_1^m + x_2^m} \left[-x_1^{m-1} f_1 - \zeta^{1-j} x_2^{m-1} \left(\frac{x_1}{x_2} \right)^{j-1} f_2 \right] \end{aligned}$$

So equation 1 is

$$\frac{\partial f_1}{\partial x_1} = \frac{(n/2) c_0}{x_1^{n/2} - x_2^{n/2}} \left[-x_1^{n/2-1} f_1 + x_2^{n/2-1} \left(\frac{x_1}{x_2} \right)^{j-1} f_2 \right] + \frac{(n/2) c_1}{x_1^{n/2} + x_2^{n/2}} \left[-x_1^{n/2-1} f_1 - \zeta^{1-j} x_2^{n/2-1} \left(\frac{x_1}{x_2} \right)^{j-1} f_2 \right]$$

Equation 2, Summation 1:

$$
\begin{aligned}
c_0 \sum_{\ell=0}^{m-1} \frac{\langle \varepsilon_1 - \xi^\ell \varepsilon_2, \varepsilon_2^\vee \rangle}{x_1 - \xi^\ell x_2} \left[-f_1 + \xi^{j\ell} f_2\right] &= \frac{c_0 f_1}{x_1^m - x_2^m} \sum_{\ell=0}^{m-1} \xi^\ell \prod_{r-0, r\neq \ell}^{m-1} (x_1 - \xi^\ell x_2) \\
&\quad - \frac{c_0 f_2}{x_1^m - x_2^m} \sum_{\ell=0}^{m-1} \xi^{(j+1)\ell} \prod_{r=0, r\neq \ell}^{m-1} (x_1 - \xi^\ell x_2) \\
&= \frac{c_0 f_1 \left(m x_2^{m-1}\right)}{x_1^m - x_2^m} - \frac{c_0 f_2 \left(m x_2^{m-1}\right)}{x_1 m - x_2^m} \left(\frac{x_1}{x_2}\right)^j \\
&= \frac{m c_0 x_2^{m-1}}{x_1^m - x_2^m} \left[f_1 - \left(\frac{x_1}{x_2}\right)^j f_2\right]
\end{aligned}
$$

Equation 2, Summation 2:

$$
\begin{aligned}
c_1 \sum_{\ell=0}^{m-1} \frac{\langle \varepsilon_1 - \xi^\ell \zeta \varepsilon_2, \varepsilon_2^\vee \rangle}{x_1 - \xi^\ell \zeta x_2} \left[-f_1 + \xi^{j\ell} \zeta f_2\right] &= \frac{c_1 \zeta f_1}{x_1^m - z_2^m} \sum_{\ell=0}^{m-1} \xi^\ell \prod_{r=0, r\neq \ell}^{m-1} (x_1 - \xi^r z_2) \\
&\quad - \frac{c_1 \zeta^2 f_2}{x_1^m - z_2^m} \sum_{\ell=0}^{m-1} \xi^{(j+1)\ell} \prod_{r=0, r\neq \ell}^{m-1} (x_1 - \xi^r x_2) \\
&= \frac{x_1 \zeta f_1 \left(m z_2^{m-1}\right)}{x_1^m + x_2^m} - \frac{c_1 \xi f_2}{x_1^m + x_2^m} \left(\frac{x_1}{z_2}\right)^j \left(m z_2^{m-1}\right) \\
&= \frac{m c_1 z_2^{m-1}}{x_1^m + x_2^m} \left[\zeta f_1 - \xi \left(\frac{x_1}{z_2}\right)^j f_2\right] \\
&= \frac{m c_1 x_2^{m-1}}{x_1^m + x_2^m} \left[-f_1 + \zeta \left(\frac{x_1}{\zeta x_2}\right)^j f_2\right] \\
&= \frac{m c_1 x_2^{m-1}}{x_1^m + x_2^m} \left[-f_1 + \zeta^{1-j} \left(\frac{x_1}{x_2}\right)^j f_2\right]
\end{aligned}
$$

So equation 2 is

$$
\frac{\partial f_1}{\partial x_2} = \frac{(n/2) c_0 x_2^{n/2-1}}{x_1^{n/2} - x_2^{n/2}} \left[f_1 - \left(\frac{x_1}{x_2}\right)^j f_2\right] + \frac{(n/2) c_1 x_2^{n/2-1}}{x_1^{n/2} + x_2^{n/2}} \left[-f_1 + \zeta^{1-j} \left(\frac{x_1}{x_2}\right)^j f_2\right]
$$

Equation 3, Summation 1:

$$
\begin{aligned}
c_0 \sum_{\ell=0}^{m-1} \frac{\langle \varepsilon_1 - \xi^\ell \varepsilon_2, \varepsilon_1^\vee \rangle}{x_1 - \xi^\ell x_2} \left[-f_2 + \xi^{-j\ell} f_1\right] &= -\frac{c_0 f_2}{x_1^m - x_2^m} \sum_{\ell=0}^{m-1} \prod_{r=0, r\neq \ell}^{m-1} (x_1 - \xi^r x_2) \\
&\quad + \frac{c_0 f_1}{x_1^m - x_2^m} \sum_{\ell=0}^{m-1} \xi^{-j\ell} \prod_{r=0, r\neq \ell}^{m-1} (x_1 - \xi^r x_2) \\
&= -\frac{c_0 f_2 \left(m x_1^{m-1}\right)}{x_1^m - x_2^m} + \frac{c_0 f_1 \left(m x_1^{m-1}\right)}{x_1^m - x_2^m} \left(\frac{x_2}{x_1}\right)^j \\
&= \frac{m c_0 x_1^{m-1}}{x_1^m - x_2^m} \left[\left(\frac{x_2}{x_1}\right)^j f_1 - f_2\right]
\end{aligned}
$$

Equation 3, Summation 2:

$$c_1 \sum_{\ell=0}^{m-1} \frac{\langle \varepsilon_1 - \xi^\ell \zeta \varepsilon_2, \varepsilon_1^\vee \rangle}{x_1 - \xi^\ell z_2} \left[-f_2 + \xi^{-j\ell}\zeta^{-1} f_1\right] = -\frac{c_1 f_2}{x_1^m - z_2^m} \sum_{\ell=0}^{m-1} \prod_{r=0, r\neq \ell}^{m-1} (x_1 - \xi^r z_2) + \frac{c_1 \zeta^{-1} f_1}{x_1^m - z_2^m} \sum_{\ell=0}^{m-1} \xi^{-j\ell} \prod_{r=0, r\neq\ell}^{m-1} (x_1 - \xi^r z_2)$$

$$= -\frac{c_1 f_2 \left(m x_1^{m-1}\right)}{x_1^m + x_2^m} + \frac{c_1 \zeta^{-1} f_1 \left(m x_1^{m-1}\right)}{x_1^m + x_2^m} \left(\frac{z_2}{x_1}\right)^j$$

$$= \frac{m c_1 x_1^{m-1}}{x_1^m + x_2^m} \left[\zeta^{j-1} \left(\frac{x_2}{x_1}\right)^j f_1 - f_2\right]$$

And now we have equation 3:

$$\frac{\partial f_2}{\partial x_1} = \frac{(n/2) c_0 x_1^{n/2-1}}{x_1^{n/2} - x_2^{n/2}} \left[\left(\frac{x_2}{x_1}\right)^j f_1 - f_2\right] + \frac{(n/2) c_1 x_1^{n/2-1}}{x_1^{n/2} + x_2^{n/2}} \left[\zeta^{j-1} \left(\frac{x_2}{x_1}\right)^j f_1 - f_2\right]$$

Equation 4, Summation 1:

$$c_0 \sum_{\ell=0}^{m-1} \frac{\langle \varepsilon_1 - \xi^\ell \varepsilon_2, \varepsilon_2^\vee \rangle}{x_1 - \xi^\ell x_2} \left[-f_2 + \xi^{-j\ell} f_1\right] = \frac{c_0 f_2}{x_1^m - x_2^m} \sum_{\ell=0}^{m-1} \xi^\ell \prod_{r=0, r\neq\ell}^{m-1} (x_1 - \xi^r x_2)$$

$$- \frac{c_0 f_1}{x_1^m - x_2^m} \sum_{\ell=0}^{m-1} \xi^{-(j-1)\ell} \prod_{r=0, r\neq\ell}^{m-1} (x_1 - \xi^r x_2)$$

$$= \frac{c_0 f_2 \left(m x_2^{m-1}\right)}{x_1^m - x_2^m} - \frac{c_0 f_1 \left(m x_1^{m-1}\right)}{x_1^m - x_2^m} \left(\frac{x_2}{x_1}\right)^{j-1}$$

$$= \frac{m c_0}{x_1^m - x_2^m} \left[-x_1^{m-1} \left(\frac{x_2}{x_1}\right)^{j-1} f_1 + x_2^{m-1} f_2\right]$$

Equation 4, Summation 2:

$$c_1 \sum_{\ell=0}^{m-1} \frac{\langle \varepsilon_1 - \xi^\ell \zeta \varepsilon_2, \varepsilon_2^\vee \rangle}{x_1 - \xi^\ell z_2} \left[-f_2 + \xi^{-j\ell} \zeta^{-1} f_1\right] = \frac{c_1 \zeta f_2}{x_1^m - z_2^m} \sum_{\ell=0}^{m-1} \xi^\ell \prod_{r=0, r\neq\ell}^{m-1} (x_1 - \xi^\ell z_2)$$

$$- \frac{c_1 f_1}{x_1^m - z_2^m} \sum_{\ell=0}^{m-1} \xi^{-(j-1)\ell} \prod_{r=0, r\neq\ell}^{m-1} (x_1 - \xi^\ell z_2)$$

$$= \frac{c_1 \zeta f_2 \left(m z_2^{m-1}\right)}{x_1^m + x_2^m} - \frac{c_1 f_1 \left(m x_1^{m-1}\right)}{x_1^m + x_2^m} \left(\frac{z_2}{x_1}\right)^{j-1}$$

$$= \frac{m c_1}{x_1^m + x_2^m} \left[-\zeta^{j-1} x_1^{m-1} \left(\frac{x_2}{x_1}\right)^{j-1} f_1 - x_2^{m-1} f_2\right]$$

So our last equation is

$$\frac{\partial f_2}{\partial x_2} = \frac{(n/2) c_0}{x_1^{n/2} - x_2^{n/2}} \left[-x_1^{n/2-1} \left(\frac{x_2}{x_1}\right)^{j-1} f_1 + x_2^{n/2-1} f_2\right] + \frac{(n/2) c_1}{x_1^{n/2} + x_2^{n/2}} \left[-\zeta^{j-1} x_1^{n/2-1} \left(\frac{x_2}{x_1}\right)^{j-1} f_1 - x_2^{n/2-1} f_2\right]$$

B Background: Topology

Hecke algebra can be defined in terms of fundamental groups of certain spaces, so here we will briefly recall the basics. A standard reference is [Hat00].

Let X be a path connected topological space, and fix $p \in X$. A loop based at p is a continuous map $\gamma : [0,1] \longrightarrow X$ such that $\gamma(0) = \gamma(1) = p$. Two loops γ_1, γ_2 based at p are said to be homotopic if there exists a continuous map

$$F : [0,1] \times [0,1] \longrightarrow X$$

such that $\gamma_1(t) = F(0,t)$ and $\gamma_2(t) = F(1,t)$ for all $t \in [0,1]$, and $F(u,0) = F(u,1) = p$ for all $u \in [0,1]$. In this case, we say $\gamma_1 \simeq \gamma_2$. Homotopy equivalence is an equivalence relation on the set of loops in X based at p.

Given two loops γ_1, γ_2 based at p, we define the concatenation $\gamma_2\gamma_1 : [0,1] \longrightarrow X$ to be the loop based at p which is the path obtained by traversing γ_1, and then γ_2.

Moreover, given a loop γ at p, define there is a loop γ^{-1} at p obtained by traversing γ in the reverse direction.

Further, set $e : [0,1] \longrightarrow X$ be the constant path at p. The following facts are true for any loop γ at p:

$$\gamma\gamma^{-1} \simeq \gamma^{-1}\gamma \simeq e \qquad e\gamma \simeq \gamma e \simeq \gamma$$

Finally, it can be shown that the homotopy class of the concatenation of two loops based at p depends only on their own homotopy classes. This means that if $[\gamma]$ is the equivalence class of loops based at p which are homotopic to γ, then the following operation is well defined for any loops γ_1, γ_2 based at p:

$$[\gamma_2] \cdot [\gamma_1] = [\gamma_2\gamma_1]$$

Let $\pi_1(X,p)$ be the set of all loops based at p identified upto homotopy equivalence class. Then $\pi_1(X,p)$ is a group under the above operation with $[\gamma]^{-1} = \left[\gamma^{-1}\right]$ and identity $[e]$. This is called the fundamental group of the space X. Provided that X is path connected, choosing a different basis point other than p will yield an isomorphic group, so in some cases we will also omit specifying the basepoint. We will omit the square bracket and implicitly mean γ to be the homotopy class of the loop γ.

Recall that as per our definition, a homotopy leaves the endpoints fixed throughout the deformation. If we allow our homotopies to move around the endpoints in between, then it turns out that γ_1 and γ_2 are conjugate elements of $\pi_1(X,p)$ if and only if γ_1 and γ_2 are homotopic in this weaker sense where the endpoints are free to move.

Let $\phi : X \longrightarrow Y$ be a continuous map between path connected topological spaces X, and let $p \in X$. Then ϕ induces a homomorphism of fundamental groups:

$$\Phi : \pi_1 (X, p) \longrightarrow \pi_1 (Y, \phi (p)) , [\gamma] \longmapsto [\phi \circ \gamma]$$

If ϕ is a homeomorphism, then Φ becomes an isomorphism.

C Background: Group Theory

Here, we assume basic knowledge of basic group theory, and introduce the notions of reflection groups, Coxeter groups, and Artin groups.

C.1 Reflection Groups and Shephard-Todd Classification

Let K be a field, either $\mathbb{R}$ or $\mathbb{C}$, and let $\mathfrak{h}$ be a non-zero vector space over K of dimension $\dim \mathfrak{h} = n$. A hyperplane $H \subseteq \mathfrak{h}$ is a subspace of dimension $n-1$. A linear map $T \in GL(\mathfrak{h})$ is called a reflection if $\ker(T - \mathrm{Id}_{\mathfrak{h}}) = n-1$. That is, T fixes a hyperplane in $\mathfrak{h}$.

A reflection group $W \subseteq GL(\mathfrak{h})$ is a group such that the set of reflection elements $T \subseteq W$ generates the whole group. If $K = \mathbb{R}$, then W is called a real reflection group, and if $K = \mathbb{C}$, then W is called a complex reflection group. From now, all reflection groups are finite.

The complex reflection groups were completely classifed by Shephard-Todd in 1954 [ST54] as follows.

Let $r, p, n \in \mathbb{Z}_{\geq 1}$ with r divisible by p. The group $G(r, p, n)$ is defined as the $n \times n$ matrices with complex entries satisfying the following:

1. Exactly one entry of each column and each row is non-zero
2. The non-zero entries are all powers of $\zeta = e^{2\pi \mathbf{i}/r}$ and
3. If P is the product over all the non-zero entries, then $P^{r/p} = 1$

The result of Shephard-Todd is that every complex reflection group is one of $G(r, p, n)$, or one of 34 exceptional cases.

Some examples of complex reflection groups are $\mathfrak{S}_n = G(1, 1, n)$, the symmetric group on n elements, $G(n, 1, 1)$, the cyclic group of order n, and $G(n, n, 2)$, the dihedral group of order $2n$.

C.2 Coxeter and Artin Groups

The standard reference for Coxeter groups is [Hu90]. Given a set $S = \{s_1, s_2, \cdots, s_N\}$, let $\mathcal{F}_S$ be the free group generated by S. A Coxeter matrix $M = [m_{i,j}]_{i,j=1}^{N}$ is a symmetric $N \times N$ matrix such that $m_{i,i} = 1$, and

$m_{i,j} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ when $i \neq j$. The Coxeter and Artin groups to this information are given by the presentations

$$W_M = \langle s_1, s_2, \cdots, s_N \mid (s_i s_j)^{m_{i,j}} = 1, \text{ for } i, j \in \{1, 2, \cdots, N\} \rangle$$
$$A_M = \langle s_1, s_2, \cdots, s_N \mid \underbrace{s_i s_j s_i \cdots}_{m_{i,j} \text{ factors}} = \underbrace{s_i s_j s_i \cdots}_{m_{i,j} \text{ factors}}, \text{ for } i, j \in \{1, 2, \cdots, N\}, i \neq j \rangle$$

As we are about to see, the dihedral groups as an example of a Coxeter group. This way, the Coxeter groups can be though of as a generalisation of dihedral groups.

After some examples, we will show how the Artin groups arise from the Coxeter groups via topology. This will allow us to define the Hecke algebra of a wide class of complex reflection groups using just generators and relations.

Next, we show that several of the complex reflection groups we want to study can be expressed as Coxeter groups. As hinted before, the dihedral group is a prototypical example, so we will start there.

Dihedral Groups as Coxeter Groups

The dihedral group $D_n = G(n, n2)$ is the complex reflection group

$$D_n = \left\{ \begin{pmatrix} \zeta^k & 0 \\ 0 & \zeta^{-k} \end{pmatrix}, \begin{pmatrix} 0 & \zeta^k \\ \zeta^{-k} & 0 \end{pmatrix} \mid 0 \leq k < n \right\}$$

Let $S = \{s_0, s_1\}$, and M be a 2×2 matrix with entries 1 on the diagonal, and entries n off the diagonal. This is a Coxeter matrix, and the Coxeter group specified by S and M is

$$W_M = \langle s_0, s_1 \mid s_0^2 = (s_0 s_1)^n = (s_1 s_0)^n = s_1^2 = 1 \rangle$$

The map $W_M \longrightarrow D_n$

$$s_0 \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad s_1 \longmapsto \begin{pmatrix} 0 & \zeta \\ \zeta^{-1} & 0 \end{pmatrix}$$

induces an isomorphism between W_M and D_n, so we have a presentation of the dihedral group as a Coxeter group:

$$D_n \cong \langle s_0, s_1 \mid s_0^2 = (s_0 s_1)^n = (s_1 s_0)^n = s_1^2 = 1 \rangle$$

Symmetric Groups as Coxeter Groups

Let $S = \{s_1, s_2, \cdots, s_n\}$, and let the Coxeter matrix M be defined by

$$m_{i,j} = \begin{cases} 2 & \text{, if } |i-j| > 1 \\ 3 & \text{, if } j = i+1 \\ 1 & \text{, if } i = j \end{cases}$$

The Coxeter group specified by this data turns out to be isomorphic to symmetric group $\mathfrak{S}_n$ if one thinks of s_k as the transposition in $\mathfrak{S}_n$ exchanging k and $k+1$.

C.3 Coxeter Groups as Real Reflection Groups

Notice that the Coxeter group presentations of the dihedral and symmetric groups has a faithful representation as a complex reflection group acting on a space the dimension of the size of the generating set. This makes one wonder whether the class of Coxeter groups and the class of complex reflection groups are really the same.

The answer is in the negative: the cyclic group, although a complex reflection group, cannot in general be written as a Coxeter group. This is because a necessary condition for a group to be a Coxeter group is that the set of elements of the group which have order 2 must generate the group. For odd cyclic groups, there are no non-trivial involutions, and for even cylic groups, there is a single involution, and the generated group only has two elements. So the only cyclic group which is a Coxeter group is the one of order 2, which is of course just the symmetric group $\mathfrak{S}_2$.

The complex reflection groups which we have demonstrated to be Coxeter groups are more special than a generic complex reflection group. The symmetric group of order $n!$ not only acts on $\mathbb{C}^n$ by permuting coordinates, but furthermore, restricting to $\mathbb{R}^n$ gives a representation as a real reflection group. Likewise, after a change of basis, the dihedral group also acts on $\mathbb{R}^2$ as a real reflection group

$$\begin{pmatrix} 0 & \zeta^k \\ \zeta^{-k} & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} \cos\left(\frac{2k\pi}{n}\right) & \sin\left(\frac{2k\pi}{n}\right) \\ \sin\left(\frac{2k\pi}{n}\right) & -\cos\left(\frac{2k\pi}{n}\right) \end{pmatrix} \qquad \begin{pmatrix} \zeta^k & 0 \\ 0 & \zeta^{-k} \end{pmatrix} \longrightarrow \begin{pmatrix} \cos\left(\frac{2k\pi}{n}\right) & -\sin\left(\frac{2k\pi}{n}\right) \\ \sin\left(\frac{2k\pi}{n}\right) & \cos\left(\frac{2k\pi}{n}\right) \end{pmatrix}$$

There is no representation of the cyclic group of order greater than 2 as a real reflection group since the reflection elements can only have order 2.

This leads one to conjecture that perhaps any finite Coxeter group is a real reflection group. This is indeed the case [Hu90, Section 5.3].

More precisely, given a generating set S with Coxeter matrix M of size N, and finite Coxeter group W_M, we have a real reflection reflection group $W \subseteq GL\left(\mathbb{R}^N\right)$ with $W_M \cong W$.

C.4 Artin Groups as Fundamental Groups

Given a finite Coxeter group W_M specified by the generating set $S = \{s_1, \cdots, s_N\}$ and Coxeter matrix M, the group W_M acts on $\mathfrak{h}_0 = \mathbb{R}^N$ as a real reflection group. We can extend this action to the complex vector space $\mathfrak{h} = \mathbb{C}^N$, so that W_M acts as a complex reflection group with a finite set of reflection hyperplanes $\mathcal{A}$.

Let $W \subseteq GL(\mathfrak{h})$ be a complex reflection group isomorphic to W_M and let $H_i \subseteq \mathfrak{h}$ be of the reflection element of W corresponding to s_i. Set

$$\mathfrak{h}^{\text{reg}} = \mathfrak{h} - \bigcup_{H \in \mathcal{A}} H$$

the topological space $\mathfrak{h}^{\text{reg}}$ is path connected. Furthermore, the only points of $\mathfrak{h}$ which are mapped into the union of the reflection hyperplanes by any $w \in W$ are those already in the union of the hyperplanes. This means that the restriction of each $w \in W$ defines a valid map. Hence, we have the topological space $\mathfrak{h}^{\text{reg}}/W$ of the orbits of the W_M-action on $\mathfrak{h}^{\text{reg}}$. Since $\mathfrak{h}^{\text{reg}}$ was path connected, so is the quotient $\mathfrak{h}^{\text{reg}}/W$, so has a well defined fundamental group $\pi_1\left(V^{\text{reg}}/W\right)$ based at any $x_0 \in \mathfrak{h}^{\text{reg}}/W$.

The fundamental group of $\mathfrak{h}^{\text{reg}}/W$ turns out to be the Artin group with the same specifications as the Coxeter group with which we began [Bri71].

The way this happens is as follows. Pick any point p in one of the connected components of $\mathfrak{h}_0$ whose walls are the real hyperplanes $H_i \cap \mathfrak{h}_0$ for $1 \leq i \leq N$.

Write $\mathfrak{h} = H_i \oplus L_i$ where H_i is the hyperplane fixed by s_i, and L_i is the one-dimensional eigenspace of s_i corresponding to the eigenvalue -1. We can write then $p = v_i + v_i^{\perp}$ where $v_i \in H_i$ and $v_i^{\perp} \in L_i$. The straight line between p and $s_i(p)$ can be slightly deformed into another path $\gamma_i : [0,1] \longrightarrow \mathfrak{h}^{\text{reg}}$ which avoids $H_i \cap \mathfrak{h}_0$

$$\gamma_i\left(t\right) = \begin{cases} v_i + \left(1 - 2t\right) v_i^{\perp} & , \text{if } t \notin \left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right] \\ v_i + 2\varepsilon \mathbf{i} \exp\left(\frac{1}{2\varepsilon}\left(t - \frac{1}{2}\right)\pi \mathbf{i}\right) v_i^{\perp} & , \text{if } t \in \left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right] \end{cases}$$

where $\varepsilon \in (0, 1/2)$ is sufficiently small.

Then the images of the γ_i in the orbit space $\mathfrak{h}^{\text{reg}}/W$ as loops based at the orbit of p will be the generators of the fundamental group, and the map between $\pi_1\left(\mathfrak{h}^{\text{reg}}/W, p\right)$ and A_M defined by

$$\phi : A_M \longrightarrow \pi_1\left(\mathfrak{h}^{\text{reg}}/W, p\right), s_i \longrightarrow q \circ \gamma_i$$

is an isomorphism, where $q : \mathfrak{h}^{\text{reg}} \longrightarrow \mathfrak{h}^{\text{reg}}/W$ is the quotient map.

D Background: Representation Theory

Here, we will recall some classical results on the representation theory of finite groups. All claims in this section can be found in [FH91]. Here will classify the irreducible representations for the cylic and dihedral, and then use them in the main section of this report to derive the differential equations for the monodromy representations for the Hecke algebra.

The following results are well known:

Lemma. *(Schur) Let (π, V) and (ρ, W) be irreducible representations of a finite group G. Then if $f : V \to W$ is a homomorphism of representations, either f is invertible, or $f = 0$. Moreover, if $(\pi, V) = (\rho, W)$ then $f = \lambda Id_V$ for some $\lambda \in \mathbb{C}$.*

Proof. It is easily checked that the subspaces $\operatorname{Im} f \subseteq W$ and $\ker f \subseteq V$ are invariant subspaces under the G-action. Then by irreducibility of V, W, either $\ker f = \{0\}, \operatorname{Im} f = W$, or $\ker f = V, \operatorname{Im} f = \{0\}$.

If f is a self-map, then it has an eigenvalue $\lambda \in \mathbb{C}$ by algebraic closure. This implies $f - \lambda \mathrm{Id}_V$ is not invertible, so the previous part tells us $f - \lambda \mathrm{Id}_V = 0$. □

A corollary of the above is that the elements of the center of G are represented as $\lambda \mathrm{Id}_V$ since they are self maps which commute with all the other actions.

Given a representation (π, V) of G, the character χ_V is the map

$$\chi_V : G \longrightarrow \mathbb{C}, g \longmapsto \operatorname{Tr}(\pi(g))$$

We say $f : G \to \mathbb{C}$ is a class function if it is equal on the conjugacy classes of G. Clearly, the characters are class functions. We introduce a complex inner product on the vector space of class functions:

$$\langle f_1 \mid f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}$$

This is well defined as the trace is basis independent, and is preserved under isomorphisms of representations. We have:

Theorem. *Let (π, V) be a representation of G. Then $\langle \chi_V \mid \chi_V \rangle = 1$ if and only if (π, V) is irreducible.*

Theorem. *Let (π, V) be an irreducible representation of G, and (ρ, W) any representation of G. Then $\langle \chi_V \mid \chi_W \rangle$ is a positive integer, which tells us how many times (π, V) appears in the decomposition of (ρ, W) as a direct sum of irreducible representations.*

Theorem. *If $\{(\pi_i, V_i)\}_{i=1}^{N}$ are all the irreducible representations of G, then*

$$\sum_{i=1}^{N} (dim\, V_i)^2 = |G|$$

Theorem. *The irreducible representations of a group G are in bijection with the conjugacy classes of G.*

The last two theorems tell us that for Abelian groups, all irreducible representations are one-dimensional. This is because there are $|G|$ so so we have a sum of $|G|$ positive integers equal to $|G|$ which can only imply that each dimension is 1.

Using the above results, we will classify the irreducible representations of cyclic, dihedral groups.

D.1 Classification of Cyclic Group Representations

Let $W = \mathbb{Z}/n\mathbb{Z}$ be the cyclic group of order n. From the theorems in the previous part, W being abelian means that all irreducible representations are one-dimensional, and there are n of them. If the elements of W are $\{k \mid 0 \leq k < n\}$, then it is easily seen that the exhaustive list of irreducible representations is

$$E^j = \mathbb{C} \text{ , with } kz = \exp\left(\frac{2\pi jk\mathbf{i}}{n}\right) z \text{ for } z \in E^j,\ k \in W$$

indexed by $0 \leq j < n$.

D.2 Classification of Dihedral Group Representations

Think of the dihedral group D_n as the Coxeter group

$$D_n \cong \langle s_0, s_1 \mid s_0^2 = (s_0 s_1)^n = (s_1 s_0)^n = s_1^2 = 1 \rangle$$

To specify a representation, we have to only specify the action of the generators s_0, s_1.

The two-dimensional representations of D_n are (π_j, V) indexed by $0 < j < n/2$ where $V = \mathbb{C}^2$, and π_j is defined as

$$\pi_j(s_0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \pi_j(s_1) = \begin{pmatrix} 0 & \zeta^j \\ \zeta^{-j} & 0 \end{pmatrix}$$

where $\zeta = e^{2\pi \mathbf{i}/n}$. This extends to the entire group. The matrices for the reflection elements are

$$\left\{ \begin{pmatrix} 0 & \zeta^{jk} \\ \zeta^{-jk} & 0 \end{pmatrix} \mid 0 \leq k < n \right\}$$

and the matrices for the rotation elements are

$$\left\{ \begin{pmatrix} \zeta^{jk} & 0 \\ 0 & \zeta^{-jk} \end{pmatrix} \mid 0 \leq k < n \right\}$$

By computing the characters and taking the inner products between them, it can be easily checked using character theory that these are all distinct irreducible characters. The sum over the squares of the dimensions of these representations is $2n - 2$ if n is odd, and $2n - 4$ when n is even.

In the odd case, 2 one-dimensional representations would complete the sum. These are the one dimensional modules $(\rho_1, \mathbb{C})$ and $(\rho_2, \mathbb{C})$ defined by the matrices

$$\rho_1(s_0) = \rho_1(s_1) = \begin{pmatrix} 1 \end{pmatrix}$$
$$\rho_2(s_0) = \rho_2(s_1) = \begin{pmatrix} -1 \end{pmatrix}$$

In the even case, we need four one-dimensional representations. These are the above two and two additional representations $(\rho_3, \mathbb{C})$ and $(\rho_4, \mathbb{C})$ defined by

$$\rho_3(s_0) = \begin{pmatrix} 1 \end{pmatrix} \qquad \rho_3(s_0) = \begin{pmatrix} -1 \end{pmatrix}$$
$$\rho_4(s_0) = \begin{pmatrix} -1 \end{pmatrix} \qquad \rho_4(s_0) = \begin{pmatrix} -1 \end{pmatrix}$$

Again, it can be checked from the characters that all of these are distinct. This completes the classification.

D.3 Classification of Symmetric Group Representations

It is well-known that two permutations from $\mathfrak{S}_n$ are conjugate if and only if they have the same cycle type. Any cycle type can be written as a finite sequence of non-increasing positive integers which sum up to n, so the conjugacy classes are indexed by partitions of n.

Thus, the irreducible representations of $\mathfrak{S}_n$ are in bijection with the partitions of n. There is a way to explitly write them down for any given partition, however, for the purpose of this project, we will only focus on a few important ones. Therefore, we only refer to [FH91] for a detailed analysis of the irreducible representations of $\mathfrak{S}_n$.

The most obvious representation of $\mathfrak{S}_n$ is the trivial representation where $\mathfrak{S}_n$ acts on $\mathbb{C}$ as the identity map.

The next representation is the sign representation $(\rho, \mathbb{C})$ which is defined for any $\sigma \in \mathfrak{S}_n$ as follows:

$$\rho(\sigma) : \mathbb{C} \longrightarrow \mathbb{C}, z \longmapsto \operatorname{sgn}(\sigma) z$$

where

$$\operatorname{sgn}(\sigma) = \begin{cases} 1 & \text{, if } \sigma \text{ is the product of evenly many transpositions} \\ -1 & \text{, if } \sigma \text{ is the product of oddly many transpositions} \end{cases}$$

Now for the permutation and standard representations. Let $E = \mathbb{C}^n$ with some choice of basis $\{e_1, e_2, \cdots, e_n\}$. The permutation representation (π, E) is

$$\pi(\sigma) e_k = e_{\sigma(k)}$$

However, notice that

$$\pi(\sigma) \left(\sum_{\ell=1}^{n} e_\ell \right) = \sum_{\ell=1}^{n} e_\ell$$

That is, the one-dimensional span of $e_1 + e_2 + \cdots + e_n$ is fixed pointiwse under the permutation action of $\mathfrak{S}_n$ which means that the permuation representation is is reducible. This is of course a copy of the one-dimensional trivial representation in the n-dimensinoal permutation representation. The complementary representation of dimension $n - 1$ is called the standard representation, and it turns out to be irreducible.