AMSI VACATION RESEARCH SCHOLARSHIPS 2021–22

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Knots & Combinatorics

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1 Prelude

1.1 Acknowledgements

I would like to thank Jian He for his ongoing support throughout the summer; thank you for patiently explaining to me while I struggled to understand the concepts and for holding many, many meetings with me, both in person and online. I have been very privileged to have you as a mentor and I would have been absolutely clueless without you. I would also like to thank Daniel Mathews, first for his enthusiasm towards knot theory and as a lecturer, it is you who got me interested in this project in the first place. I would also like to thank him for his delightful guidance, providing support both in meetings and out despite the busy period this project was done in.

1.2 Abstract

Dehn surgery is a fundamental method of constructing 3-manifolds. This project centres around the 3d index, an invariant introduced by Dimofte, Gaiotto and Gukov, associated with a suitable ideal triangulation of a 3-manifold with torus boundary components. The 3d index has since been extended to closed 3-manifolds obtained via Dehn surgery, further Gang has conjectured that the 3d index is an indicator of whether or not the resulting manifold obtained by Dehn surgery on a knot has hyperbolic geometry. We focus on the numerical verification of this conjecture, specifically the conjectural vanishing of the 3d index for a (1,0)-filling of the figure eight knot complement, which we compute to be zero up to degree 5.

1.3 Statement of Authorship

McQuire developed most of the code used to calculate the tetrahedron index, open 3d index and closed 3d index for the figure eight knot complement (included in the appendix) under the supervision and guidance of He, written in MATLAB. Some aspects of the code are written by He. McQuire wrote this report and also made the figures and tables included. The report was proofread by He and Mathews.

2 Introduction

Knot theory is a well established field of mathematics, with origins in the late 1700s [1]. The field has applications in areas such as DNA modelling and statistical mechanics [1], as a result, some major results in knot theory, including the focus of this report, emerge from fields such as physics, rather than purely from mathematics. One of the central concerns of knot theory is the classification of knots and methods used to distinguish between different knots. A key tool for this is knot invariants, properties which are the same for all equivalent classes of knots (the notion of equivalence is discussed in the appendix). Examples of knot invariants include the crossing number, defined as the minimum number of crossings of a given knot, and knot polynomials, polynomials which are computed from properties of the knot or its complement and result in invariants of the manifold. [1]

Definition 2.1. A *knot* is an embedding of the circle in any 3-manifold. In this project we focus on knots in the 3-sphere. A *link* is a collection of knots which do not intersect.

Definition 2.2. Given an embedding of a knot in the 3-sphere, the *knot complement* is all the points in the 3-sphere not contained in the knot. The resulting space is a *manifold*, a space which locally looks like Euclidean space.

This report centres around an object introduced by three physicists Dimofte, Gaiotto and Gukov, the 3d index, which arose from their work on the low energy limit of gauge theory with $N = 2$ supersymmetry [2]. The 3d index, defined for 3-manifolds with r tori boundary components, is a collection of power series, one for each choice of a peripheral curve on each of the r boundary tori. It is a partially defined function since it does not necessarily converge (refer to [3] and [4] for more details) and is associated with a suitable triangulation (precise statements given in [5]). It is a formal Laurent series in $q^{\frac{1}{2}}$ and, loosely speaking, is defined as the infinite sum over integer weights attached to edges in an ideal triangulation of a 3-manifold [4]. The 3d index is known to be a topological invariant of oriented cusped hyperbolic 3-manifolds, and is predicted to be a topological invariant of the underlying 3-manifold [3], however, this is not known in general [4].

Definition 2.3. Manifolds can be characterised as the result of gluing polyhedra together by pairing up their faces in a particular way. These polyhedra can be decomposed into tetrahedra, giving what is known as a *triangulation*. An *ideal triangulation* is one which uses ideal tetrahedra; tetrahedra with vertices which are 'ideal points', points at infinity rather than in the interior of the hyperbolic space [6].

Proposed by Gang and Yonekura [7], the 3d index can be extended to closed manifolds obtained via Dehn fillings on knot/link complements.

Definition 2.4. *Dehn surgery* is a method of constructing manifolds by the means of cutting and pasting a solid torus. Given a manifold N, identified with a link L, whose boundary components are tori, we remove an open tubular neighbourhood of L from N, and glue in solid tori such that the meridian of the i-th torus is mapped to the curve $x\mu + y\lambda$ on the torus boundary [8], where μ and λ are the meridian and longitudinal curves, and x and y are relatively prime integers. When specifying a particular x and y, we will refer to this as a (x, y)-filling. The process of gluing in the tori is referred to as *Dehn filling*; a Dehn surgery consists of removing an open tubular neighbourhood together with a Dehn filling.

It is known that the (x, y)-filling of a knot complement has hyperbolic geometry for all x, y except for finitely many exceptions [9], however these exceptions can be hard to detect. Gang has conjectured that the 3d index should provide an effective way to detect these exceptions, depending on whether or not the 3d index of the resulting manifold is a power series [10]. In particular, Gang has conjectured that if a (1,0)-filling is performed, that is, if one glues in a solid torus exactly in a manner such that one recovers $\mathbb{S}^3$, then the 3d index is zero [10]. The main focus of this report is the numerical verification of this conjecture, in particular, for the figure eight knot complement.

The following is an outline of this report. In section 3 we introduce the tetrahedron index, an object required to define the 3d index, and provide some example calculations. We then define the 3d index in section 4 (also referred to as the open 3d index throughout), motivating the definition with some examples using the figure eight knot complement. In section 5, we introduce the 3d index for closed manifolds (also referred to throughout as the closed 3d index). Using Garoufalidis' work [5] on the minimum degree of the tetrahedron index, we prove the following theorem: in order to be certain of the closed 3d index of a (1,0)-filling of the figure eight knot complement up to degree n, one must sum each of the relevant open 3d indices up to $e = \max(2n,7)$. Gang's conjecture is then discussed where we present the numerical verification of the (1,0)-filling of the figure eight knot complement up to degree 5, computed using a program written in MATLAB (included in the appendix). Finally, future directions for research are suggested in section 6.

3 The tetrahedron Index

Definition 3.1. The tetrahedron index is defined $I_\Delta : \mathbb{Z}^2 \to \mathbb{Z}[[q^{1/2}]]$ by the equation[1] [5]

$$I_\Delta(m, e) = \sum_{n=(-e)_+}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)-(n+\frac{1}{2}e)m}}{(q)_n (q)_{n+e}}.$$

[1] The parameters m and e are named after magnetic and electric charges [5].

Here $(q)_n$ are q-pochhammer symbols defined by $(q)_n = \prod_{i=1}^{n}(1-q^i)$ for $n \geq 0$ and $(q)_0 = 1$ by convention. Note that $(-e)_+ = \max(0, -e)$. We can consider the tetrahedron index as a formal infinite power series by considering

$$\frac{1}{1-q^n} = \sum_{k=0}^{\infty} q^{nk} = 1 + q^n + q^{2n} + q^{3n} + \dots.$$

There is no obvious reason that the tetrahedron index should converge, nor that it should have no negative powers, however these turn out to be indeed the case [5]. The tetrahedron index satisfies a three fold symmetry [5],

$$I_\Delta(m,e) = (-q^{\frac{1}{2}})^{-e} I_\Delta(e, -e-m) = (-q^{\frac{1}{2}})^{m} I_\Delta(-e-m, m).$$

Example 3.2. Some example calculations of the tetrahedron are listed (computed using the code included in the appendix).

$$I_\Delta(0{,}0) = 1 - q - 2q^2 - 2q^3 - 2q^4 + q^6 + 5q^7 + 7q^8 + 11q^9 + 13q^{10} + 16q^{11} + \dots$$

$$I_\Delta(1, -1) = \sqrt{q}(-1 + q^2 + 2q^3 + 3q^4 + 3q^5 + 3q^6 + q^7 - q^8 - 5q^9 - 9q^{10} - 15q^{11} + \dots)$$

$$I_\Delta(2{,}2) = q^5 + q^6 + 2q^7 + 2q^8 + 3q^9 + 2q^{10} + 2q^{11} - 2q^{13} - 6q^{14} - 10q^{15} - 16q^{16} \dots$$

4 The 3d Index

4.1 Evaluation at 0

The 3d index can be evaluated along any peripheral curve on the torus boundary. In order to motivate the definition of the 3d index, we first consider the case where there are no peripheral curves. Let T be an ideal triangulation of a manifold N triangulated with n tetrahedra, with a boundary consisting of r tori. Note that the number of edges in T is equal to n, this can be derived from a simple Euler characteristic calculation. Given $\mathbf{k} = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$, each edge class is assigned an integer weight k_i (where an edge class consists of all the edges in each tetrahedron which get glued to a single edge in the manifold). This results in a weight attached to each edge in each tetrahedron in T. For each tetrahedron j, we label the three pairs of opposite edges a_j, b_j, c_j such that the labelled edges appear in the order a_j, b_j, c_j counterclockwise around a vertex when viewed from outside the tetrahedron, as in Figure 1.

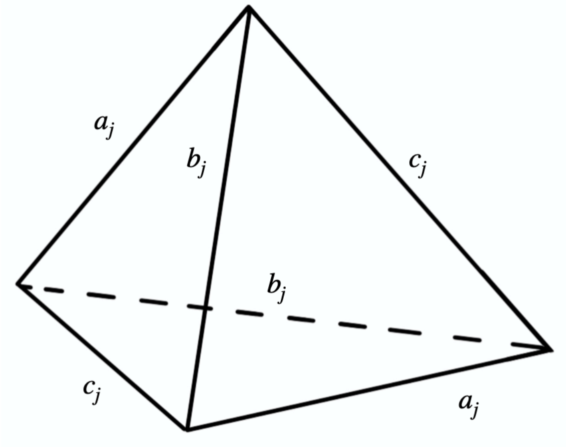

Figure 1. Labelling of edges in tetrahedron j. Opposite pairs of edges are labelled a_j, b_j, c_j.

Let $a_j(\mathbf{k}), b_j(\mathbf{k}), c_j(\mathbf{k})$ be the sums of the weights associated with the edges labelled a_j, b_j, c_j respectively. We now introduce an alternative, more symmetric notation for the tetrahedron index. For integers $a, b, c,$ write

$J_\Delta(a,b,c) = (-q^{\frac{1}{2}})^{(-b)} I_\Delta(b-c, a-b) = (-q^{\frac{1}{2}})^{(-c)} I_\Delta(c-a, b-c) = (-q^{\frac{1}{2}})^{(-a)} I_\Delta(a-b, c-a)$

[4]. This leads to the following.

Definition 4.1.1. The 3d index evaluated at 0 (meaning there are no peripheral curves) is defined by the equation [4]

$$I_N(\mathbf{0})(x) = \sum_{\mathbf{k}\in\mathbb{Z}^{n-r}\subset\mathbb{Z}^n} q^{\sum_i k_i} \prod_{j=1}^{n} J_\Delta(a_j(\mathbf{k}), b_j(\mathbf{k}), c_j(\mathbf{k})).$$

Here, we are summing over the sublattice $\mathbb{Z}^{n-r} \subset \mathbb{Z}^n$, which is equivalent to setting r of the integer weights k_i to be zero. In the case of a manifold with a single torus boundary, any set of $n-1$ edges can be used, but in general the set of $n-r$ edges must be chosen carefully (refer to [3] for details).

Example 4.1.2. We demonstrate the above process for the figure eight knot complement. On the next page is shown a diagram of an ideal triangulation of the figure eight knot complement, where the faces of the tetrahedra with matching labels and edges with the same thickness are glued together. There are two tetrahedra, as shown in Figure 2, thus $n = 2$. Note that the figure eight knot complement has a boundary consisting of a single torus, so we can choose either edge to be zero. We label a_j, b_j, c_j as shown in Figure 2 and let us assign an integer weight k to the thick edges and 0 to the thin edges.

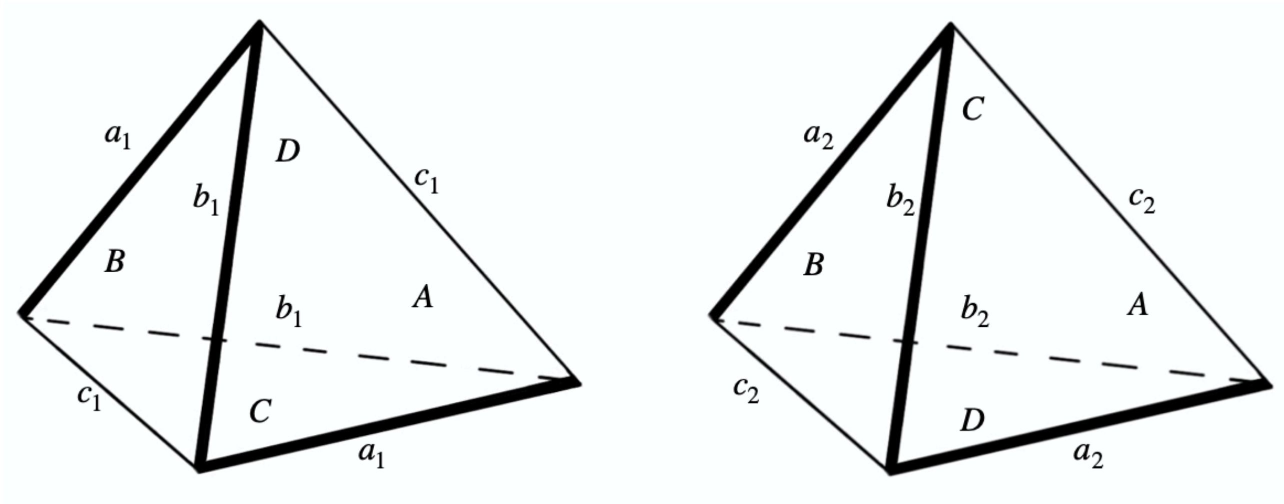

Figure 2. Ideal triangulation of figure eight knot complement.

Then

$$a_1(k) = 2k, b_1(k) = k, c_1(k) = 0, a_2(k) = 2k, b_2(k) = k, c_2(k) = 0.$$

This yields

$$I_N(\mathbf{0}) = \sum_{k\in\mathbb{Z}} q^k J_\Delta(2k, k, 0)^2 = \sum_{k\in\mathbb{Z}} I_\Delta(k, k)^2 = 1 - 2q + 3q^2 + 2q^3 + 8q^4 + 18q^5 + 18q^6 + 14q^7 + \dots$$

4.2 Evaluation Along a General Curve

We now consider the general case. Given an oriented, non-contractible multicurve γ on the boundary δN of the manifold, we deform γ so that it is a union of disjoint oriented normal arcs in each triangle of $T_{\delta N}$ (a normal arc being a simple arc which connects two distinct sides of a triangle). Each normal arc can be given a sign based on the orientation of δN and the orientation of γ. In particular, we say the sign is positive if the arc winds counterclockwise around the vertex of a triangle, when viewed from outside δN. The sign is negative otherwise. We define

$$a_j(\gamma) = \text{signed count of arcs in } \gamma \text{ around } a_j$$

$$b_j(\gamma) = \text{signed count of arcs in } \gamma \text{ around } b_j$$

$$c_j(\gamma) = \text{signed count of arcs in } \gamma \text{ around } c_j,$$

where a_j, b_j, c_j are defined as in section 4.1. These definitions are illustrated in Example 4.2.2.

Definition 4.2.1. The 3d index along γ is defined as

$$I_N(\gamma)(q) = \sum_{\mathbf{k}\in\mathbf{Z}^{n-r}\subset\mathbf{Z}^n} q^{\sum_i k_i} \prod_j J_\Delta(a_j(\mathbf{k}) + a_j(\gamma), b_j(\mathbf{k}) + b_j(\gamma), c_j(\mathbf{k}) + c_j(\gamma)).$$

Again, the summation is over the sublattice $\mathbb{Z}^{n-r} \subset \mathbb{Z}^n$, which has been shown to depend only on the homology class of γ [4].

Example 4.2.2. Let N be the figure eight knot complement. Then N can be triangulated with two tetrahedra, its boundary consisting of a single torus (as described in Example 4.1.2). We demonstrate how to derive the explicit formula for the open 3d index of the figure eight knot complement.

The diagram below shows the triangulation induced on the torus boundary.

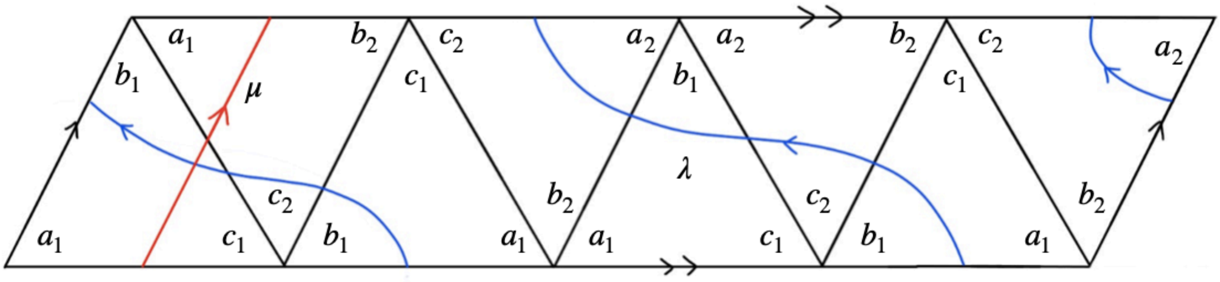

Figure 3. Triangulation of torus boundary of figure eight knot complement.

The red curve denotes the meridian μ and the blue curve denotes the longitude λ. We count the number of times each arc winds around the edges a_j, b_j, c_j, accounting for the sign (where we give a positive sign if the arc winds counterclockwise around a vertex, when viewed from outside δN). This yields the following.

Table 1. Signed count of meridian and longitudinal curves for figure eight knot complement.

	a_1	b_1	c_1	a_2	b_2	c_2
μ	0	0	-1	1	0	0
λ	0	0	0	-2	0	2
$x\mu + y\lambda$	0	0	$-x$	$x-2y$	0	$2y$

Hence,

$$I_N(x,y)(q) = \sum_{k\in\mathbb{Z}} q^k J_\Delta(2k,k,-x)J_\Delta(2k+x-2y,k,2y) = \sum_{k\in\mathbb{Z}} I_\Delta(k+x,k)I_\Delta(k-2y,k+x-2y)$$

Here, we can let y be a half-integer since it always appears in the formula as $2y$, thus the 3d index is well-defined although the pair (x,y) no longer represent a curve on the torus boundary. Thus,

$$I_N(x,y)(q) = \sum_{k\in\mathbb{Z}} I_\Delta(k+x,k)I_\Delta(k-y,k+x-y).$$

Example 4.2.3. Below are some example calculations of the 3d index for the figure eight knot complement.

$$I_N(1,0)(q) = -2q - 2q^2 + 2q^3 + 8q^4 + 16q^5 + 16q^6 + 19q^7 - 14q^8 - 52q^9 - 102q^{10} - 154q^{11} + \dots$$
$$I_N(0,1)(q) = q^3 + 2q^4 + 5q^5 + 2q^6 - 3q^7 - 16q^8 - 32q^9 - 52q^{10} - 67q^{11} - 64q^{12} + \dots$$
$$I_N(1,-4)(q) = q^{10} + 3q^{11} + 7q^{12} + 12q^{13} + 20q^{14} + 27q^{15} + 35q^{16} + 35q^{17} + 26q^{18} - 3q^{19} - 55q^{20} + \dots$$

Note that -4 is a half-integer in $I_N(1,-4)(q)$.

5 The 3d Index for Closed Manifolds

We now turn to the extension of the 3d index to closed 3-manifolds. Let N be a 3-manifold with a single torus boundary component. Denote the meridian and longitude of the torus boundary by μ and λ respectively (here, we take N to be a knot complement in $\mathbb{S}^3$ to ensure that μ and λ are well defined). Denote the closed 3-manifold obtained by Dehn filling along the boundary cycle $x\mu + y\lambda$ as $M = N_{x\mu+y\lambda}$, where x, y are necessarily taken coprime.

Definition 5.1. The the 3d index for $N_{x\mu+y\lambda}$ is defined by the following equation [10]

$$I_{N_{x\mu+y\lambda}}(q) = \sum_{(m,e)\in\mathbb{Z}^2} K(m,e,x,y;q) I_N(e,m)$$

where $K(m,e,x,y;q) = \frac{1}{2}(-1)^{rm+2se}[\delta_{xm+2ye,0}(q^{\frac{rm+2se}{2}} + q^{-\frac{rm+2se}{2}}) - \delta_{xm+2ye,-2} - \delta_{xm+2ye,2}]$.

Here, r and s are integers such that $yr - xs = 1$. The choice of (r,s) is not unique, it is in fact invariant under the shift $(r,s) \mapsto (r+x, s+y)$ [10]. Observe that the function $K(m,e,x,y;q)$ means the summation is over three parallel lines $xm + 2ye = 0$, $xm + 2ye = -2$ and $xm + 2ye = 2$.

Gang has made the following conjecture about the 3d index [10].

Conjecture 5.2.

$$I_M(q) = I_{N_{x\mu+y\lambda}}(q) = \begin{cases} \text{Infinite series beginning with } 1 - \dots, & \text{if } M \text{ is hyperbolic} \\ 0,1, \text{ or } \infty \text{ (does not converge)}, & \text{if } M \text{ is non-hyperbolic} \end{cases}$$

Moreover, if M is a Lens space, then $I_M(q) = 0$.

Thus, a (1,0)-filling should produce S^3 for any knot complement, with the resulting 3d index being 0. In this case, $x = 1$ and $y = 0$, so taking $r = 0$ and $s = -1$ yields

$$K(m,e,1,0;q) = \frac{1}{2}[\delta_{m,0}(q^{-e} + q^e) - \delta_{m,-2} - \delta_{m,2}].$$

Putting this together and recalling that e is a half-integer gives

$$2I_{N_\mu} = \sum_e (q^{-e/2} + q^{e/2}) I_N(e,0) - \sum_e I_N(e,-2) - \sum_e I_N(e,2).$$

In order to verify Gang's calculations of the closed 3d index for a (1,0)-filling of the figure eight knot complement, we prove the following theorem, which gives an indication of how many terms we should compute to be certain of our result.

Theorem 5.3. Let N be the figure eight knot complement. In order to be certain of I_{N_μ} up to degree n, one must compute $I_N(e,0)$, $I_N(e,-2)$, and $I_N(e,2)$ up to $e = \max(2n,7)$.

To prove this theorem, we first prove the following lemmas, which immediately imply the result.

Lemma 5.4. For all $|e| \geq 7$, the minimum degree of $I_N(e,0)$ is at least $|e|$.

Proof. Recall that $I_N(e,0) = \sum_e I_\Delta(k+e,k) I_\Delta(k,k+e)$.

Thus, to prove Lemma 5.4, we show that the minimum degree of $I_\Delta(k+e,k)I_\Delta(k,k+e)$ is at least $|e|$ for sufficiently large e. To do this, we use Garoufalidis's work on the minimum degree of the tetrahedron index [5], summarised in the diagram below.

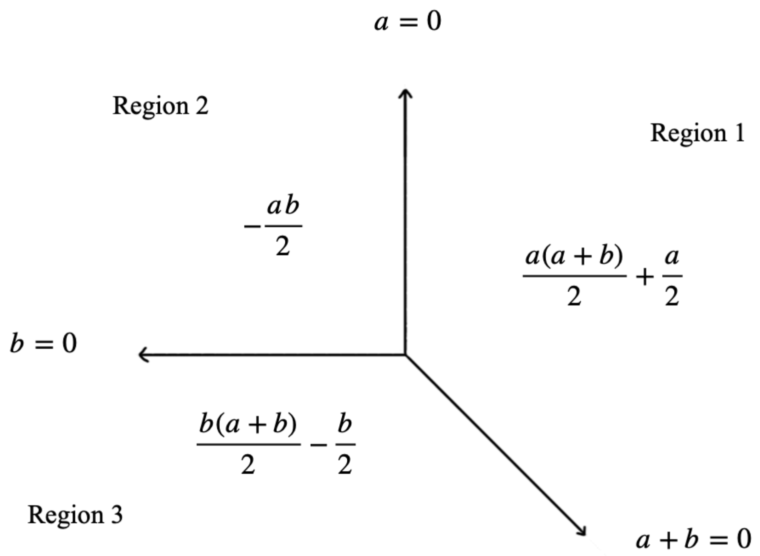

Figure 4. The minimum degree of the tetrahedron index $I_\Delta(a,b)$ [5]. The minimum degree depends on which of the three regions (a,b) lies in, which we refer to as Region 1, 2 and 3, as labelled above.

Now, writing $I_\Delta(a,b)$, we can consider the summation in $I_N(e,0)$ as over the two lines $b = a+e$ and $b = a-e$, where each term is the product of $I_\Delta(a,b)$ with $I_\Delta(b,a)$. We will hereby refer to the line $b = a - e$ as Line 1 and the line $b = a + e$ as Line 2. A diagram of the situation is depicted below.

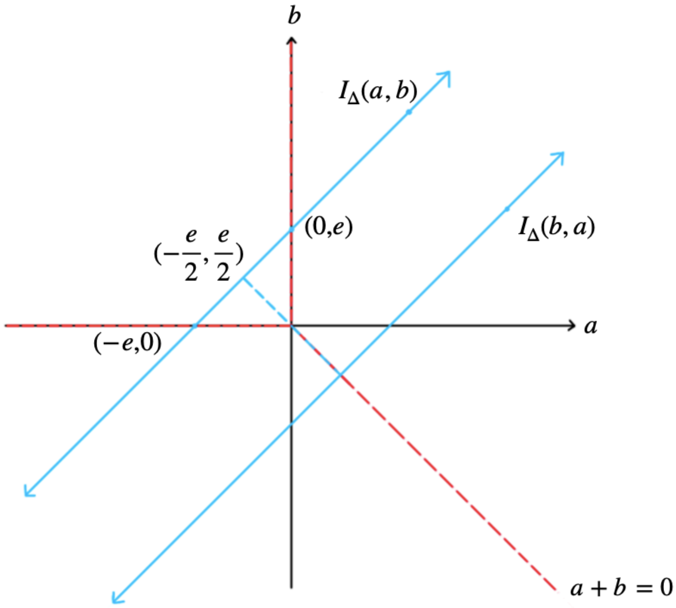

Figure 5. There are four cases: Case 1: $I_\Delta(a,b)$ and $I_\Delta(b,a)$ both are in Region 1; Case 2: $I_\Delta(a,b)$ is in Region 2 while $I_\Delta(b,a)$ is in Region 1; Case 3: $I_\Delta(a,b)$ is in Region 2 while $I_\Delta(b,a)$ is in Region 3; Case 4: $I_\Delta(a,b)$ and $I_\Delta(b,a)$ both lie in Region 3.

There are four relevant cases, depicted in Figure 5. Due to the symmetry of the situation, it suffices to only consider the cases 1 and 2.

Case 1: $a \geq 0, b \geq e$. Both Line 1 and 2 are in Region 1, so the minimum degree of $I_N(e,0)$ is given by

$$\delta_1 = \frac{a(a+b)}{2} + \frac{a}{2} + \frac{b(a+b)}{2} + \frac{b}{2} = \frac{(a+b)^2}{2} + \frac{a+b}{2} \geq \frac{b^2}{2} + \frac{b}{2} > |e|$$

since $b \geq e$ and $b \geq 1$.

Case 2: $-\frac{e}{2} \leq a \leq 0, \frac{e}{2} \leq b \leq e$. Now, Line 1 is in Region 1 while Line 2 is in Region 2. Thus, the minimum degree is

$$\delta_2 = -\frac{ba}{2} + \frac{b(a+b)}{2} + \frac{b}{2} = \frac{b^2}{2} + \frac{b}{2}.$$

This function is a quadratic in b so within the region we are considering, it achieves its minimum at $b = \frac{e}{2}$. Thus,

$$\delta_2 \geq \frac{b^2}{8} + \frac{b}{4} > |e| \text{ for all } e \geq 7.$$

The stated result follows.

Analogous results can be proven for $I_N(e,-2)$ and $I_N(e,2)$. The proofs are omitted here and included in the appendix.

Lemma 5.5. For all $|e| \geq 7$, the minimum degree of $I_N(e,-2)$ is at least $|e|/2$.

Lemma 5.6. For all $|e| \geq 7$, the minimum degree of $I_N(e,2)$ is at least $|e|/2$.

Observe that Lemma 5.4 implies the minimum degree of $(q^{-e/2}+q^{e/2})I_N(e,0)$ is at least $|e|/2$ for $|e| \geq 7$. Thus, Lemmas 5.4, 5.5 and 5.6 immediately imply Theorem 5.3.

Next we give an indication of how many terms must be computed in each open 3d index.

Lemma 5.7. Let N be the figure eight knot complement. In order to be certain of $I_N(x,y)$ up to degree n, one should sum for $|k| \leq 2n+2|x|$.

Proof. Recall that

$$I_N(x,y) = \sum_k I_\Delta(k+x,k)I_\Delta(k-y,k+x-y).$$

Since $I_\Delta(m,e)$ has non-negative degrees, the minimum degree of $I_N(x,y)$ is at least the minimum degree of $I_\Delta(k+x,k)$. Observe that if $k > 2|x|$ or $k < -2|x|$, then the minimum degree of $I_\Delta(k+x,k)$ is in Region 1 or 3 respectively. First suppose $k > 2|x|$. Then the minimum degree of $I_\Delta(k+x,k)$ is given by

$$\delta_1 = \frac{(k+x)(2k+x)}{2} + \frac{k+x}{2} > \frac{k}{2}.$$

Now suppose $k < -2|x|$. Then the minimum degree is

$$\delta_2 = \frac{k(2k+x)}{2} - \frac{k}{2} > \frac{|k|}{2}.$$

The statement follows.

Using the MATLAB code included in the appendix, we calculated the closed 3d index for a (1,0)-filling of the figure eight knot complement. Our result is a power series with the first term $-6q^{14}$. Here we give the specific inputs used in this calculation:

```
>> A=pochhammer_matrix(150,150,150);B=tetmatrix(-100,100,50,A);filledIndexfig8(1,0,30,50,B)
```

Figure 6. Inputs used to compute I_{N_μ}, where N is figure eight knot complement.

Note that pochhammer_matrix(150,150,150) computes the product of 150 pochhammer symbols, tetmatrix(-100,100,50,A) computes the tetrahedron index for $m \in [-100,0]$, $e \in [0,100]$, summing from $n = 0$ to $n = 50$ in each tetrahedron index, filledIndexfig8(1,0,30,50,B) computes the closed 3d index for the figure eight knot complement, summing over $m \in [-30,30]$, $e \in [-30,30]$, also summing from $k = -30$ to 30 in each open 3d index, using 50 terms in each open 3d index. The appendix contains more details on the input values.

Since we have summed $e \in [-30,30]$, Theorem 5.3 implies I_{N_μ} is correct up to degree 15, provided each of the open 3d indices are accurate up to degree 15. Now, each of the open 3d indices $I_N(e,0), I_N(e,-2),$ and $I_N(e,2)$ are summed from $k = -30$ to 30, thus by Lemma 5.7 we have that each of the open 3d indices are correct up to degree 5 for $|e| \leq 10$. Observe that the minimum degree of $I_N(e,0), I_N(e,-2),$ and $I_N(e,2)$ are at least 5 for $e > 10$ by Lemmas 5.4-5.6. Since the first term in our result was $-6q^{14}$, we have the following theorem.

Theorem 5.8. Let N be the figure eight knot complement. Then I_{N_μ} is zero up to degree 5.

The first input pochhammer_matrix(150,150,150) took approximately 6 hours to compute. Contrastingly, the second and third input took anywhere from 10-30 seconds to compute, indicating that once one has stored a large amount of pochhammer symbols, the 3d indices can be computed relatively quickly.

Our results provide some verification of Gang's conjecture, however little can be said on why the cancellation indeed occurs.

6 Discussion & Conclusions

The major result of this project is that to be certain up to degree n of the (1,0)-filling of the figure eight knot complement I_{N_μ}, where N is the figure eight knot complement, one must compute each of the open 3d indices $I_N(e,0), I_N(e,-2),$ and $I_N(e,2)$ up to $e = \max(2n,7)$ (Theorem 5.3). Using this, we have computed that I_{N_μ} is zero up to degree 5. Although this is a small step towards the numerical verification of Gang's conjecture, the result shows support for the conjectural vanishing of a Lens space. The computation was limited by the computing power available; the use of a more capable machine may be able to verify the conjecture to a much higher degree for the figure eight knot complement. Likewise, the bounds for the summation required to be certain of $I_N(e,0), I_N(e,-2),$ and $I_N(e,2)$ up to degree n given in this report are quite conservative; improved bounds could allow the conjecture to be verified to a higher degree.

In terms of direct avenues forward from this report, extending the MATLAB code used to compute the closed 3d index for a figure eight knot complement to other knot complements would allow further calculations on (x, y)-filling, to thus numerically verify Gang's conjecture for a variety of knots. Although the code can be run in reasonable time once a matrix of q-pochhammer symbols are stored, the use of symbolic calculations in the two functions fig8index and filledIndexfig8 (used to calculate the open and closed 3d indices for the figure eight knot complement) are thought to be making the computations slow. Using matrix computations throughout would potentially improve the efficiency. Moreover, the MATLAB code is currently capable of computing any (x, y)-filling of the figure eight knot complement; performing analysis similar to that used to prove Theorem 5.3 on the closed 3d index of the figure eight knot complement may allow similar results to be proven for general (x, y)-fillings.

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7 Appendix

7.1 Proof of Lemmas 5.5 & 5.6.

Lemma 5.5. For all $|e| \geq 7$, the minimum degree of $I_N(e,-2)$ is at least $|e|/2$.

Proof. Recall that

$$I_N(e,-2) = \sum_e I_\Delta(k+e,k)I_\Delta(k+2,k+e+2).$$

Write $I_\Delta(a,b)$, then this summation can be considered as one over the two lines $b = a + e$ and $b = a - e$, where each term is the product of $I_\Delta(a,b)$ with $I_\Delta(b+2,a+2)$. Again, we refer to the line $b = a - e$ as Line 1 and the line $b = a + e$ as Line 2. First consider the case $e \geq 0$. A diagram for this situation is depicted below.

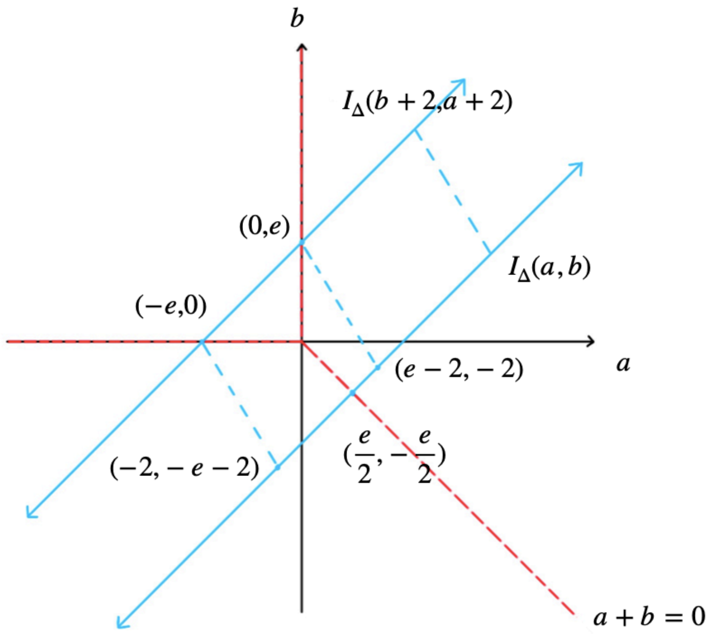

Figure 7. The key transitions from Region 1 to Region 2, and Region 2 to Region 3 occur along Line 2 at the points $(0,e)$ and $(-e,0)$ respectively, corresponding to the points $(e-2,-2)$ and $(-2,-e-2)$ on Line 1. The transition from Region 1 to Region 3 occurs along Line 1 at the point $(\frac{e}{2},-\frac{e}{2})$.

Observe that if $e \geq 4$,then $e - 2 \geq \frac{e}{2}$, so the transition from Region 1 to Region 2 along Line 2 occurs before the transition from Region 1 to Region 3 along Line 1. Since we seek to prove Lemma 5.5 for $e \geq 7$, it suffices to only consider the case $e \geq 4$.

Case 1: $a \geq e-2, b \geq -2$. In this region, both Line 1 and Line 2 lie in Region 1. Thus, the minimum degree of $I_N(e,-2)$ is given by

$$\delta_1 = \frac{a(a+b)}{2} + \frac{a}{2} + \frac{(b+2)(a+b+4)}{2} + \frac{b+2}{2} = \frac{a^2}{2} + \frac{b^2}{2} + ab + \frac{3}{2}a + \frac{7}{2}b + 5.$$

Observe that $\nabla\delta_1 = (a+b+\frac{3}{2}, b+a+\frac{7}{2}) = \mathbf{0}$ has no solutions so δ_1 has no critical points.

Thus, its minimum is achieved at the boundary, $(a,b) = (e-2,-2)$. Hence,

$$\delta_1 \geq \frac{e^2}{2} - \frac{5}{2}e + 3 > \frac{|e|}{2} \text{ for all } e \geq 5.$$

Case 2: $\frac{e}{2} \leq a \leq e-2, -\frac{e}{2} \leq b \leq -2$. Now, Line 1 still lies in Region 1 but Line 2 is in Region 2. Thus, the minimum degree is

$$\delta_2 = \frac{a(a+b)}{2} + \frac{a}{2} - \frac{(a+2)(b+2)}{2} = \frac{a^2}{2} - \frac{a}{2} - b - 2.$$

This function again has no critical points and varies quadratically in a and linear in b, so its minimum is achieved at $(a,b) = (\frac{e}{2},-2)$. Thus, $\delta_2 \geq \frac{e^2}{8} - \frac{e}{4} > \frac{|e|}{2}$ for all $e \geq 7$.

Case 3: $-2 \leq a \leq \frac{e}{2}$, $-e-2 \leq b \leq -\frac{e}{2}$. Now, Line 1 is in Region 3 and Line 2 is in Region 2, so the minimum degree is

$$\delta_3 = \frac{b(a+b)}{2} - \frac{b}{2} - \frac{(a+2)(b+2)}{2} = \frac{b^2}{2} - \frac{3}{2}b - a - 2.$$

This function again has no critical points and varies quadratically in b and linear in a, so its minimum is at $(a,b) = (\frac{e}{2}, -\frac{e}{2})$. So $\delta_4 \geq \frac{e^2}{8} + \frac{e}{4} - 2 > \frac{|e|}{2}$ for all $e \geq 6$.

Case 4: $a \leq -2, b \leq -e-2$. Both lines are now in Region 3, so the minimum degree is

$$\delta_4 = \frac{b(a+b)}{2} - \frac{b}{2} + \frac{(a+2)(a+b+4)}{2} - \frac{a+2}{2} = \frac{a^2}{2} + \frac{b^2}{2} + ab + \frac{5}{2}a + \frac{b}{2} + 3.$$

Again, this function has no critical points so the minimum occurs at the boundary,

$$\delta_4 \geq \frac{e^2}{2} + \frac{7}{2}e + 5 > \frac{|e|}{2} \text{ for all } e \geq 0.$$

The case for $e \leq 0$ can be done analogously and are in fact symmetric. The statement follows.

Lemma 5.6. For all $|e| \geq 7$, the minimum degree of $I_N(e,2)$ is at least $|e|/2$.

Proof. This proof can be rigorously done analogously to the proof of Lemma 5.5, again realising that we are summing over Line 1 ($b = a - e$) and Line 2 ($b = a + e$), however this time taking the products of $I_\Delta(a, b)$ with $I_\Delta(b-2, a-2)$. Alternatively, we can note that by symmetry to $I_N(e, -2)$, the cases $e \geq 0$ of $I_N(e,2)$ correspond to the cases $e \leq 0$ in $I_N(e, -2)$ and visa versa, so Lemma 5.5 implies Lemma 5.6.

7.2 Equivalence of Knots

The notion of equivalence of knots can be characterised by their knot diagrams and the following theorem.

Theorem 7.2.1. If two knots K and K' have the same knot diagram, then they are equivalent [1].

This notion of equivalence satisfies the definition of an equivalence relation; it is transitive, symmetric and reflexive. In general, it is difficult to determine when two knots lie in the same equivalence class or are in fact equivalent; much of knot theory focuses on developing techniques which can be used to make this decision.

Reidemeister moves are also used to characterise equivalence.

Definition 7.2.2. A *Reidemeister move* is an operation which can be performed on a knot diagram without altering the knot itself, and consists of one of the following moves.

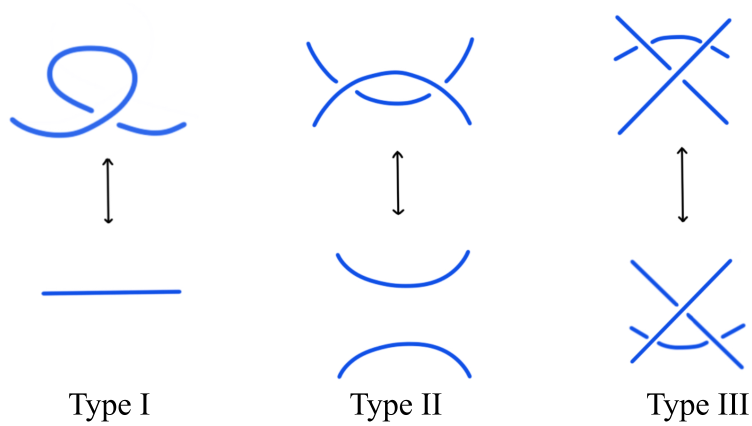

Figure 8. Reidemeister moves. Type I: twist and untwist in either direction, type 11: move one loop completely over another, type 111: move a string completely over or under a crossing [1].

Reidemeister moves correspond to the simplest changes when a knot is deformed. We also have the following theorem.

Theorem 7.2.3. If two knots are equivalent, then they are related by a sequence of Reidemeister moves [1].

7.3 Hyperbolic Manifolds

Here, some background theory on hyperbolic manifolds is provided.

Definition 7.3.1. A compact 3-manifold M is said to have a (complete) hyperbolic structure if $M - \delta M$ has a complete Riemannian metric that is locally isometric to $\mathbb{H}^3$ [9].

Hyperbolic geometry does not satisfy the parallel axiom and has many representations, including the Poincare disk model and the Klein model. For example, in the Poincare disk model, which models 2-dimensional hyperbolic geometry, hyperbolic straight lines appear as arcs of circles orthogonal to the boundary of the disk. There are a variety of important results concerning hyperbolic manifolds, including that most knot complements have hyperbolic geometry, specifically,

Theorem 7.3.2. Let N be the exterior of a knot K in the 3-sphere. Then N has hyperbolic structure if and only if K is neither a satellite knot nor a torus knot [9].

Here, a torus knot is a knot which lies on the surface of an unkotted torus in $\mathbb{R}^3$, while a satellite knot is a knot which contains an incompressible, non boundary-parallel torus in its complement.

7.4 Code used to Calculate 3d Index

All code is written in MATLAB. Note that the function tetindex uses the 3-fold symmetry of the tetrahedron index to compute the tetrahedron index from the indices stored via tetmatrix.

```
function [A] = qseries(i,k)
% Computes the first k terms of power series 1/(1-q^i)
% Gives result as matrix
if i==0
    A=[1,zeros(1,k-1)];
else
    A=zeros(1,k);
    for j=1:k
        A(2*i*(j-1)+1)=1;
    end
end
end
```

```
function [B]=qpochhammer(n,k)
% Computes 1/(q)_n with k terms in each power series 1/(1-q^i)
% Gives result as matrix
if n==0
    B=qseries(0,k);
else
    B=qseries(1,k);
    for i=2:n
        C=qseries(i,k);
        B=conv(B,C);
    end
end
end
```

```
function [matrix]=pochhammer_matrix(p,l,k)
% Computes matrix with entries as products of pochhammer symbols.
% Computes the products up to (q)_p*(q)_p, with up to q^l in each product.
% Each (q)_n is calculated with k terms in each power series 1/(1-q^i).
% matrix(i,:,j) gives (q)_(i-1)*(q)_(j-1)
matrix=zeros(p+1,2*l+1,'int64');
for i=1:p+1
    if i==1
        entryi=qpochhammer(0,k);
        matrix(1,1:min(2*l+1,length(entryi)),1)=entryi(1:min(2*l+1,length(entryi)));
    else
        entryi=conv(prev_entryi,qseries(i-1,k));
        matrix(i,1:min(2*l+1,length(entryi)),1)=entryi(1:min(2*l+1,length(entryi)));
    end
    prev_entryj=entryi;
    for j=2:p+1
        entryj=conv(prev_entryj,qseries(j-1,k));
        matrix(i,1:min(2*l+1,length(entryj)),j)=entryj(1:min(2*l+1,length(entryj)));
        prev_entryj=entryj;
    end
    prev_entryi=entryi;
end
end
```

```
function [matrix]=tetmatrix(mval,eval,l,qMatrix)
% Computes tetrahedron index in quadrant with m<=0 and e>=0 (thus, a
% negative m values is expected and a positive e value).
% Gives result as matrix.
% Input argument qMatrix is expected to a be matrix of products of
% pochhammer symbols.
% Sums from n=0 to n=l.
matrix=zeros(-mval+1,'int64');
len=size(qMatrix,2);
for m=mval:0
    for e=0:eval
        index=zeros(1,len,'int64');
        for n=0:l
            a_n=(-1)^n*qMatrix(n+1,:,n+e+1);
```

```
                shift=min(n*(n+1)-(2*n+e)*m,len);
                a_n=[zeros(1,shift),a_n(1:len-shift)];
                index=index+a_n;
            end
            matrix(-m+1,1:length(index),e+1)=index;
        end
    end
end
```

```
function [index]=tetindex(m,e,tetMatrix)
% gives the tetrahedron index I(m,e), caculated from the 3-fold symmetry
% using the indices stored in tetMatrix.
if e>=0 && m<=0
    index=tetMatrix(-m+1,:,e+1);
elseif e>=0 && m>=0
    index=tetMatrix(-(-e-m)+1,:,m+1);
    shift=min(m,length(index));
    index=(-1)^m*[zeros(1,shift),index(1:length(index)-shift)];
elseif e<=0 && m<=0
    index=tetMatrix(-e+1,:,-e-m+1);
    shift=min(-e,length(index));
    index=(-1)^(-e)*[zeros(1,shift),index(1:length(index)-shift)];
elseif e<=0 && m>=0 && abs(e)>=m
    index=tetMatrix(-e+1,:,-e-m+1);
    shift=min(-e,length(index));
    index=(-1)^(-e)*[zeros(1,shift),index(1:length(index)-shift)];
else
    index=tetMatrix(-(-e-m)+1,:,m+1);
    shift=min(m,length(index));
    index=(-1)^m*[zeros(1,shift),index(1:length(index)-shift)];
end
end
```

```
function [index2]=fig8index(x,y,n,l,tetMatrix)
% Calculates the 3d index for the figure 8 knot complement along the curve
% x*mu+y/2*lambda. Sums from -n to n. Gives the first l terms.
% Requires matrix with tetrahedron indices as input.
% The result is a series in x^1/2.
index=zeros(1,size(tetMatrix,2));
for k=-n:n
    term=conv(tetindex(k-x,k,tetMatrix),tetindex(k+y,k-x+y,tetMatrix));
    index=index+term(1:length(index));
end
syms x;
index2=0;
for i=1:(l+1)
    index2=index2+index(i)*x^((i-1));
end
end
```

```
function [filledIndex]=filledIndexfig8(p,q,n,l,tetMatrix)
% Calculates the 3d index of the figure eight knot complement
% obtained by performing a Dehn filling along the boundary cycle p*mu+q*lambda.
% Sums over -n to n for values of m and e. Also sums over -n to n for each
% open index, taking l terms in each open index.
% Requires matrix with tetrahedron indices as input.
% The result is a series in x^1/2.
[G,r,s] = gcd(q,-p);
filledIndex=0;
syms x;
for m=-n:n
    for e=-n:n
        if p*m+2*q*e==0||p*m+2*q*e==-2||p*m+2*q*e==2
            index=fig8index(e,m,n,l,tetMatrix);
            if p*m+2*q*e==0
                term=expand((x^(r*m+2*s*e)+x^(-r*m-2*s*e))*index);
            end
            if p*m+2*q*e==-2
                term=(-1)*index;
            end
            if p*m+2*q*e==2
                term=(-1)*index;
            end
            term=(1/2)*(-1)^(r*m+2*s*e)*term;
            filledIndex=filledIndex+term;
        end
    end
end
filledIndex=expand(filledIndex);
end
```